Acta Math. Univ. Comenianae
Vol. LXXXV, 1 (2016), pp. 97–105

ON A GENERALIZATION OF SOME THEOREMS ON THE SMOOTHNESS OF THE SUM OF TRIGONOMETRIC SERIES

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Abstract. In this paper, consider the trigonometric series

$$\sum_{m\in\mathbb{Z}} c_m \,\mathrm{e}^{\mathrm{i}\, mx},$$

where $(c_m)_{\in\mathbb{Z}}$ is a sequence of complex numbers such that

$$\sum_{m\in\mathbb{Z}} |m|^{r-1}|c_m| < +\infty, \qquad (r = 1, 2, \dots).$$

Then the $(r-1)$-th derivative of the trigonometric series converges absolutely and uniformly. If we denote the sum function of such trigonometric series by $f(x)$, then its $(r-1)$-th derivative $f^{(r-1)}(x)$ is obviously a continuous one. We give sufficient conditions in terms of some means of $(c_m)_{\in\mathbb{Z}}$ to ensure that $f(x)$ belongs to one of the classes $W^r(\alpha)$ or $w^r(\alpha)$ for $0 < \alpha \le 2$. The results of Krizsán and Móricz obtained in [**1**] and those of Zygmund obtained in [**2**] are particular results of ours.

1. Introduction and Known Results

Let $f\colon \mathbb{T} := [-\pi, \pi) \to \mathbb{C}$ be a periodic function. The following classes of functions can be found in [**1**] and [**2**]:

1. For some $\alpha > 0$, f belongs to the Lipschitz class $\mathrm{Lip}(\alpha)$ if

$$|\triangle f(x;h)| := |f(x+h) - f(x)| \le Ch^{\alpha} \qquad \text{for all } x \text{ and } h > 0, \tag{1}$$

   where the positive constant C depends only on f.

2. For some $\alpha > 0$, f belongs to the little Lipschitz class $\mathrm{lip}(\alpha)$ if

$$\lim_{h\to 0} \frac{|f(x+h) - f(x)|}{h^{\alpha}} = 0 \qquad \text{uniformly in } x.$$

3. For some $\alpha > 0$, a continuous function f belongs to the Zygmund class $\mathrm{Zyg}(\alpha)$ if

$$|\triangle^2 f(x;h)| := |f(x+h) - 2f(x) + f(x-h)| \le Ch^{\alpha} \tag{2}$$

   for all x and $h > 0$, where the positive constant C depends only on f.

Received October 16, 2014.
2010 *Mathematics Subject Classification.* Primary 26A16, 42A16.
Key words and phrases. Trigonometric series; absolute convergence; Lipschitz classes; Zygmund classes; smooth functions.

4. For some $\alpha > 0$, a continuous function f belongs to the little Zygmund class $\mathrm{zyg}(\alpha)$ if
$$\lim_{h\to 0} \frac{|f(x+h) - 2f(x) + f(x-h)|}{h^\alpha} = 0 \quad \text{uniformly in } x.$$

One can find that (see, for example, [**7**, pages 43 and 44]) a function f may be non measurable in Lebesgue's sense and still satisfies the condition
$$f(x+h) - 2f(x) + f(x-h) = 0 \qquad \text{for all } x \text{ and } h > 0,$$
which reveals the reason why in the definitions of the classes $\mathrm{Zyg}(\alpha)$ and $\mathrm{zyg}(\alpha)$ the continuity of the function f is required.

As pointed out in [**1**], every continuous periodic function is bounded, and thus it is enough to require the fulfillment of conditions (1) and (2) for all $0 < h < 1$ and obviously for all $x \in \mathbb{T}$ (this fact is considered throughout this paper). In the same paper, the interested reader can find the relations between classes $\mathrm{Lip}(\alpha)$, $\mathrm{lip}(\alpha)$ and $\mathrm{Zyg}(\alpha)$, $\mathrm{zyg}(\alpha)$, respectively (those are not recalled here).

Let $(c_m)_{m\in\mathbb{Z}} \in \mathbb{C}$ be such that
$$\sum_{m\in\mathbb{Z}} |c_m| < +\infty. \tag{3}$$
Then the trigonometric series
$$\sum_{m\in\mathbb{Z}} c_m \,\mathrm{e}^{\mathrm{i}\, mx}, \quad x \in \mathbb{T}, \tag{4}$$
converges absolutely and uniformly, and we denote its sum-function by $f(x)$.

Krizsán and Móricz [**1**] gave sufficient conditions in terms of certain means of $(c_m)_{m\in\mathbb{Z}}$ to ensure that the sum-function $f(x)$ of the trigonometric series of the complex form $\sum_{m\in\mathbb{Z}} c_m \,\mathrm{e}^{\mathrm{i}\, mx}$ belongs to one of Zygmund classes $\mathrm{Zyg}(\alpha)$ or $\mathrm{zyg}(\alpha)$ for some $\alpha \in (0, 2]$ proving the following theorems:

Theorem 1.1 ([**1**]). *Let $(c_m)_{m\in\mathbb{Z}} \in \mathbb{C}$. If for some $\alpha \in (0, 2]$ we have,*
$$\frac{1}{M^{2-\alpha}} \sum_{|m|\le M} m^2 |c_m| \le C_\alpha \qquad \text{for all } M = 1, 2, \dots, \tag{5}$$
where C_α is a positive constant, then the series (4) converges absolutely and uniformly, and its sum-function $f(x) \in \mathrm{Zyg}(\alpha)$.

Theorem 1.2 ([**1**]). *Let $(c_m)_{m\in\mathbb{Z}} \in \mathbb{C}$. If for some $\alpha \in (0, 2)$ we have,*
$$\lim_{M\to\infty} \frac{1}{M^{2-\alpha}} \sum_{|m|\le M} m^2 |c_m| = 0, \tag{6}$$
then $f(x) \in \mathrm{zyg}(\alpha)$.

Theorem 1.3 ([**1**]). *Let $(c_m)_{m\in\mathbb{Z}} \in \mathbb{C}$ be such that*
$$\sum_{m\in\mathbb{Z}} |mc_m| < +\infty, \tag{7}$$
then $f(x) \in \mathrm{Lip}(1)$.

Let the function f has derivatives of order r, $(r = 1, 2, \dots)$. We say that for some $\alpha > 0$, f belongs to the class $\mathrm{W}^r(\alpha)$ if

$$|\triangle^2 f^{(r-1)}(x;h)| := |f^{(r-1)}(x+h) - 2f^{(r-1)}(x) + f^{(r-1)}(x-h)| \le Ch^{\alpha} \tag{8}$$

for all x and $h > 0$, where the positive constant C depends only on f.

We say that for some $\alpha > 0$, f belongs to the class $\mathrm{w}^r(\alpha)$ if

$$\lim_{h \to 0} \frac{|f^{(r-1)}(x+h) - 2f^{(r-1)}(x) + f^{(r-1)}(x-h)|}{h^{\alpha}} = 0$$

for all x, $h > 0$ and r.

It is clear that for $r = 1$ and for some $\alpha \in (0, 2]$, we have $\mathrm{W}^1(\alpha) \equiv \mathrm{Zyg}(\alpha)$ and $\mathrm{w}^1(\alpha) \equiv \mathrm{zyg}(\alpha)$ while for $r = 1$ and $\alpha \in (0, 1]$, we obtain $\mathrm{W}^1(\alpha) \equiv \mathrm{Lip}(\alpha)$ and $\mathrm{w}^1(\alpha) \equiv \mathrm{lip}(\alpha)$. The main aim of this paper is to give sufficient conditions in terms of certain means of $(c_m)_{m\in\mathbb{Z}}$ to ensure that the sum-function $f(x)$ of the trigonometric series of the complex form $\sum_{m\in\mathbb{Z}} c_m \,\mathrm{e}^{\mathrm{i}\, mx}$ belongs to one of classes $\mathrm{W}^r(\alpha)$ or $\mathrm{w}^r(\alpha)$ for some $\alpha \in (0, 2]$. For the proof of our results, we have adopted the reasoning used by authors of the paper [**1**].

2. Helpful Lemmas

In this section, we prove some helpful statements needed for the proofs of main results and which indeed are of some interest in themselves.

Let $(c_m)_{m\in\mathbb{Z}} \in \mathbb{C}$ be such that

$$\sum_{m\in\mathbb{Z}} |m|^{r-1}|c_m| < +\infty, \qquad r \in \{1, 2, \dots\}. \tag{9}$$

Then obviously the formal derivative of order $(r-1)$ of the trigonometric series (4)

$$\sum_{m\in\mathbb{Z}} (im)^{r-1} c_m \,\mathrm{e}^{\mathrm{i}\, mx}, \qquad x \in \mathbb{T}, \tag{10}$$

converges absolutely and uniformly, and we denote its sum-function by $f^{(r-1)}(x)$.

Lemma 2.1. *Let $(c_m)_{m\in\mathbb{Z}} \in \mathbb{C}$.*

(i) *If for some $\alpha \in (0, 2]$ and $r \in \{1, 2, \dots\}$, we have*

$$\frac{1}{M^{2-\alpha}} \sum_{|m|\le M} |m|^{r+1}|c_m| \le C_{\alpha} \qquad \text{for all } M = 1, 2, \dots, \tag{11}$$

then there exists another positive constant $\widetilde{C}_{\alpha}$ such that

$$M^{\alpha} \sum_{|m|\ge M} |m|^{r-1}|c_m| \le \widetilde{C}_{\alpha} \qquad \text{for all } M = 1, 2, \dots, \tag{12}$$

and in particular,

$$\sum_{m\in\mathbb{Z}} |m|^{r-1}|c_m| < +\infty.$$

(ii) *Conversely, if condition* (12) *is satisfied for some* $\alpha \in [0,2)$ *and* $r \in \{1,2,\dots\}$, *then the condition* (11) *is satisfied as well. In particular, for* $\alpha \in (0,2)$ *and* $r \in \{1,2,\dots\}$, *conditions* (11) *and* (12) *are equivalent.*

Proof. Part (i) Denote the set $\{2^p, 2^p+1, \dots, 2^{p+1}-1\}$, $(p = 0,1,2,\dots)$ by D_p. Based on condition (11) for a given nonnegative integer p, we get

$$2^{2p} \sum_{|m|\in D_p} |m|^{r-1}|c_m| \le \sum_{|m|\in D_p} |m|^{r+1}|c_m| \le C_\alpha 2^{(p+1)(2-\alpha)},$$

and thus

$$\sum_{|m|\in D_p} |m|^{r-1}|c_m| \le 2^{2-\alpha} C_\alpha 2^{-p\alpha}, \qquad (p = 0,1,2,\dots;\ r = 1,2,\dots). \tag{13}$$

Let q be any nonnegative integer. Since $\alpha > 0$, we have

$$\begin{aligned}\sum_{|m|\ge 2^q} |m|^{r-1}|c_m| &= \sum_{p=q}^{\infty} \sum_{|m|\in D_p} |m|^{r-1}|c_m| \\ &\le 2^{2-\alpha} C_\alpha \sum_{p=q}^{\infty} 2^{-p\alpha} = 2^{2-\alpha} C_\alpha \frac{2^{-q\alpha}}{1-2^{-\alpha}}.\end{aligned} \tag{14}$$

This implies

$$2^{q\alpha} \sum_{|m|\ge 2^q} |m|^{r-1}|c_m| \le \frac{4C_\alpha}{2^\alpha - 1} =: \widetilde{C}_\alpha \qquad \text{for all } q = 0,1,2,\dots,$$

which means that (12) holds true for the subsequence $\{M = 2^q : q = 0,1,2,\dots\}$. Consequently, the truth of (12) for full sequence $\{M : M = 1,2,\dots\}$ follows easily.

Part (ii) Let p be any nonnegative integer. Then based on (12), we have

$$\frac{1}{2^{2(p+1)}} \sum_{|m|\in D_p} |m|^{r+1}|c_m| \le \sum_{|m|\in D_p} |m|^{r-1}|c_m| \le \widetilde{C}_\alpha 2^{-p\alpha},$$

that implies

$$\sum_{|m|\in D_p} |m|^{r+1}|c_m| \le 4\widetilde{C}_\alpha 2^{(2-\alpha)p} \qquad \text{for all } p = 0,1,2,\dots. \tag{15}$$

Let $q \ge 1$ be any integer. Then taking into account that $\alpha < 2$, we have

$$\begin{aligned}\sum_{|m|<2^q} |m|^{r+1}|c_m| &= \sum_{p=0}^{q-1} \sum_{|m|\in D_p} |m|^{r+1}|c_m| \\ &\le 4\widetilde{C}_\alpha \sum_{p=0}^{q-1} 2^{(2-\alpha)p} = 4\widetilde{C}_\alpha \frac{2^{(2-\alpha)q}-1}{2^{2-\alpha}-1}.\end{aligned} \tag{16}$$

Whence form (16), we clearly obtain

$$\frac{1}{2^{(2-\alpha)q}-1}\sum_{|m|\le 2^q-1}|m|^{r+1}|c_m|\le \frac{4\widetilde{C}_\alpha}{2^{2-\alpha}-1},\qquad (q=1,2,\dots). \tag{17}$$

It is not difficult to calculate that for $0\le\alpha<2$ and $r=1,2,\dots$, the limit

$$\lim_{q\to\infty}\frac{2^{(2-\alpha)q}-1}{(2^q-1)^{(2-\alpha)}}=1$$

holds true. It means that there exists a constant γ_α depending only on α such that

$$\frac{1}{(2^q-1)^{(2-\alpha)}}\le\frac{\gamma_\alpha}{2^{(2-\alpha)q}-1}\qquad \text{for all } q=1,2,\dots.$$

Consequently, based on (17), we obtain

$$\frac{1}{(2^q-1)^{(2-\alpha)}}\sum_{|m|\le 2^q-1}|m|^{r+1}|c_m|\le\frac{4\widetilde{C}_\alpha\gamma_\alpha}{2^{2-\alpha}-1}=:C_\alpha. \tag{18}$$

By that the proof of this Lemma is completed in the special case $M=2^q-1$, while for the general case when $M=1,2,\dots$, it follows easily. □

Remark. Lemma 2.1 fails to be true at the end points. Indeed, for $\alpha=0$ and $(c_m)_{m\in\mathbb{Z}\setminus\{0\}}=(1/(m^r))_{m\in\mathbb{Z}\setminus\{0\}}$, $r\in\{1,2,\dots\}$, we have

$$\frac{1}{M^{2-\alpha}}\sum_{|m|\le M}|m|^{r+1}|c_m|\le 2,$$

while

$$M^\alpha\sum_{|m|\ge M}|m|^{r-1}|c_m|=\infty.$$

On the other hand for, $\alpha=2$ and $(c_m)_{m\in\mathbb{Z}\setminus\{0\}}=(1/(m^{r+2}))_{m\in\mathbb{Z}\setminus\{0\}}$, $r\in\{1,2,\dots\}$, the condition (11) is not satisfied, but the condition (12) is.

Lemma 2.2. *Let* $(c_m)_{m\in\mathbb{Z}}\in\mathbb{C}$.

(i) *If for some* $\alpha\in(0,2]$ *and* $r\in\{1,2,\dots\}$ *we have,*

$$\lim_{M\to\infty}\frac{1}{M^{2-\alpha}}\sum_{|m|\le M}|m|^{r+1}|c_m|=0, \tag{19}$$

then we have

$$\lim_{M\to\infty}M^\alpha\sum_{|m|\ge M}|m|^{r-1}|c_m|=0 \tag{20}$$

as well.

(ii) *Conversely, if condition* (20) *is satisfied for some* $\alpha\in[0,2)$ *and* $r\in\{1,2,\dots\}$, *then the condition* (19) *is satisfied as well. In particular, for* $\alpha\in(0,2)$ *and* $r\in\{1,2,\dots\}$, *conditions* (19) *and* (20) *are equivalent.*

Proof. Part (i) Based on (19), for every $\varepsilon > 0$, there exists an integer $p_0 = p_0(\varepsilon) \geq 0$ such that

$$\frac{1}{M^{2-\alpha}} \sum_{|m| \leq M} |m|^{r+1} |c_m| < \varepsilon \qquad \text{for all } M \geq 2^{p_0}.$$

Let $q \geq p_0$ be an integer. Reasoning in the same way as for (13) and (14), we have obtained

$$\sum_{|m| \geq 2^q} |m|^{r-1} |c_m| \leq 2^{2-\alpha} \varepsilon \frac{2^{-q\alpha}}{1 - 2^{-\alpha}},$$

and thus we have

$$2^{q\alpha} \sum_{|m| \geq 2^q} |m|^{r-1} |c_m| \leq \frac{4\varepsilon}{2^\alpha - 1} \qquad \text{for all } q \geq p_0.$$

Taking into account that $\varepsilon > 0$ is arbitrary, the condition (20) holds true for the subsequence $\{M = 2^q : q = 1, 2, \dots\}$. The truth of (20) for the full sequence $\{M : M = 1, 2, \dots\}$ follows easily.

Part (ii) Let the condition (20) be satisfied. Then for every $\varepsilon > 0$, there exists an integer $\tilde{p}_0 = \tilde{p}_0(\varepsilon) \geq 0$ such that

$$M^\alpha \sum_{|m| \geq M} |m|^{r-1} |c_m| < \varepsilon \qquad \text{for all } M \geq 2^{\tilde{p}_0}.$$

Let $q > \tilde{p}_0$ be any integer. Reasoning in the same way as for (15) and (16), we have obtained

$$\sum_{|m| < 2^q} |m|^{r+1} |c_m| \leq 4\varepsilon \frac{2^{(2-\alpha)q} - 1}{2^{2-\alpha} - 1}$$

as well as for (17) and (18), it follows that

$$\frac{1}{(2^q - 1)^{(2-\alpha)}} \sum_{|m| \leq 2^q - 1} |m|^{r+1} |c_m| \leq \frac{4\varepsilon\gamma_\alpha}{2^{2-\alpha} - 1} \qquad \text{for all } q > \tilde{p}_0.$$

Taking into account that $\varepsilon > 0$ is arbitrary, the condition (19) holds true for the subsequence $\{M = 2^q - 1 : q = 1, 2, \dots\}$. The truth of (19) for the full sequence $\{M : M = 1, 2, \dots\}$ follows easily. □

3. Main Results

First we prove the following main result.

Theorem 3.1. *Let $(c_m)_{m \in \mathbb{Z}} \in \mathbb{C}$. If for some $\alpha \in (0, 2]$ and $r = 1, 2, \dots,$ we have*

$$\frac{1}{M^{2-\alpha}} \sum_{|m| \leq M} m^{r+1} |c_m| \leq C_\alpha \qquad \textit{for all } M = 1, 2, \dots, \tag{21}$$

where $C_{\alpha,r}$ is a positive constant, then the series

$$\sum_{m\in\mathbb{Z}} (\mathrm{i}\, m)^{r-1} c_m \,\mathrm{e}^{\mathrm{i}\, mx}, \qquad x \in \mathbb{T}, \tag{22}$$

converges absolutely and uniformly, and for the sum-function of the series (4)*, $f(x) \in W^r(\alpha)$ holds.*

Proof. Let $x \in \mathbb{T}$, $0 < h < 1$ be arbitrary and $r \in \{1, 2, \dots\}$. By (10), we have

$$\begin{aligned}\triangle^2 f^{(r-1)}(x; 2h) &= \sum_{m\in\mathbb{Z}} (\mathrm{i}\, m)^{r-1} c_m \,\mathrm{e}^{\mathrm{i}\, mx} \left(\mathrm{e}^{\mathrm{i}\, m2h} - 2 + \mathrm{e}^{-\mathrm{i}\, m2h}\right)\\ &= -4 \sum_{m\in\mathbb{Z}} (\mathrm{i}\, m)^{r-1} c_m \,\mathrm{e}^{\mathrm{i}\, mx} \sin^2 mh,\end{aligned}$$

and thus

$$\begin{aligned}\frac{|\triangle^2 f^{(r-1)}(x; 2h)|}{(2h)^\alpha} &\le \frac{2^{2-\alpha}}{h^\alpha} \left(\sum_{|m|\le M} + \sum_{|m|>M}\right) |m|^{r-1} |c_m| \sin^2 mh\\ &=: S_1 + S_2,\end{aligned} \tag{23}$$

where

$$M := \left[\frac{1}{h}\right], \qquad 0 < h < 1, \tag{24}$$

and $[\,\cdot\,]$ denotes the integer part of a real number.

Based on (11) and (24), we obtain

$$\begin{aligned} S_1 &\le \frac{2^{2-\alpha}}{h^\alpha} \sum_{|m|\le M} |m|^{r-1} |c_m| (mh)^2\\ &\le \left(\frac{2}{M}\right)^{2-\alpha} \sum_{|m|\le M} |m|^{r+1} |c_m| \le 2^{2-\alpha} C_\alpha.\end{aligned} \tag{25}$$

In the sequel using part (i) of the Lemma 2.1 and (24), we have

$$\begin{aligned} S_2 &\le \frac{2^{2-\alpha}}{h^\alpha} \sum_{|m|>M} |m|^{r-1} |c_m|\\ &\le 2^{2-\alpha} (M+1)^\alpha \sum_{|m|\ge M+1} |m|^{r-1} |c_m| \le 2^{2-\alpha} \widetilde{C}_\alpha.\end{aligned} \tag{26}$$

Finally, (23) along with (25) and (26) implies

$$|\triangle^2 f^{(r-1)}(x; 2h)| \le 4h^\alpha \left(C_\alpha + \widetilde{C}_\alpha\right) \qquad \text{for all } x \in \mathbb{T} \text{ and } 0 < h < 1.$$

Last estimate verifies that $f(x) \in \mathrm{W}^r(\alpha)$. □

Theorem 3.2. *Let $(c_m)_{m\in\mathbb{Z}} \in \mathbb{C}$. If for some $\alpha \in (0,2)$ and $r = 1, 2, \dots,$ we have*

$$\lim_{M\to\infty} \frac{1}{M^{2-\alpha}} \sum_{|m|\le M} m^{r+1}|c_m| = 0, \tag{27}$$

then $f(x) \in w^r\,(\alpha)$.

Proof. This time we use Lemma 2.2 for the proof of this theorem that runs along the same lines as in the proof of the Theorem 3.1. We omit it and leave details to the interested reader. □

Theorem 3.3. *Let $r = 1, 2, \dots$ and $(c_m)_{m\in\mathbb{Z}} \in \mathbb{C}$ be such that*

$$\sum_{m\in\mathbb{Z}} |m^r c_m| < +\infty, \tag{28}$$

then $f^{(r-1)}(x) \in \mathrm{Lip}(1)$.

Proof. Let $x \in \mathbb{T}$, $0 < h < 1$ be arbitrary and $r \in \{1, 2, \dots\}$. By (10), we have

$$\begin{aligned}\triangle f^{(r-1)}(x;2h) &= \sum_{m\in\mathbb{Z}} (\mathrm{i}\, m)^{r-1} c_m\, \mathrm{e}^{imx} \left(\mathrm{e}^{\mathrm{i}\, m2h} - 1\right)\\ &= \sum_{m\in\mathbb{Z}} (\mathrm{i}\, m)^{r-1} c_m\, \mathrm{e}^{\mathrm{i}\, m(x+h)} \left(\mathrm{e}^{\mathrm{i}\, mh} - \mathrm{e}^{-\mathrm{i}\, mh}\right)\\ &= 2\sum_{m\in\mathbb{Z}} \mathrm{i}^r\, m^{r-1} c_m\, \mathrm{e}^{\mathrm{i}\, m(x+h)} \sin mh.\end{aligned} \tag{29}$$

Again let M be defined by

$$M := \left[\frac{1}{h}\right], \quad 0 < h < 1.$$

Then based on definition of M and (29), we obtain

$$\begin{aligned}\frac{|\triangle^2 f^{(r-1)}(x;2h)|}{h} &\le \frac{2}{h}\left(\sum_{|m|\le M} + \sum_{|m|>M}\right) |m|^{r-1}|c_m||\sin mh|\\ &\le 2\sum_{|m|\le M} |m|^r|c_m| + 2(M+1)\sum_{|m|\ge M+1} |m|^{r-1}|c_m|\\ &\le 2\sum_{m\in\mathbb{Z}} |m|^r|c_m| < +\infty,\end{aligned} \tag{30}$$

due to the assumption of the theorem. The latest estimate verifies that $f^{(r-1)}(x) \in \mathrm{Lip}(1)$. The proof is completed. □

Remark. Let $\alpha = 1$, then the condition (12) follows from the condition (28), but conversely, in general it fails to be true. For instance, taking into consideration the sequence $(c_m)_{m\in\mathbb{Z}\smallsetminus\{0\}} = (1/(m^{r+1}))_{m\in\mathbb{Z}\smallsetminus\{0\}}$, $r \in \{1, 2, \dots\}$ and $\alpha = 1$, we verify easily that the condition (12) holds true, but the condition (28) does not.

Remark. Note that putting $r = 1$ in Theorems 3.1–3.3, we immediately obtain Theorems 1.1–1.3 proved in [**1**].

Remark. Putting $r = 1$ and $\alpha = 1$ in Theorems 3.1–3.3, we obtain some results proved in [**2**].

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