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ON PROJECTIVE RICCI CURVATURE OF MATSUMOTO METRICS

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Abstract. In this paper, we study Finsler metrics with weak, isotropic and flat projective Ricci curvature (briefly, **PRic**-curvature). First, we prove a rigidity result that shows that for a complete Finsler manifold, inequality $\mathbf{PRic} \geq \mathbf{Ric}$ holds if and only if $\mathbf{S} = 0$. Then, we show that the Mstsumoto metric is of weak **PRic**-curvature if and only if it is a **PRic**-flat metric. We characterize projective Ricci flat Matsumoto metrics with constant length one-forms. In this case, we show that the Matsumoto metric reduces to a Ricci flat metric. Finally, we prove that a Matsumoto metric is **PRic**-reversible if and only if it is **PRic**-quadratic.

1. Introduction

One of the important problems in Finsler geometry is to understand the geometric meanings of various Riemannian and non-Riemannian quantities and their impacts on the global geometric structures (see [**7, 8, 15**]). The flag curvature $\mathbf{K} = \mathbf{K}(x, y, P)$ is a natural extension of the sectional curvature $\mathbf{K} = \mathbf{K}(x, P)$ in Riemannian geometry which tells us how curved the Finsler manifold is at a point [**18**]. The Ricci curvature is defined as the trace of Riemannian curvature. The well-known Ricci tensor was introduced by G. Ricci. The Ricci curvature tensor represents the amount by which the volume of a small wedge of a geodesic ball in a curved Riemannian manifold deviates from that of the standard ball in Euclidean space.

The Ricci curvature has deep relation with the S-curvature. The S-curvature $\mathbf{S} = \mathbf{S}(x, y)$ is an important non-Riemannian quantity which was constructed by Z. Shen for given comparison theorems on Finsler manifolds. It is interesting to consider the geometric quantities derived from Ricci curvature and S-curvature. In [**11**], Shen considered the projective spray $\widetilde{\mathbf{G}}$ associated with a given spray $\mathbf{G}$ on an n-dimensional manifold M which is defined by $\mathbf{G}$ and its S-curvature $\mathbf{S}$ as follows:

$$\widetilde{\mathbf{G}} = \mathbf{G} + \frac{2\mathbf{S}}{n+1}\mathbf{Y},$$

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where $\mathbf{Y} := y^i \frac{\partial}{\partial y^i}$ is the vertical radial field on TM. Then $\widetilde{\mathbf{G}}$ is a projectively invariant. It is easy to see that the Ricci curvature $\widetilde{\mathbf{Ric}}$ of $\widetilde{\mathbf{G}}$ is given by

$$\widetilde{\mathbf{Ric}} = \mathbf{Ric} + \frac{n-1}{n+1}\mathbf{S}_{\|i}y^i + \frac{n-1}{(n+1)^2}\mathbf{S}^2,$$

where "$\|$ " denotes the horizontal covariant derivative with respect to the Berwarld connection of $\mathbf{G}$.

Let (M, F) be an n-dimensional Finsler manifold. Recently, Cheng-Shen-Ma [**5**] defined the concept of projective Ricci curvature for a Finsler metric F by

$$\mathbf{PRic} = \mathbf{Ric} + \frac{n-1}{n+1}\mathbf{S}_{|i}y^i + \frac{n-1}{(n+1)^2}\mathbf{S}^2, \tag{1}$$

where "$|$ " denotes the horizontal covariant derivative with respect to the Berwarld connection of F. It is easy to show that if two Finsler metrics are pointwise projectively related Finsler metrics on a manifold with a fixed volume form, then their projective Ricci curvature are equal. In other words, the projective Ricci curvatures is projective invariant with respect to a fixed volume form. Also, the projective Ricci curvature is actually a kind of weighted Ricci curvatures [**9, 14**]. However, the projective Ricci curvature can be defined for a Finsler metric F and an independent volume form $\mathrm{d}V$. Recently, Shen-Sun [**13**] consider the projective Ricci curvature for a pair $(F, \mathrm{d}V)$ not F only with $\mathrm{d}V = \mathrm{d}V_F$. In this paper, we prove the following rigidity result for the complete Finsler manifolds.

Theorem 1.1. *Let (M, F) be a complete Finsler manifold. Then* $\mathbf{PRic} \geq \mathbf{Ric}$ *if and only if* $\mathbf{S} = 0$.

A Finsler metric F on an n-dimensional manifold M is called weak projective Ricci curvature, weak **PRic**-curvature for short, if $\mathbf{PRic} = (n-1)[3\theta + \kappa F]F$, where $\theta = \theta_i(x)y^i$ is a 1-form and $\kappa = \kappa(x)$ is a scalar function on M. If $\theta = 0$, then F is called isotropic projective Ricci curvature or briefly isotropic **PRic**-curvature. F is called constant projective Ricci curvature if $\mathbf{PRic} = (n-1)cF^2$, where c is a real constant. If $c = 0$, then F is called a projective Ricci flat metric or **PRic**-flat metric. In [**5**], Cheng-Shen-Ma characterized projective Ricci flat Randers metrics. Later, Cheng and the second author wrote the modification to this paper and corrected the results [**3**]. In [**28**], Zhu and Zhang studied the projective Ricci curvature and characterized projective Ricci flat spherically symmetric Finsler metrics.

In order to find the Finsler metrics with weak (and isotropic) Ricci curvature, we consider the Matsumoto metric. The Matsumoto metric was first introduced by Matsumoto in order to study the time it takes to negotiate any given path on a hill side. It is the Matsumoto's slope-of-a-mountain metric. A slope, the graph of a function $z = f(x, y)$, of the earth surface is regarded as a two-dimensional Finsler space with the fundamental function $F(z, y, \dot{x}, \dot{y}) = \alpha^2/(c_1\alpha - c_2\beta)$, where c_1 and c_2 are non-zero real constants, $\alpha^2 := \dot{x}^2 + \dot{y}^2 + (\dot{x}f_x + \dot{y}f_y)^2$ and $\beta := \dot{x}f_x + \dot{y}f_y$. Here, α is the usual induced Riemannian metric and β is a derived form, $\beta(x, \mathrm{d}x) = \mathrm{d}f(x, y)$. The two constants c_1 and c_2 are such that one can walk

c_1 meters per minute on the horizontal plane and $2c_2$ is equal to the acceleration of falling. Aikou-Hashiguchi-Yamauchi generalized and normalized the above metric as follows:

$$F = \frac{\alpha^2}{\alpha - \beta}.$$

Many authors have studied this metric from different perspectives (see [**1, 10, 17, 22, 23, 26, 27**]).

In this paper, we study Matsumoto metrics with weak (and isotropic) Ricci curvature and prove the equivalency of these notions. In this case, the metrics are actually projective Ricci flat Finsler metrics.

Theorem 1.2. *The Matsumoto metric is of weak* **PRic***-curvature if and only if it is a* **PRic***-flat metric.*

As a natural application, we characterize projective Ricci flat Matsumoto metrics with constant length one-forms. We show that these metrics are Ricci-flat metrics.

Theorem 1.3. *Let $F = \alpha^2/(\alpha - \beta)$ be a non-Riemannian Matsumoto metric on a manifold M of dimension $n \geq 3$. Suppose that β has constant length with respect to α. Then F is of isotropic projective Ricci curvature $\mathbf{PRic} = (n-1)\kappa F^2$ for a scalar function $\kappa = \kappa(x)$ on M if and only if α is Ricci-flat and β is parallel with respect α. In this case, F is a Ricci-flat metric.*

A Finsler metric (F, dV) on a manifold M is called **PRic**-reversible if $\mathbf{PRic}(y) = \mathbf{PRic}(-y)$. $(F, \mathrm{d}V)$ is called **PRic**-quadratic if its **PRic**-curvature is quadratic in $y \in T_xM$. Finally, we consider **PRic**-reversible and **PRic**-quadratic Matsumoto metric. Then, we prove the following.

Theorem 1.4. *The Matsumoto metric is* **PRic***-reversible if and only if it is* **PRic***-quadratic.*

2. Preliminaries

Let (M, F) be an n-dimensional Finsler manifold. A global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate system (x^i, y^i), for TM_0, is given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4} g^{il} \Big[\frac{\partial^2 (F^2)}{\partial x^k \partial y^l} y^k - \frac{\partial (F^2)}{\partial x^l} \Big], \qquad y \in T_xM. \tag{2}$$

The vector field $\mathbf{G}$ is called the associated spray to (M, F) and G^i are called the spray coefficients.

For a Finsler metric $F = F(x, y)$ on an n-dimensional manifold M, the Busemann-Hausdorff volume form $\mathrm{d}V_F = \sigma_F(x)\mathrm{d}x^1 \ldots \mathrm{d}x^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbf{B}^n(1))}{\mathrm{Vol}\Big\{ (y^i) \in R^n \;\Big|\; F\Big(y^i \frac{\partial}{\partial x^i}|_x \Big) < 1 \Big\}}.$$

Let G^i denote the geodesic coefficients of F in the same local coordinate system. The S-curvature can be defined by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x,y) - y^i \frac{\partial}{\partial x^i}\Big[\ln \sigma_F(x)\Big], \tag{3}$$

where $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|x \in T_xM$. The Finsler metric is said to be of isotropic S-curvature if

$$\mathbf{S} = (n+1)cF,$$

where $c = c(x)$ is a scalar function on M (see [**6, 18**]).

For $y \in T_xM_0$, the Riemann curvature is a family of linear transformation $\mathbf{R}_y : T_xM \to T_xM$ which is defined by $\mathbf{R}_y(u) := R^i{}_k(y)u^k \frac{\partial}{\partial x^i}$, where

$$R^i{}_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{4}$$

The family $\mathbf{R} := \{\mathbf{R}_y\}_{y \in TM_0}$ is called the Riemann curvature.

The Ricci curvature $\mathbf{Ric}(x,y)$ is the trace of the Riemann curvature defined by $\mathbf{Ric}(x,y) := R^m{}_m(x,y)$. A metric F on an n-dimensional manifold M is called a weakly Einstein metric if

$$\mathbf{Ric} = (n-1)\Big(\kappa + \frac{3\theta}{F}\Big)F^2, \tag{5}$$

where $\kappa = \kappa(x)$ is a scalar function and $\theta = \theta_i(x)y^i$ is a 1-form on M. If $\theta = 0$, then F is called an Einstein metric.

Let (M,F) be an n-dimensional Finsler manifold. The projective Ricci curvature of F is defined by

$$\mathbf{PRic} = \mathbf{Ric} + \frac{n-1}{n+1}\mathbf{S}_{|i}y^i + \frac{n-1}{(n+1)^2}\mathbf{S}^2, \tag{6}$$

where "| " denotes the horizontal covariant derivative with respect to the Berwarld connection of F (see [**5**]). F is called weak projective Ricci curvature if

$$\mathbf{PRic} = (n-1)\Big[\frac{3\theta}{F} + \kappa\Big]F^2,$$

where $\theta = \theta_i(x)y^i$ is a 1-form and $\kappa = \kappa(x)$ is a scalar function on M. If $\theta = 0$, then F is called isotropic projective Ricci curvature $\mathbf{PRic} = (n-1)\kappa F^2$. If $\kappa =$ constant, then F is called constant projective Ricci curvature.

3. Examples

In this section, we give some examples of projective Ricci flat, constant, isotropic, or weak projective Ricci curvature Finsler metrics.

Example 3.1. Every Ricci-flat Kropina metric is a Berwald metric (see [**25**]). Berwald metrics have vanishing S-curvature. Thus a Ricci-flat Kropina metric satisfies $\mathbf{PRic} = 0$.

Example 3.2. Let $\alpha_1 = \sqrt{a_{ij}(x)y^iy^j}$ and $\alpha_2 = \sqrt{\bar{a}_{ij}(x)y^iy^j}$ be two Ricci-flat Riemannian metrics on the manifolds M_1 and M_2, respectively. Consider the following 4-th root metric

$$F := \sqrt[4]{\alpha_1^4 + 2c\alpha_1^2\alpha_2^2 + \alpha_2^4}.$$

This is a Ricci-flat ($\mathbf{Ric} = 0$) and Berwald metric on $M := M_1 \times M_2$. Thus F is a non-Riemannian Finsler metric with projective Ricci flat curvature $\mathbf{PRic} = 0$.

Below, we give two well-known Finsler metrics which have constant projective Ricci curvature.

Example 3.3. Given a Finsler metric Φ and a vector field $\mathbf{v}$ on a manifold M, define a function $F\colon TM \to [0, \infty)$ by

$$\Phi\Big(\frac{\mathbf{y}}{F(\mathbf{y})} - \epsilon\mathbf{v}_p\Big) = 1, \qquad \mathbf{y} \in T_pM, \tag{7}$$

where ϵ is a constant. F is a Finsler metric when ϵ is small. Now, express the spherical metric in a radial form $\Phi(\mathbf{y}) = \sqrt{u^2 + \sin^2(r)v^2}$, where $\mathbf{y} = u\frac{\partial}{\partial r} + v\frac{\partial}{\partial \theta} \in T_{(r,\theta)}\big((0,\infty) \times \mathbb{S}^1\big)$. Take $\mathbf{v} = \frac{\partial}{\partial \theta} \in T_{(r,\theta)}\big((0,\infty) \times \mathbb{S}^1\big)$ and define F by (7). We obtain

$$F = \frac{\sqrt{\Big(1 - \varepsilon^2\sin^2(r)\Big)u^2 + \sin^2(r)v^2} - \varepsilon\sin^2(r)v}{1 - \varepsilon^2\sin^2(r)}. \tag{8}$$

F satisfies that $\mathbf{K} = 1$ and $\mathbf{S} = 0$. Thus F has constant projective Ricci curvature with $\kappa = 1$.

Example 3.4. Denote generic tangent vectors on S^3 as

$$u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

The Finsler function for Bao-Shen's Randers space is given by

$$F(x, y, z; u, v, w) = \alpha(x, y, z; u, v, w) + \beta(x, y, z; u, v, w),$$

with

$$\alpha = \frac{\sqrt{K(cu - zv + yw)^2 + (zu + cv - xw)^2 + (-yu + xv + cw)^2}}{1 + x^2 + y^2 + z^2},$$

$$\beta = \frac{\pm\sqrt{K-1}\,(cu - zv + yw)}{1 + x^2 + y^2 + z^2},$$

where $K > 1$ is a real constant. The family of Randers metrics on S^3 constructed by Bao-Shen satisfies $\mathbf{S} = 0$. Since these metrics are of constant flag curvature $\mathbf{K}$, then $\mathbf{Ric} = 2\mathbf{K}F^2$. Thus Bao-Shen's metrics have constant projective Ricci curvature with $\kappa = \mathbf{K} =$ constant.

Now, we give an example of isotropic projective Ricci curvature.

Example 3.5. Every Einstein Finsler metric $\mathbf{Ric} = (n-1)\lambda F^2$, $\lambda = \lambda(x)$, with vanishing S-curvature is of isotropic projective Ricci curvature. It is remarkable that every Einstein Kropina metric has vanishing S-curvature (see [**25**]). Thus an Einstein Kropina metric has isotropic projective Ricci curvature $\kappa = \lambda$.

Here, we present some Finsler metrics of weak projective Ricci curvature.

Example 3.6. For an constant number $a \in R^n$, let us define the Randers metric $F := \alpha + \beta$ by

$$\alpha := \frac{\sqrt{(1-|a|^2|x|^4)|y|^2 + (|x|^2\langle a,y\rangle - 2\langle a,x\rangle\langle x,y\rangle)^2}}{1-|a|^2|x|^4},$$
$$\beta := \frac{|x|^2\langle a,y\rangle - 2\langle a,x\rangle\langle x,y\rangle}{1-|a|^2|x|^4}.$$

This Randers metric satisfies

$$\mathbf{S} = (n+1)cF \qquad \text{and} \qquad \mathbf{Ric} = (n-1)(3c_0F + \rho F^2),$$

where

$$c := \langle a,x\rangle, \qquad c_0 := c_{x^m}y^m, \qquad \rho := 3\langle a,x\rangle^2 - 2|a|^2|x|^2.$$

(See [**4**]). Then

$$\mathbf{PRic} = (n-1)\Big[\frac{4c_0}{F} + c^2 + \rho\Big]F^2.$$

Thus F is of weak projective Ricci curvature with $\theta = 4c_0/3$ and $\kappa = c^2 + \rho$.

Example 3.7. Every two-dimensional Finsler manifold (M,F) is of scalar flag curvature. It is proved that Finsler surface of isotropic S-curvature $\mathbf{S} = 3c(x)F$ has the following flag curvature

$$\mathbf{K} = \frac{3c_{x^m}y^m}{F} + \rho,$$

where $\rho = \rho(x)$ is a scalar function on M [**2**]. In this case, F satisfies $\mathbf{Ric} = 2cF$, and then it is of weak projective Ricci curvature $\theta = 4c_0/3$ and $\kappa = \rho + c^2$.

Example 3.8. Let $F = \alpha+\beta$ be a Randers metric on an n-dimensional manifold M of weak isotropic flag curvature

$$\mathbf{K} = \frac{3c_{x^m}y^m}{F} + \rho,$$

where $c = c(x)$ and $\sigma = \sigma(x)$ are scalar function on M. In [**12**], Shen-Yildrim proved that F is of isotropic S-curvature $\mathbf{S} = (n+1)c(x)F$ if and only if it has weak isotropic flag curvature. Thus F is of weak projective Ricci curvature with $\theta = 4c_0/3$ and $\kappa = \rho + c^2$.

4. Proof of Theorem 1.1

Suppose that F is of isotropic S-curvature $\mathbf{S} = (n+1)cF$, where $c = c(x)$ is a scalar function on M. In this case, we get

$$\mathbf{S}_{|i}y^i = (n+1)c_0F,$$

where $c_0 := c_{|i}y^i = c_{x^i}y^i$. Thus, we have

$$\mathbf{PRic} = \mathbf{Ric} + (n-1)[c_0 + c^2F]F.$$

In this case, one can show that F is of isotropic projective Ricci curvature

$$\mathbf{PRic} = (n-1)\lambda(x)F$$

if and only if it is a weakly Einstein metric

$$\mathbf{Ric} = (n-1)\Big(\mu + \frac{3\theta}{F}\Big)F^2, \tag{9}$$

with $\theta = -c_0/3$ and $\mu = F^{-1}\lambda - c^2$.

A Finsler manifold (M,F) is called complete if any unit speed geodesic $c\colon [a,b] \to M$ can be extended to a geodesic defined on $\mathbb{R}$. Now, we are going to prove the Theorem 1.1.

Proof of Theorem 1.1. Let M be an n-dimensional manifold. Fix an arbitrary vector $\mathbf{y} \in T_xM_0$ and let $c = c(t)$ denote the geodesic of F with $\dot{c}(0) = y$. Since the manifold is complete, then c is defined for $-\infty < t < +\infty$. Let

$$\mathbf{S}(t) := \frac{1}{n+1}\mathbf{S}\big(\dot{c}(t)\big).$$

Then

$$\mathbf{S}' = \frac{1}{n+1}\mathbf{S}_{|i}\big(\dot{c}(t)\big)\dot{c}^i(t).$$

By assumption, we have

$$\mathbf{S}_{|i}y^i - \frac{1}{n+1}\mathbf{S}^2 = \frac{n+1}{n-1}\Big(\mathbf{PRic} - \mathbf{Ric}\Big) \geq 0.$$

Then

$$\mathbf{S}'(t) - \mathbf{S}^2(t) \geq 0. \tag{10}$$

Let us put

$$\mathbf{S}_0(t) := \frac{\mathbf{S}(\mathbf{y})}{1 - t\,\mathbf{S}(\mathbf{y})}.$$

It is easy to see that $\mathbf{S}_0$ satisfies

$$\mathbf{S}_0'(t) - \mathbf{S}_0^2(t) = 0.$$

Let us define

$$h(t) := \exp\Big\{-\int_0^t [\mathbf{S}(s) + \mathbf{S}_0(s)]ds\Big\}\Big\{\mathbf{S}(t) - \mathbf{S}_0(t)\Big\}.$$

We get

$$h'(t) := \exp\Big\{-\int_0^t [\mathbf{S}(s)+\mathbf{S}_0(s)]ds\Big\}\Big\{\mathbf{S}'(t)-\mathbf{S}'_0(t)+\mathbf{S}_0^2(t)-\mathbf{S}^2(t)\Big\} \geq 0.$$

Also, we have $h(0)=0$. It results that

$$\begin{aligned} h(t) &\geq 0, \quad t>0,\\ h(t) &< 0, \quad t<0. \end{aligned}$$

Then, we conclude that

$$\begin{aligned} \mathbf{S}(t) &\geq \mathbf{S}_0(t), \quad t>0,\\ \mathbf{S}(t) &\leq \mathbf{S}_0(t), \quad t<0. \end{aligned}$$

Suppose that $\mathbf{S}(\mathbf{y}) \neq 0$. Now, let us put

$$t_0 := \frac{1}{\mathbf{S}(\mathbf{y})}.$$

If $\mathbf{S}(\mathbf{y})>0$, then $t_0>0$ and we get

$$\mathbf{S}(\dot{c}(t_0)) \geq \lim_{t\to t_0^-} \mathbf{S}_0(t) = \infty$$

which is impossible. If $\mathbf{S}(\mathbf{y})<0$, then $t_0<0$ and

$$\mathbf{S}(\dot{c}(t_0)) \leq \lim_{t\to t_0^-} \mathbf{S}_0(t) = -\infty.$$

This case is impossible, also. Then $\mathbf{S}(\mathbf{y})=0$ for any $\mathbf{y}\in T_xM$. By (6), it follows that $\mathbf{PRic}=\mathbf{Ric}$. The converse is trivial. □

The completeness condition in Theorem 1.1 cannot be replaced by positively complete or negatively complete. See the following.

Example 4.1. A Finsler metric F satisfying $F_{x^k} = FF_{y^k}$ is called a Funk metric. The standard Funk metric on the Euclidean unit ball $B^n(1)$ is denoted by Θ and defined by

$$\Theta(x,y) := \frac{\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}}{1-|x|^2} + \frac{\langle x,y\rangle}{1-|x|^2}, \qquad y\in T_x\mathbb{B}^n(1)\simeq\mathbb{R}^n, \tag{11}$$

where $\langle,\rangle$ and $|\cdot|$ denote the Euclidean inner product and norm on $\mathbb{R}^n$, respectively. Funk metric is a non-Riemannian positively complete Finsler metric. The spray coefficients of F are given by $G^i=\frac{1}{2}Fy^i$. Funk metric satisfy

$$\mathbf{S} = \frac{(n+1)}{2}F, \qquad \text{and} \qquad \mathbf{Ric} = -\frac{(n-1)}{4}F^2 < 0.$$

Since $F_{|i}=0$, then

$$\mathbf{S}_{|i} = 0$$

which implies that $\mathbf{PRic}=0$. Therefore, $\mathbf{PRic}\geq\mathbf{Ric}$ while $\mathbf{S}\neq 0$.

5. Proof of Theorem 1.2

In this section, we are going to prove Theorem 1.2. First, we compute the projective Ricci curvature of a Matsumoto metric.

For an (α, β)-metric $F := \alpha\phi(s)$, $s = \beta/\alpha$, let us define $b_{i;j}$ by

$$b_{i;j}\theta^j := \mathrm{d}b_i - b_j\theta_i^j,$$

where $\theta^i := \mathrm{d}x^i$ and $\theta_i^j := \Gamma^j_{ik}\mathrm{d}x^k$ denote the Levi-Civita connection form of α. Let

$$r_{ij} := \frac{1}{2}\Big[b_{i;j} + b_{j;i}\Big], \qquad s_{ij} := \frac{1}{2}\Big[b_{i;j} - b_{j;i}\Big],$$

$$\begin{aligned}
&r^i{}_j := a^{im}r_{mj}, && s^i{}_j := a^{im}s_{mj}, && r_j := b^m r_{mj}, && s_j := b^m s_{mj},\\
&q_{ij} := r_{im}s^m_{\ j}, && t_{ij} := s_{im}s^m_{\ j}, && q_j := b^i q_{ij} = r_m s^m_{\ j}, && t_j := b^i t_{ij} = s_m s^m_{\ j},\\
&r_{i0} := r_{ij}y^j, && s_{i0} := s_{ij}y^j, && r_{00} := r_{ij}y^i y^j, && r := b^i r_i,\\
&r_0 := r_j y^j, && s_0 := s_j y^j. && &&
\end{aligned}$$

For an (α, β)-metrics, the form β is said to be Killing (resp., closed) form if $r_{ij} = 0$ (resp., $s_{ij} = 0$). β is said to be a constant Killing form if it is a Killing form and has constant length with respect to α, equivalently $r_{ij} = 0$ and $s_i = 0$ (see [**16, 20, 21, 24**]).

Let $F = \alpha^2/(\alpha - \beta)$ be a Matsumoto metric on an n-dimensional manifold M. Suppose that G^i and $\bar{G}^i$ denote the geodesic coefficients of F and α, respectively. Then G^i and $\bar{G}^i$ are related by

$$G^i = \bar{G}^i - \frac{\alpha}{A_1}s^i{}_0 + \frac{1}{2\alpha A_1 A_2}(2\alpha s_0 + A_1 r_{00})\Big[(2A_1 + 1)y^i - 2\alpha b^i\Big], \tag{12}$$

where

$$A_1 = A_1(s) := 2s - 1, \qquad A_2 = A_2(s, b) := 3s - 2b^2 - 1.$$

For a Matsumoto metric $F = \frac{\alpha^2}{\alpha - \beta}$, the S-curvature is given by

$$\begin{aligned}
\mathbf{S} = {} & 2\frac{s_0}{{A_1}^2} + 6\,\frac{\left(b^2 - s^2\right)s_0}{A_1\,{A_2}^2} - 2\,\frac{s s_0}{A_1\,A_2} + 4\,\frac{\left(b^2 - s^2\right)s_0}{{A_1}^2 A_2} + \frac{(n+1)\,(4\,s - 1)\,s_0}{A_1\,A_2}\\
& + 3\,\frac{\left(b^2 - s^2\right)r_{00}}{\alpha\,{A_2}^2} + 1/2\,\frac{(n+1)\,(4\,s - 1)\,r_{00}}{\alpha\,A_2} - 2\,\frac{r_0}{{A_2}^2} + \Lambda\,\left(r_0 + s_0\right),
\end{aligned} \tag{13}$$

where

$$\Lambda := \frac{f'(b)}{bf(b)}. \tag{14}$$

Lemma 5.1. *The Ricci curvature of a Matsumoto metric $F = \frac{\alpha^2}{\alpha - \beta}$ is given by*

$$\mathbf{Ric} \;=\; \frac{1}{4\alpha^2(\alpha - 2\beta)^3(\alpha - 3\beta + 2b^2\alpha)^4}\sum_{k=0}^{11} d_k\alpha^k, \tag{15}$$

where $\mathbf{Ric} := R^m_{\ m}$ *and* d_k, $(k = 0, 1, \ldots, 11)$, *are as follows:*

$$\begin{cases} d_0 & := -288(8n-11)r_{00}^2\beta^7, \\ & \vdots \\ d_{11} & := -4(1+2b^2)^3\Big[(1+2b^2)t^m_{\ m} + 4s_m s^m\Big]. \end{cases}$$

All the coefficients of d_i *are listed in* [**23**].

Lemma 5.2. *Let* $F = \frac{\alpha^2}{\alpha-\beta}$ *be a Matsumoto metric on an* n*-dimensional manifold* M*. Then the projective Ricci curvature of* F *is given by*

$$\mathbf{PRic} = \frac{1}{\alpha^2\,(n+1)^2\,(\alpha-2\,\beta)^3\,(\alpha-3\,\beta+2\,b^2\alpha)^4}\sum_{k=0}^{11} t_k\alpha^k, \tag{16}$$

where

$$\begin{cases} t_0 & := 72(n^2+5n+6)\beta^7 r_{00}^2, \\ & \vdots \\ t_{11} & := -(n+1)^2(1+2b^2)^3\big[(1+2b^2)t^m_{\ m} + 4s_m s^m\big]. \end{cases}$$

All the coefficients of t_k, $(k = 0, 1, \ldots, 11)$, *are polynomials and other coefficients of* t_k *can be calculated by maple program if necessary.*

Proof. According to the definition, the projective Ricci curvature is given by

$$\mathbf{PRic} = \mathbf{Ric} + \frac{n-1}{n+1}\mathbf{S}_{|m}y^m + \frac{n-1}{(n+1)^2}\mathbf{S}^2. \tag{17}$$

By (12), we have

$$\begin{aligned} G^i_m y^m = \bar{G}^i_m y^m - \frac{2\alpha}{A_1}s^i_{\ 0} + \frac{1}{\alpha A_1 A_2}(2\alpha s_0 + A_1 r_{00})(2A_1+1)y^i \\ - \frac{2}{A_1A_2}(2\alpha s_0 + A_1 r_{00})b^i. \end{aligned} \tag{18}$$

Thus

$$\begin{aligned} \mathbf{S}_{|m}y^m &= y^m\frac{\partial \mathbf{S}}{\partial x^m} - G^l_m y^m \frac{\partial \mathbf{S}}{\partial y^l} \\ &= \mathbf{S}_{;m}y^m + \Big[\frac{2\alpha}{A_1}s^m_{\ 0} - \frac{1}{\alpha A_1 A_2}(2\alpha s_0 + A_1 r_{00})(2A_1+1)y^m \\ &\quad + \frac{2}{A_1A_2}(2\alpha s_0 + A_1 r_{00})b^m\Big]\frac{\partial \mathbf{S}}{\partial y^m} \\ &= \mathbf{S}_{;m}y^m + \frac{2\alpha}{A_1}s^m_{\ 0}\mathbf{S}_{y^m} \\ &\quad - \frac{1}{A_1A_2}(2\alpha s_0 + A_1 r_{00})\Big[\alpha^{-1}(2A_1+1)\mathbf{S} - 2b^m\mathbf{S}_{y^m}\Big]. \end{aligned} \tag{19}$$

The following holds

$$b^2_{;m} = 2(r_m + s_m), \quad s_{;m} = \frac{r_{0m} + s_{0m}}{\alpha}, \quad s_{y^m} = \frac{\alpha b_m - s y_m}{\alpha^2}.$$

Consequently,

$$
\begin{aligned}
&b^2_{;0} = 2(r_0 + s_0), && s_{;0} = \frac{r_{00}}{\alpha}, && \alpha_{;0} = 0, && \Lambda_{;0} = \Lambda_0,\\
&s^m{}_0 s_{y^m} = \frac{s_0}{\alpha}, && s^m{}_0 (r_{00})_{y^m} = 2s^m_0 r_{0m} = 2q_{00}, && \Lambda_{y^m} = 0, &&\\
&b^m s_{y^m} = \frac{b^2 - s^2}{\alpha}, && b^m (r_{00})_{y^m} = 2r_0, && b^m \alpha_{y^m} = s, && b^m (r_0)_{y^m} = r.
\end{aligned}
$$

By (13), we get

$$
\begin{aligned}
\mathbf{S}_{;m} y^m = {} & \frac{1}{2\alpha(\alpha - 3\beta + 2b^2\alpha)^3(\alpha - 2\,\beta)^2} \\
& \times \Big(16\, b^6\, \alpha^6\, \Lambda\, r_{0;0} + \cdots - 72\,(1 + 2\,n)\,\beta^5\, r_{00;0}\Big),
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
s^m{}_0 \mathbf{S}_{y^m} = {} & \frac{1}{2\,\alpha\,(\alpha - 3\beta + 2b^2\alpha)^3(\alpha - 2\beta)^2} \\
& \times \Big(8\,(1 + 2\,b^2)\,b^4\,\alpha^6 \Lambda q_0 + \cdots - 144\,(1 + 2\,n)\,q_{00}\beta^5\Big),
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
b^m \mathbf{S}_{y^m} = {} & \frac{1}{2\,\alpha^3(\alpha - 3\beta + 2b^2\alpha)^3(\alpha - 2\beta)^2} \\
& \times \Big(8\,r\,(1 + 2\,b^2)\,b^4\,\alpha^8 \Lambda + \cdots + 72\,(1 + 2\,n)\beta^6 r_{00}\Big).
\end{aligned} \tag{22}
$$

Substituting (20), (21), and (22) into (19) yields

$$
\begin{aligned}
\mathbf{S}_{|m} y^m = {} & -\frac{1}{2\alpha^2(\alpha - 3\beta + 2b^2\alpha)^4(\alpha - 2\beta)^2} \\
& \times \Big(32\,(1 + 2\,b^2)\,b^6\,\alpha^9 \Lambda\, q_0 + \cdots + 432\,(1 + 2\,n)\beta^6 r^2_{00}\Big).
\end{aligned} \tag{23}
$$

By using (13), (15), (23), and Maple program, we obtain (16). □

Now, we remark the following.

Lemma 5.3 ([**26**])**.** *Let* $B_1 := \frac{\beta - \alpha}{\alpha}$ *and* $B_2 := \frac{\beta^2 - b^2\alpha^2}{\alpha^2}$. *Then*

(1) B_1 *and* $\dfrac{\alpha - 3\beta + 2b^2\alpha}{\alpha}$ *are relatively prime polynomials in* y *if and only if* $b \neq 1$.

(2) B_2 *and* $\dfrac{\alpha - 3\beta + 2b^2\alpha}{\alpha}$ *are relatively prime polynomials in* y *if and only if* $b \neq 1$.

(3) B_2 *or* $\dfrac{\alpha - 3\beta + 2b^2\alpha}{\alpha}$ *and* $\frac{\alpha - 2\beta}{\alpha}$ *are relatively prime polynomials in* y *if and only if* $b \neq 1/2$.

Proof of Theorem 1.2. Let M be an n-dimensional manifold and F be a Finsler metric on M with weak **PRic**-curvature

$$
\mathbf{PRic} = (n - 1)\Big(\frac{3\theta}{F} + \kappa\Big)F^2,
$$

where $\theta = \theta_i(x)y^i$ is a 1-form and $\kappa = \kappa(x)$ is a scalar function on M. Then by (16), we have

$$
\begin{aligned}
&(n-1)\Big[\frac{3\theta\alpha^2}{\alpha-\beta}+\frac{\kappa\alpha^4}{(\alpha-\beta)^2}\Big]\\
&=\frac{1}{\alpha^2\,(n+1)^2\,(\alpha-3\,\beta+2\,b^2\alpha)^4\,(\alpha-2\,\beta)^3}\sum_{k=0}^{11}t_k\alpha^k. \qquad (24)
\end{aligned}
$$

Multiplying $(\alpha-\beta)^2$ on both side (24) yields

$$
\begin{aligned}
&(n-1)\Big[3\theta(\alpha-\beta)+\kappa\alpha^2\Big]\alpha^2\\
&=\frac{(\beta-\alpha)^2}{\alpha^2\,(n+1)^2\,\,(\alpha-3\,\beta+2\,b^2\alpha)^4\,(\alpha-2\,\beta)^3}\sum_{k=0}^{11}t_k\alpha^k. \qquad (25)
\end{aligned}
$$

For the Matsumoto metric, we have $b<\frac{1}{2}$, which implies that $\alpha-3\beta+2b^2\alpha$ cannot be divided by $\beta-\alpha$ from Lemma 5.3. Obviously, $\alpha-2\beta$ and α cannot be divided by $\beta-\alpha$ either. Thus $\kappa\alpha^4$ must be divided by $\beta-\alpha$. This is impossible unless $\kappa=0$. From this and (25), $\theta\alpha^2$ is divided by $\beta-\alpha$, which is equivalent to θ is divided by $\beta-\alpha$. This is impossible unless $\theta=0$. Then F reduces to a **PRic**-flat metric. The converse is obvious. This completes the proof. □

6. Proof of Theorem 1.3

To prove Theorem 1.3, we need the following lemma

Lemma 6.1. *Let $F=\frac{\alpha^2}{\alpha-\beta}$ be a* **PRic***-flat non-Riemannian Matsumoto metric on a manifold M of dimension $n\geq 3$. Then β is a conformal 1-form with respect to α, i.e., there is a function $\sigma=\sigma(x)$ on M such that $r_{00}=\sigma\alpha^2$.*

Proof. Let us assume that $\mathbf{PRic}=0$, or equivalently,

$$\alpha^2\,(n+1)^2\,\big(\alpha-3\,\beta+2\,b^2\alpha\big)^4\,(\alpha-2\,\beta)^3\,\mathbf{PRic}=0.$$

By (16), we obtain

$$\sum_{k=0}^{11}t_k\alpha^k=0. \qquad (26)$$

By (26), we obtain the following fundamental equations:

$$
\begin{cases}
0 &= t_0+t_2\alpha^2+t_4\alpha^4+t_6\alpha^6+t_8\alpha^8+t_{10}\alpha^{10},\\
0 &= t_1+t_3\alpha^2+t_5\alpha^4+t_7\alpha^6+t_9\alpha^8+t_{11}\alpha^{10}.
\end{cases} \qquad (27)
$$

From the first equation of (27), we know that α^2 divides t_0. Since α^2 is an irreducible polynomial in y and β^7 factors into linear terms, then α^2 divides r_{00}^2. Thus $r_{00}=\sigma\alpha^2$ for some scalar function $\sigma=\sigma(x)$ on M. □

Lemma 6.2 ([**27**]). *Let $F = \frac{\alpha^2}{\alpha-\beta}$ be a non-Riemannian Matsumoto metric on a manifold M of dimension $n \geq 3$. Then S-curvature vanishes if and only if β is a constant Killing form.*

Proof of Theorem 1.3. Let F be a Matsumoto metric on an n-dimensional manifold M. In [**27**], the authors proved that for a non-Riemannian Matsumoto metric F of constant killing 1-form β, F is Einstein with scalar function $\lambda = \lambda(x)$ on M if and only if α is Ricci-flat and β is parallel with respect α. In this case, F is Ricci-flat. By Lemma 6.2, we conclude that $\mathbf{PRic} = \mathbf{Ric}$ in the case of constant killing 1-form. Thus, in this case, one can get the same results as in [**27**] for isotropic projective Ricci curvature. So, to complete the proof, we just have to prove that β is of a constant killing 1-form.

Assume that F is of an isotropic projective Ricci curvature. By Theorem 1.2 and Lemma 6.1, $r_{00} = \sigma\alpha^2$. Thus $r_i = \sigma b_i$. Since the length of β is constant with respect to α, then we have $0 = (b^2)_{|i} = 2(r_i + s_i)$, i.e., $r_i + s_i = 0$. Hence we get

$$\sigma b_i + s_i = 0.$$

Contracting both sides of it with b^i yields that $\sigma = 0$. Therefore, $r_{00} = 0$ and $s_i = 0$, i.e., β is a constant Killing 1-form.

Conversely, if α is Ricci-flat and β is parallel with respect to α, then the length of β with respect to α is constant. It follows that F is Einstein as well as isotropic projective Ricci curvature. This completes the proof. $\square$

7. Proof of Theorem 1.4

Let F be a Matsumoto metric on an n-dimensional manifold M. The sufficiency is obvious. We only need to prove the necessity. Assume that the projective Ricci curvature of F is reversible, i.e.,

$$\mathbf{PRic}(y) = \mathbf{PRic}(-y). \tag{28}$$

Then by contracting both sides of (28) with $\alpha^2(n+1)^2(\alpha+2\beta)^3(\alpha+3\beta+2b^2\alpha)^4$ and by a quite long computational procedure using Maple program, we obtain

$$\sum_{k=0}^{11} h_k\alpha^k = 0, \tag{29}$$

where h_k, $k = 0, 1, \ldots, 11$ are as follows:

$$\begin{cases} h_0 & := 72(n^2+5n+6)\beta^7 r_{00}^2, \\ & \vdots \\ h_{11} & := (n+1)^2(1+2b^2)^3[(1+2b^2)t^m_{\ m} + 4s_m s^m]. \end{cases}$$

From (29), we obtain the following fundamental equations:

$$\begin{cases} 0 & = h_0 + h_2\alpha^2 + h_4\alpha^4 + h_6\alpha^6 + h_8\alpha^8 + h_{10}\alpha^{10}, \\ 0 & = h_1 + h_3\alpha^2 + h_5\alpha^4 + h_7\alpha^6 + h_9\alpha^8 + h_{11}\alpha^{10}. \end{cases} \tag{30}$$

All of coefficients h_k are polynomials and other coefficients of h_k can be calculated by maple program if necessary. From the first equation of (30), we know that α^2 divides h_0. Since α^2 is an irreducible polynomial in y and β^7 factors into linear terms, it must be the case that α^2 divides r_{00}^2. Thus $r_{00} = \sigma\alpha^2$ for some function $\sigma = \sigma(x)$, i.e., β is a conformal form with respect to α. In this case, the following holds

$$\begin{array}{llll} r_{00} = \sigma\alpha^2, & r_{ij} = \sigma a_{ij}, & r_{0i} = \sigma y_i, & r_i = \sigma b_i, \\ r = \sigma b^2, & r^i{}_j = \sigma\delta^i{}_j, & r_{0i}s^i{}_0 = q_{00} = 0, & r_{0i}s^i = \sigma s_0, \\ r_0 = \sigma\beta, & s^i{}_0 r_i = q_0 = \sigma s_0, & r_{00;i} = \sigma_i\alpha^2, & r_{00;0} = \sigma_0\alpha^2, \\ r^i{}_i = n\sigma, & r_{0;0} = \sigma_0\beta + \sigma^2\alpha^2, & & \end{array}$$

where $y_i := a_{ij}y^j$.

Substituting all of these into the first equation of (30) and dividing both sides by common factor α^2, we obtain

$$h_0^{'} + h_2^{'}\alpha^2 + h_4^{'}\alpha^4 + h_6^{'}\alpha^6 + h_8^{'}\alpha^8 = 0, \tag{31}$$

where

$$\begin{aligned} h_0' :=\ & -648\Big\{(n-1)\Lambda^2(\beta\sigma + s_0)^2 \\ & + (n^2-1)\big[\Lambda(\beta\sigma_0 + s_{0;0}) + \Lambda_0(\beta\sigma + s_0)\big] + (n+1)^2({}^{\alpha}\mathbf{Ric} - \mathbf{PRic})\Big\}\beta^7 \end{aligned}$$

and other coefficients of $h_i^{'}$ can be calculated by maple program if necessary. From (31), we know that α^2 divides $h_0^{'}$. Thus

$$\begin{aligned} k\alpha^2 = & (n-1)\Lambda^2(\beta\sigma + s_0)^2(\beta\sigma_0 + s_{0;0}) \\ & + (n^2-1)\big[\Lambda + \Lambda_0(\beta\sigma + s_0)\big] + (n+1)^2({}^{\alpha}\mathbf{Ric} - \mathbf{PRic}), \end{aligned} \tag{32}$$

where $k = k(x)$ is a scalar function on M. By (32), we get

$$\begin{aligned} \mathbf{PRic} = & \frac{1}{(n+1)^2}\Big\{(n-1)\Lambda^2(\beta\sigma + s_0)^2 + (n^2-1)\big[\Lambda(\beta\sigma_0 + s_{0;0}) \\ & + \Lambda_0(\beta\sigma + s_0)\big] - k\alpha^2\Big\} + {}^{\alpha}\mathbf{Ric} \end{aligned}$$

which shows that F is **PRic**-quadratic. □

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