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η-RICCI SOLITON AND ALMOST η-RICCI SOLITON ON ALMOST coKÄHLER MANIFOLDS

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ABSTRACT. The aim of this paper is to study η-Ricci soliton and almost η-Ricci soliton in the context of smooth almost coKähler manifold for which Reeb vector field ξ is Killing and for ξ belongs to (κ, μ)-nullity distribution. For a (κ, μ)-almost coKähler metric manifold M, we prove that if M is non-coKähler and g is gradient η-Ricci soliton, then M is η-Einstein with $\lambda = 0$. Next we prove that if g is an η-Ricci soliton on M with $\lambda + \mu' \leq 0$, then M is coKähler. Further we show that, M is η-Einstein if and only if V is a strict infinitesimal contact transformation. Finally, we prove that if the non-coKähler (κ, μ)-almost coKähler manifold M admits an almost η-Ricci soliton with $V = \rho\xi$ or $V = Df$, then M is η-Einstein. We construct the suitable example which justifies our results.

1. INTRODUCTION

Let M be a Riemannian manifold. A Ricci soliton on M is the case of choosing a smooth vector field V (if any) satisfying the soliton equation

$$S(Y,Z) + \frac{1}{2}(\pounds_V g)(Y,Z) + \lambda g(Y,Z) = 0 \tag{1}$$

for any $Y, Z \in \mathcal{X}(M)$ and for some soliton constant $\lambda \in R$, where $\pounds_V$ denotes the Lie derivative along the vector field V and S is the Ricci tensor. Moreover, Ricci soliton was first introduced by R. S. Hamilton [**10**], in 1982. The Ricci soliton (g, V, λ) is said to be shrinking, steady or expanding according to the soliton constant λ appearing in (1), satisfying $\lambda < 0$, $\lambda = 0$, or $\lambda > 0$, respectively. During the last two decades, the Ricci soliton has been studied by many mathematicians ([**19**], [**7**], [**20**], [**21**], [**22**], [**14**], [**13**]). In equation (1), by allowing the soliton constant λ to become a smooth variable function on M, Pigola et al. [**16**] studied gradient Ricci almost soliton on M. In [**15**], Perelman proved that potential vector

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field of a Ricci soliton on a compact manifold is the gradient of a potential function $-f$. But this result need not be true for an almost Ricci soliton case.

In general, Ricci soliton represents a generalization of Einstein metric on M. By adding the term $\mu'\eta \otimes \eta$ for μ' a real constant and η a 1-form in equation (1), we obtain the η-Ricci soliton introduced by Cho and Kimura [**8**]. Moreover, in [**5**], C. Calin and M. Crasmareanu treated this on Hopf hypersurfaces in complex hypersurfaces. So the η-Ricci soliton on M is defined as follows:

A Riemannian metric g is said to be an *η-Ricci soliton* on M, if there exits a smooth vector field V called the potential vector field such that

$$S + \frac{1}{2}\pounds_V g + \lambda g + \mu'\eta \otimes \eta = 0, \tag{2}$$

where λ and μ' are the soliton constants, η is the 1-form and $\pounds_V g$ denotes the Lie derivative of g along the vector field V. Next from the Poincare lemma, we know the relation

$$g(Y, \nabla_Z Df) = g(Z, \nabla_Y Df). \tag{3}$$

Therefore, if V is gradient of f, then condition (2) turns into

$$QY + \nabla_Y Df + \lambda Y + \mu'\eta(Y)\xi = 0 \tag{4}$$

for all $Y \in TM$, this is called a gradient η-Ricci soliton. If λ and μ' appearing in equations (2) and (4), are in $C^\infty(M)$, then g is said to be an almost η-Ricci soliton. The notion of an almost η-Ricci soliton has been studied in many contexts: on paracontact manifolds ([**12**], [**4**]), on Lorentzian manifolds ([**3**], [**2**]).

The paper is structured as as follows. After introduction part for soliton condition in section 1, in the section 2, we recall the notion and properties of an almost coKähler manifolds with having Killing ξ vector for (κ, μ)-nullity distribution. We prove some basic properties on it. Section 3 is devoted to the study of an η-Ricci soliton on (κ, μ)-almost coKähler manifolds. First we consider the gradient η-Ricci soliton in a non-coKähler case and prove the constancy of a function f with showing $\lambda = 0$. Next for $\lambda + \mu' \leq 0$ case, we prove the non-existence of an η-Ricci soliton in a non-coKähler (κ, μ)-almost coKähler manifold M. Further, we show that M is η-Einstein if and only if V is strict infinitesimal contact transformation. In section 4, we consider an almost η-Ricci soliton for $V = \xi$ and $V \perp \xi$ cases in an almost coKähler manifold with parallel ξ vector. So we arrive at the condition $\lambda + \mu' = 0$. In the last section, we prove that if M is a non-coKähler (κ, μ)-almost coKähler manifold and admits either gradient almost η-Ricci soliton or almost η-Ricci soliton $(g, \rho\xi)$, then M is η-Einstein.

2. Almost coKähler manifolds

An odd dimensional smooth manifold M having 1-form η, vector field ξ, and endomorphism φ such that

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1 \tag{5}$$

is called as an almost contact manifold. It is well known that in an almost contact manifold, equation (5) implies $\eta \circ \varphi = 0$, $\varphi\xi = 0$, and there always exists a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = (g - \eta \otimes \eta)(X, Y), \tag{6}$$

for all X, $Y \in TM$. Then the manifold M together with g is said to be an almost contact metric manifold. In the study of an almost contact metric manifold, another class of an almost contact metric manifold and an odd analogy of Kähler manifold whose 1-form η and 2-form Φ defined as $\Phi(X, Y) = g(X, \varphi Y)$ satisfying $d\eta = 0$ and $d\Phi = 0$ are said to be an almost coKähler manifolds. (In recent few years, it has been studied by many authors, e.g., [**9**], [**11**]). If an almost coKähler manifold M is normal, then M is said to be coKähler. An almost coKähler manifold with parallel Reeb vector field ξ was considered and studied in [**1**]. As like contact, a three dimensional almost coKähler manifold with parallel ξ vector is always a coKähler. If M has a parallel vector ξ, then $R(X, Y)\xi = 0$ which gives:

$$Q\xi = 0. \tag{7}$$

On an almost coKähler manifold M, for a symmetric operator h and asymmetric operator φ, we have the following conditions from [**18**]

$$h\varphi = -\varphi h, \qquad h\xi = 0,$$

$$\mathrm{trace}(h) = \mathrm{div}\,\xi = 0, \qquad \nabla_X \xi = h' = h\varphi X, \tag{8}$$

$$\nabla_\xi \varphi = 0. \tag{9}$$

In an almost coKähler manifold M, if the vector field ξ belongs to (κ, μ)-nullity distribution, i.e.,

$$R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\} \tag{10}$$

for all X and $Y \in TM$, where $(\kappa, \mu) \in R^2$, then such type of manifold is defined as (κ, μ)-almost coKähler manifold.

Thus from [**6**], we have the following conditions on (κ, μ)-almost coKähler manifolds:

$$Q\xi = 2n\kappa\xi, \tag{11}$$

$$\nabla_\xi h = \mu h', \tag{12}$$

$$h^2 = \kappa\varphi^2. \tag{13}$$

Clearly, from the above equation, we can conclude that $\kappa \leq 0$ and M is coKähler if and only if $\kappa = 0$. For non-coKähler (κ, μ)-almost coKähler manifold ($\kappa < 0$), we have

$$QY = \mu hY + 2n\kappa\eta(Y)\xi, \tag{14}$$

$$(\nabla_Y h)Z = -\kappa g(\varphi Y, Z)\xi + \mu\eta(Y)h'Z + \kappa\eta(Z)\varphi Y, \tag{15}$$

$$(\nabla_Y \varphi)Z = g(hY, Z)\xi - \eta(Z)hY. \tag{16}$$

And moreover, if M is three dimensional, then its Ricci operator is given by

$$QY = \Big\{\frac{r}{2} - \kappa\Big\}Y + \Big\{3\kappa - \frac{r}{2}\Big\}\eta(Y)\xi + \mu hY. \tag{17}$$

From Yano [**23**], for a Riemannian manifold M, we have

$$2g((\pounds_V\nabla)(X,Y),Z) = (\nabla_X\pounds_V g)(Y,Z) + (\nabla_Y\pounds_V g)(X,Z) - (\nabla_Z\pounds_V g)(X,Y), \tag{18}$$

and

$$(\pounds_V R)(X,Y)Z = (\nabla_X\pounds_V\nabla)(Y,Z) - (\nabla_Y\pounds_V\nabla)(X,Z). \tag{19}$$

If M holds equation(2), then the condition (18) turns into

$$\begin{aligned} g((\pounds_V\nabla)(X,Y),Z) = & -g((\nabla_X Q)Y,Z) - g((\nabla_Y Q)X,Z) \\ & + g((\nabla_Z Q)X,Y) - \mu'(\nabla_X\eta\otimes\eta)(Y,Z) \\ & - \mu'(\nabla_Y\eta\otimes\eta)(X,Z) + \mu'(\nabla_Z\eta\otimes\eta)(X,Y). \end{aligned} \tag{20}$$

A Riemannian manifold M is said to be η-Einstein if it satisfies the condition

$$QZ = aZ + b\eta(Z)\xi \tag{21}$$

for all $Z \in TM$, where a and b are smooth functions on M.

Next from K. Yano [**17**], we have the following definition.

Definition 2.1. In an almost contact manifold, a vector field V is said to be an infinitesimal contact transformation if it satisfies the relation

$$\pounds_V\eta = \sigma\eta, \tag{22}$$

where σ is the smooth function on M. Further, V is said to be strict infinitesimal contact transformation if and only if $\sigma = 0$.

Now we recall one important result of B. C. Montano, et. al., [**6**].

Theorem 2.2. *Any compact Ricci-flat almost coKähler manifold is coKähler.*

Next, in the following propositions, we prove certain basic properties of an almost coKähler manifold.

Proposition 2.3. *Let M be an almost coKähler manifold. If ξ is Killing, then the derivative of a scalar curvature in the direction of ξ is zero.*

Proof. From relation (7), since ξ is parallel, we get

$$(\nabla_X Q)\xi = 0. \tag{23}$$

Contracting the expression (23) over X, we get $\xi r = 0$. □

Proposition 2.4. *If M is an η-Einstein almost coKähler manifold with Killing Reeb vector field ξ, then for* $\dim M > 3$*, the scalar curvature r is constant on M.*

Proof. If M is an almost coKähler for which equations (21) and (7) hold, then we have $a+b=0$ and $r=2na$. Further, taking covariant differentiation of (21) along an arbitrary vector field Y yields

$$(\nabla_Y Q)Z = (Ya)Z - (Ya)\eta(Z)\xi - ag(Y, h\varphi Z)\xi - a\eta(Z)h\varphi Y. \tag{24}$$

Contracting the above relation over Y with respect to an orthonormal basis and making use of (8) results

$$\frac{Zr}{2} = (Za) - (\xi a)\eta(Z). \tag{25}$$

As we know, $Zr = 2nZa$ and from the above Propostion 2.3, we have $\xi r = 0$, which together yield these in procceding relation provides

$$(n-1)Zr = 0. \tag{26}$$

□

Hence for $\dim M > 3$, r is constant on M.

Proposition 2.5. *If M is a three dimensional (κ,μ)-almost coKähler manifold, then the scalar curvature r is constant in the direction of ξ.*

Proof. Let M be a three dimensional almost coKähler. If the Reeb vector ξ belongs to the (κ,μ)-nullity distribution, then from equation (17) and by the fact $Q\xi = 2\xi$, we obtain

$$(\nabla_Y Q)\xi = \nabla_Y Q\xi - Q\nabla_Y\xi = \Big\{3\kappa - \frac{r}{2}\Big\}\nabla_Y\xi - h\nabla_Y\xi. \tag{27}$$

Use of equations (8) and (13) in (27), and then contraction over Y results $\xi r = 0$. □

3. η-Ricci soliton on (κ,μ)-almost coKähler manifolds

Theorem 3.1. *If a non-coKähler (κ,μ)-almost coKähler metric g is a gradient η-Ricci soliton, then the soliton function f is a constant and soliton is steady with $\mu' \neq 0$.*

Proof. We assume that non-coKähler (κ,μ)-almost coKähler metric g is a gradient η-Ricci soliton. Then by condition (4), the well known expression $R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ is calculated as

$$R(X,Y)Df + (\nabla_X Q)Y - (\nabla_Y Q)X + \mu'\eta(Y)\nabla_X\xi - \mu'\eta(X)\nabla_Y\xi = 0. \tag{28}$$

Contracting foregoing relation over X with respect to orthonormal basis, we get

$$2S(Y, Df) = g(Y, Dr). \tag{29}$$

Since M is non-coKähler, the scalar curvature $r = 2n\kappa$ is a constant, which shows Df is the eigen vector of Q with eigen value 0. Therefore, equation (14) for $Y = Df$ gives

$$\mu h Df + 2n\kappa\eta(Df)\xi = 0. \tag{30}$$

Next, by taking inner product of the above relation with respect to ξ, we get $\xi f = 0$.

In (28), for $Y=\xi$, scalar product with ξ results

$$g(R(X,\xi)Df,\xi)=0. \tag{31}$$

Make use of (10) and the fact $\xi f=0$ in the above relation, leads to

$$\kappa g(X,Df)=0, \tag{32}$$

which implies f is constant. Further, from equations (2) and (14), we can easily obtain the following conditions

$$r=-2n\lambda-\lambda-\mu', \tag{33}$$

$$2n\kappa+\lambda+\mu'=0, \tag{34}$$

$$r=2n\kappa. \tag{35}$$

By combining the above three equations, we find $\lambda=0$ and μ' never be zero, i.e., there is no existence of gradient Ricci soliton on a non-coKähler (κ,μ)-almost coKähler manifold. □

Theorem 3.2. *If g is a non-coKähler (κ,μ)-almost coKähler metric, then g can never be an η-Ricci soliton with soliton constants satisfying the condition $\lambda+\mu'\leq 0$.*

Proof. For a non-coKähler (κ,μ)- almost coKähler manifold M, the equation (14) yields

$$(\nabla_X Q)\xi=\mu(\nabla_X h)\xi+2n\kappa\nabla_X\xi=\mu\kappa\varphi X+2n\kappa h\varphi X \tag{36}$$

and

$$(\nabla_\xi Q)Y=\mu(\nabla_\xi h)Y=\mu^2 h\varphi X.$$

By applying these two relations in (20), we get

$$\begin{aligned} g((\pounds_V\nabla)(Y,\xi),Z)&=-g((\nabla_Y Q)\xi,Z)-g((\nabla_\xi Q)Y,Z)+g((\nabla_Z Q)\xi,Y)\\ &=g(-2\mu\kappa\varphi Y-\mu^2 h\varphi Y,Z), \end{aligned} \tag{37}$$

which gives

$$(\pounds_V\nabla)(Y,\xi)=-2\mu\kappa\varphi Y-\mu^2 h\varphi Y. \tag{38}$$

Taking the covariant derivative of $\pounds_V\nabla$ in the foregoing condition along an arbitrary vector field X provides

$$(\nabla_X\pounds_V\nabla)(\xi,\xi)=-2(\pounds_V\nabla)(\nabla_X\xi,\xi)=4\kappa\mu hX+2\mu^2\kappa\varphi^2 X. \tag{39}$$

Again, in equation (38) by putting $Y=\xi$, taking covariant derivative along the vector filed ξ, and using (15), (16), we get

$$(\nabla_\xi\pounds_V\nabla)(X,\xi)=-2\kappa\mu(\nabla_\xi\varphi)X-\mu^2(\nabla_\xi h\varphi)X=\mu^3 hX. \tag{40}$$

Using the above two conditions in the computation formula (19) for $Y=Z=\xi$, results

$$(\pounds_V R)(X,\xi)\xi=4\kappa\mu hX+2\mu^2\kappa\varphi^2 X-\mu^3 hX. \tag{41}$$

Contracting this over X yields

$$(\pounds_V S)(\xi,\xi)=-4n\mu^2\kappa. \tag{42}$$

Finally, a straightforword caluculation gives

$$-\kappa\eta(\nabla_\xi V) = \mu^2\kappa. \tag{43}$$

Now, by virtue of soliton equation, we obtain $\eta(\nabla_\xi V) = -\lambda - \mu' - 2n\kappa$. Using this in the above relation leads to

$$\kappa\{\lambda + \mu' + 2n\kappa - \mu^2\} = 0. \tag{44}$$

As M is non-coKähler, i.e., $\kappa < 0$ in (44), reduces $\lambda + \mu' = -2n\kappa + \mu^2$, showing that $\lambda + \mu' > 0$. Hence we can conclude that if $\lambda + \mu' \leq 0$, then there exists no η-Ricci soliton on M. □

Contrapositive to the above theorem, directly, we can also state the following result.

Theorem 3.3. *Let M be a (κ, μ)-almost coKähler manifold. If g is an η-Ricci soliton with $\lambda + \mu' \leq 0$, then M is a coKähler manifold.*

As consequence to Theorem 3.2, if a non-coKähler (κ, μ)-almost coKähler manifold M has Ricci soliton, then the soliton constant $\mu' = 0$ and $\lambda > 0$. Hence we can state the following corollary.

Corollary 3.4. *Let g be a non-coKähler (κ, μ)-almost coKähler metric. If g is a Ricci soliton, then the soliton is expanding.*

Theorem 3.5. *If M is a three dimensional (κ, μ)-almost coKähler manifold and it admits an η-Ricci soliton for $V = \xi$, then M is coKähler.*

Proof. If M is three dimensional and M holds soliton equation (2) for $V = \xi$, then the soliton condition in the virtue of (8) assumes the form

$$g(QY, Z) + \lambda g(Y, Z) + \mu'\eta(Y)\eta(Z) + g(h\varphi Y, Z) = 0. \tag{45}$$

Replacing Y by hY, and using (17) lead to

$$\{\frac{r}{2} - \kappa + \lambda\}g(hY, Z) + g(h\varphi Y, hZ) + \mu g(hY, hZ) = 0. \tag{46}$$

On A-asymmetrizing this gives

$$g(h^2\varphi Y, Z) = 0, \tag{47}$$

which implies $h = 0$. Therefore, M is coKähler and as well as $\lambda + \mu' = 0$. □

Theorem 3.6. *If a non-coKähler (κ, μ)-almost coKähler metric g is an η-Ricci soliton for V orthogonal to ξ, then the manifold is η-Einstein.*

Proof. Suppose M is a (κ, μ)-almost coKähler for $\kappa < 0$, and g is an η-Ricci soliton for V, orthogonal to ξ. Then from equation (2) for $Y = Z = \xi$, we obtain

$$\eta(\nabla_\xi V) = -(\lambda + \mu' + 2nk). \tag{48}$$

Since V is orthogonal to ξ, the above relation gives $\lambda + \mu' + 2nk = 0$, so this in equation (44), provides $\mu = 0$. This completes the proof. □

Lemma 3.7. *Let M be a non-coKähler (κ,μ)-almost coKähler manifold. If it admits η-Ricci soliton for $\mu^2 \neq -4n^2\kappa$, then the potential vector field V is an infinitesimal contact transformation.*

Proof. Suppose a (κ,μ)-almost coKähler manifold M is non-coKähler, then by relation (14), we can deduce

$$(\nabla_X Q)Y = \mu(\nabla_X h)Y + 2n\kappa\{g(X,h\varphi Y)\xi + \eta(Y)h\varphi X\}. \tag{49}$$

By deducing the right hand side of equation (20) with the help of foregoing condition and from (13), and (15), we get

$$\begin{aligned}(\pounds_V\nabla)(Y,Z) = &-\mu^2\{\eta(Y)h'Z + \eta(Z)h'Y\\ &- g(h'Y,Z)\xi\} - 2\mu\kappa\{\eta(Y)\varphi Z + \eta(Z)\varphi Y\}\\ &- 4n\kappa g(Y,h'Z)\xi - 2\mu' g(Y,h'Z)\xi.\end{aligned} \tag{50}$$

Therefore, using foregoing relation in finding the covarient derivative of $\pounds_V\nabla$ along an arbitrary vector X, we obtain

$$\begin{aligned}(\nabla_X\pounds_V\nabla)(Y,Z) = &-\mu^2\big\{g(X,h'Y)h'Z + g(X,h'Z)h'Y + \eta(Y)(\nabla_X h')Z\\ &+ \eta(Z)(\nabla_X h')Y - g((\nabla_X h')Y,Z)\xi - g(h'Y,Z)h'X\big\}\\ &- 2n\kappa\{g(X,h'Z)\varphi Y + \eta(Z)(\nabla_X\varphi)Y + g(X,h'Y)\varphi Z\\ &+ \eta(Y)(\nabla_X\varphi)Z\} - 4n\kappa\{g(Y,(\nabla_X h')Z)\xi + g(Y,h'Z)h'X\}\\ &- 2\mu'\{g(Y,(\nabla_X h')Z)\xi + g(Y,h'Z)h'X\}.\end{aligned} \tag{51}$$

Contracting the above relation over X with respect to an orthonormal basis gives

$$\begin{aligned}\sum_{i=1}^{2n+1} g((\nabla_{e_i}\pounds_V\nabla)(Y,Z),e_i) = &-2\mu^2 g(\varphi Y,\varphi Z) - 4n\kappa\mu^2\eta(Y)\eta(Z)\\ &+ 4\mu\kappa g(hY,Z) + 4n\kappa g(hY,Z)\\ &- \mu^3 g(hY,Z) + 2\mu\mu' g(hY,Z).\end{aligned} \tag{52}$$

Again contracting (51) over Y yields

$$\sum_{i=1}^{2n+1} g((\nabla_X\pounds_V\nabla)(e_i,Z),e_i) = 0. \tag{53}$$

Contracting computational formula (19) over X with respect to an orthonormal basis, using (52) and (53), we find

$$\begin{aligned}(\pounds_V S)(Y,Z) = &\,2\mu^2\kappa g(\varphi Y,\varphi Z) - 4n\kappa\mu^2\eta(Y)\eta(Z)\\ &+ 8\mu\kappa g(hY,Z) - \mu^3 g(hY,Z).\end{aligned} \tag{54}$$

As M is non-coKähler manifold, then by equation(14), we have

$$(\pounds_V S)(Y,Z) = \mu\{(\pounds_V g)(hY,Z) + g((\pounds_V h)Y,Z)\} + 2n\kappa(\pounds_V\eta\otimes\eta)(Y,Z). \tag{55}$$

In the foregoing relation using equation (2) on $Y = hY$, gives the following expression

$$(\pounds_V S)(Y,Z) = -2\mu^2 g(h^2Y,Z) - 2\mu\lambda g(hY,Z) + \mu g((\pounds_V h)Y,Z) + 2n\kappa\{\eta(Z)(\pounds_V\eta)Y + \eta(Y)(\pounds_V\eta)Z\}. \tag{56}$$

Next by equating (54) and (56), we obtain

$$\begin{aligned} 8\kappa\mu g(hY,Z) = {} & 4n\kappa\mu^2\eta(Y)\eta(Z) + \mu^3 g(hY,Z) \\ & - 2\mu\lambda g(hY,Z) + \mu g((\pounds_V h)Y,Z) \\ & + 2n\kappa\{\eta(Z)(\pounds_V\eta)Y + \eta(Y)(\pounds_V\eta)Z\}. \end{aligned} \tag{57}$$

By applying $Y = \xi$ and $Z = \varphi Z$ in the last equation, we get

$$\mu h' \pounds_V \xi = -2n\kappa\varphi\pounds_V\xi. \tag{58}$$

By operating h on both sides of the above relation and simplifying it, we get

$$\mu\varphi\pounds_V\xi = 2nh'\pounds_V\xi. \tag{59}$$

Finally by substituting equation(58) in equation(59), we obtain

$$(\mu^2 + 4n^2\kappa)h'\pounds_V\xi = 0. \tag{60}$$

Thus, in the above equation, $\mu^2 \neq -4n^2\kappa$ implies $h'\pounds_V\xi=0$ which gives $h\pounds_V\xi=0$. Hence using this in equation(57) for $Z = \xi$, we obtain

$$(\pounds_V\eta)Y = -\{\lambda + \mu' + 2n\kappa\}\eta(Y). \tag{61}$$

This completes the proof. □

Theorem 3.8. *If a non-coKähler (κ,μ)-almost coKähler manifold M admits an η-Ricci soliton for vector field V then M is η-Einstein if and only if V is strictly infinitesimal contact transformation.*

Proof. Let M be a non-coKähler (κ,μ)-almost coKähler manifold. If M admits η-Ricci soliton for vector field V. If M is an η-Einstein manifold, then from equation (14), we have that $\mu = 0$. So μ^2 is never equal to $-4n^2\kappa$. Hence from the above lemma and from equation (44), we obtain

$$(\pounds_V\eta) = -\{\lambda + \mu' + 2n\kappa\}\eta = \mu^2\eta. \tag{62}$$

Since $\mu = 0$, which in the above condition, implies that V is strictly infinitesimal contact transformation.

Conversely, if V is strictly infinitesimal contact transformation, then from Definition 2.1 and from equation (2) for $X = Y = \xi$, we have that, $\lambda + \mu' + 2n\kappa = 0$. Therefore, using it in (44), leads to $\mu = 0$. Hence the manifold M is η-Einstein. This finishes the proof. □

4. Almost η-Ricci soliton on an almost coKähler manifold M with $\nabla_X \xi = 0$

In this section, by considering the vector ξ parallel (i.e., ξ is Killing) on M, we obtain following results.

Theorem 4.1. *Let M be an almost coKähler manifold with $\nabla_X \xi = 0$, and M admits an almost η-Ricci soliton for $V = \xi$, then M is an η-Einstein manifold.*

Proof. Suppose M is an almost η-Ricci soliton for $V = \xi$ and M has a vector field ξ which is parallel. Then the proof of the theorem follows from equation(2) and the fact $\nabla_X \xi = 0$. Thus the Ricci curvature takes the form

$$S(X, Y) = \mu' g(X, Y) - \mu' \eta(X)\eta(Y) \tag{63}$$

for any $X, Y \in \mathcal{X}(M)$ and μ' is a smooth function on M. □

Corollary 4.2. *Let M be a compact almost coKähler manifold with Killing ξ. If M admits an almost Ricci soliton for $V = \xi$, then M is a coKähler manifold.*

Proof. If M has an almost Ricci soliton, then $\mu' = 0$. Thus, from equation (63), we have

$$S(X, Y) = 0 \tag{64}$$

for all X, $Y \in \mathcal{X}(M)$. Hence the proof of corollary follows from the Theorem 2.2. □

Next, we study an almost η-Ricci soliton on M with the potential vector field V orthogonal to ξ, i.e., $\eta(V) = 0$. Since the manifold M has parallel ξ vector, then from the relation (2), we have

$$S(Y, Z) = -\frac{1}{2}\{g(\nabla_Y V, Z) + g(Y, \nabla_Z V)\} - \lambda g(Y, Z) - \mu' \eta(Y)\eta(Z).$$

Putting $Y = Z = \xi$ in the above equation, yields

$$-\lambda - \mu' = \eta(\nabla_\xi V) = 0,$$

which implies $\lambda = -\mu'$. Hence we can state the following theorem.

Theorem 4.3. *Let M be an almost coKähler manifold with parallel vector ξ. If M admits an almost η-Ricci soliton for the potential vector field V orthogonal to ξ, then $\lambda = -\mu'$.*

Corollary 4.4. *If M is an almost coKähler manifold with $\nabla_X \xi = 0$ and admits an almost Ricci soliton (g, V, λ), where the V is orthogonal to ξ, then soliton is steady.*

5. Almost η-Ricci soliton on a non-coKähler (κ, μ)-almost coKähler manifold

Theorem 5.1. *Let M be a non-coKähler almost coKähler (κ, μ) manifold. If M admits gradient almost η-Ricci soliton or an almost η-Ricci soliton with V as collinear with the Reeb vector field ξ, then M is η-Einstein. Moreover, for $V = \rho\xi$ case, soliton becomes steady.*

Proof. For $V = Df$, from equation(4) we obtain

$$\begin{aligned} R(X,Y)Df + (\nabla_X Q)Y - (\nabla_Y Q)X + (X\lambda)Y \\ -(Y\lambda)X + (X\mu')\eta(Y)\xi - (Y\mu')\eta(X)\xi \\ +\mu'\eta(Y)h\varphi X - \mu'\eta(X)h\varphi Y = 0. \end{aligned} \tag{65}$$

Taking scalar product of above relation with ξ for $X = \varphi X$ and $Y = \varphi Y$, and using (14) and (8), we get

$$g(R(\varphi X, \varphi Y)\xi, Df) = 2\mu\kappa g(\varphi X, Y), \tag{66}$$

by using the equation (10) on simplifying the above relation, we get

$$\mu\kappa g(X, \varphi Y) = 0. \tag{67}$$

Next, if M has an almost η-Ricci soliton for $V = \rho\xi$, then the equation (2) reduces to

$$\begin{aligned} S(X,Y) + \frac{1}{2}\Big\{(X\rho)\eta(Y) + (Y\rho)\eta(X)\Big\} \\ +g(h\varphi X, Y) + \lambda g(X,Y) + \mu'\eta(X)\eta(Y) = 0. \end{aligned} \tag{68}$$

And for $X = \varphi X$, $Y = \varphi Y$ in (68), using (8) and (14), we deduce

$$-\mu g(hX, Y) - g(hX, \varphi Y) + \lambda g(\varphi X, \varphi Y) = 0. \tag{69}$$

Again, applying $X = hX$ in (69), applying (8) and (13), and contracting over X and Y, gives

$$\mu\kappa = 0. \tag{70}$$

Therefore, the non-coKähler condition in both cases gives $\mu = 0$. Moreover, contracting (69) leads to obtain the value of λ, so we get $\lambda = 0$. Hence the proof completes. □

Example 5.2. Before ending this paper, we construct a 3-dimensional (κ, μ)-almost coKähler manifold admitting η-Ricci solitons. Let we consider $M^3 = R^3$ with Cartesian coordinates (x, y, z) endowed with an orthonormal basis $\{e_1, e_2, e_3\}$ which satisfies

$$[e_1, e_2] = \alpha e_3, \qquad [e_2, e_3] = 0, \qquad [e_3, e_1] = -\alpha e_2.$$

Let η and g be the 1-form and Riemannian metric, respectively, defined by

$$\begin{aligned} &\eta(e_1) = 1, \quad \eta(e_2) = \eta(e_3) = 0, \quad \eta(X) = g(X, e_1), \qquad \text{for all } X. \\ &g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1, \\ &g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0, \end{aligned}$$

and the $(1,1)$ tensor φ defined by

$$\varphi e_1 = 0, \qquad \varphi e_2 = e_3, \qquad \varphi e_3 = -e_2.$$

From the above relations, its clear that M^3 holds $\varphi^2 X = -X + \eta(X)\xi$ and $g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$ for all X, $Y \in TM^3$. Hence M^3 has an almost contact structure.

Next, using Koszul's formula, we have

$$\nabla_{e_1} e_1 = \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = 0, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = 0,$$
$$\nabla_{e_2} e_1 = -\alpha e_3, \quad \nabla_{e_2} e_3 = \alpha e_1, \quad \nabla_{e_3} e_1 = -\alpha e_2, \quad \nabla_{e_3} e_2 = \alpha e_1.$$

Comparing the above relations with $\nabla_X \xi = h\varphi X$, we caluculate

$$he_1 = 0, \qquad he_2 = \alpha e_2, \qquad he_3 = -\alpha e_3.$$

From the formula $R(Y,Z)W = \nabla_Y \nabla_Z W - \nabla_Z \nabla_Y W - \nabla_{[Y,Z]} W$, we obtain

$$R(e_1, e_2)e_1 = \alpha^2 e_2, \qquad R(e_2, e_3)e_1 = 0, \qquad R(e_1, e_3)e_1 = \alpha^2 e_3.$$

In view of the above relations we can easily conclude that, for α being constant on M, the field e_1 belongs to $(-\alpha^2, 0)$-nullity distribution. Therefore, (φ, e_1, g, η) is a $(-\alpha^2, 0)$-almost coKähler structure on M^3. Moreover, the remaining curvature tensor is given by

$$\begin{aligned} R(e_3, e_2)e_2 &= -\alpha^2 e_3, \\ R(e_3, e_2)e_3 &= -\alpha^2 e_1, \\ R(e_1, e_2)e_2 &= -\alpha^2 e_1, \\ R(e_1, e_2)e_3 &= 0. \end{aligned}$$

We simply obtain Ricci tensor S for the above defined basis,

$$\begin{aligned} S(e_1, e_1) &= S(e_2, e_2) = S(e_3, e_3) = -2\alpha^2, \\ S(e_1, e_2) &= S(e_2, e_3) = S(e_3, e_1) = 0. \end{aligned}$$

Let $V = a_1 e_1 + a_2 e_2 + a_3 e_3$, where a_1, a_2, a_3 are the real numbers and V be the general vector field. Now from equation (2), for the potential vector field V, we have

$$\begin{aligned} S(e_1, e_1) &= -\lambda - \mu', \\ S(e_2, e_2) &= S(e_2, e_2) = -\lambda, \\ S(e_1, e_2) &= -\frac{1}{2}\alpha a_3, \\ S(e_1, e_3) &= -\frac{1}{2}\alpha a_2, \\ S(e_2, e_3) &= \alpha a_1. \end{aligned}$$

Therefore, it is clear that for $\lambda = 2\alpha^2$, $\mu' = 0$, and $\alpha a_1 = \alpha a_2 = \alpha a_3 = 0$, M admits an η-Ricci soliton for soliton field V. Now we discuss the following cases:

1. If M^3 is non-coKähler, then $\lambda + \mu' = 2\alpha^2 > 0$.
2. If $\lambda + \mu' = 0$, then $\alpha = 0$ which shows the coKähler structure on M^3.

Therefore, the above cases justify the Theorem 3.2 and Theorem 3.3, respectively.

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