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NEW PROOFS OF RESULTS CONCERNING BASES OF A LATTICE

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Abstract. Applying basic facts of linear algebra, we present new simpler and much shorter proofs of results presented by Cherednik in the paper [*The non-negative basis of a lattice*, Diskret. Mat. **26**(3) 2014, 127–135]. Recall, Cherednik proved that each lattice of dimension n in the linear space $\mathbb{R}^n$ has a basis consisting non-negative vectors, (i.e., vectors which contain only non-negative coordinates). Applying this theorem, he also showed that an arbitrary (not necessarily of the maximal dimension) lattice has such a basis if and only if it is generated by all its non-negative vectors. Next, these results are generalized for arbitrary convex cones (note that the set of all non-negative vectors is a convex cone). Finally, he showed that each lattice of dimension $n \geq 2$ in $\mathbb{R}^n$ has a basis in any translation of every convex cone of dimension n.

Take the linear space $\mathbb{R}^n$, where $\mathbb{R}$ is the field of real numbers. A *lattice* L of dimension k (where $0 \leq k \leq n$) in $\mathbb{R}^n$ is a subgroup of the abelian group $(\mathbb{R}^n, +)$, which is generated by k linearly independent (over $\mathbb{R}$) vectors $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$ which are called a basis of L. In other words, L consists of all linear combinations of $v_1, v_2, \ldots, v_n$ with integer coefficients. If $v_1, v_2, \ldots, v_n \in \mathbb{Z}^n$ (equivalently, $L \subseteq \mathbb{Z}^n$), then L is called an *integer lattice*, where $\mathbb{Z}$ is the set of integers. A lattice $L \subseteq \mathbb{R}^n$ is called *full* if its dimension equals n (see [**1**]). For example, $\mathbb{Z}^n$ is a full integer lattice in $\mathbb{R}^n$ generated by the standard basis $\varepsilon_1 = (1, 0, 0, \ldots, 0, 0), \varepsilon_2 = (0, 1, 0, \ldots, 0, 0), \ldots, \varepsilon_n = (0, 0, 0, \ldots, 0, 1)$ of $\mathbb{R}^n$. Next, a lattice of dimension 0 contains only the zero vector $\mathbf{0} = (0, 0, \ldots, 0)$ (i.e., it is a trivial lattice).

Of course, lattices in $\mathbb{R}^n$ are isomorphic with finitely generated free abelian groups, i.e., if L is a lattice of dimension k, then $L \simeq \mathbb{Z}^k$. Recall that each free abelian group has a group basis (i.e., a set of generators which are linearly independent over $\mathbb{Z}$) and all its group bases have the same cardinality that is called the rank of a group (see [**4**, Theorem 10.14, Chapter 10]). Every subgroup H of a free abelian group G is free and its rank is not greater that the rank of G (see [**4**, Theorem 10.17, Chapter 10]). If G is a finitely generated free abelian group and H is its subgroup of finite index, then there is a group basis $g_1, g_2, \ldots, g_n$ of G and positive integers $l_1, l_2, \ldots l_n \in \mathbb{N} \setminus \{0\}$ such that $l_1 g_1, l_2 g_2, \ldots, l_n g_n$ form a group basis of H (see [**4**, Theorem 10.21, Chapter 10]). Next, a subgroup H of G has

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a finite index if and only if H is a free abelian of rank n (see [**4**, Exercise 10.15, Chapter 10]). All these facts are particular cases of analogous results holding for modules over principal ideal domains (see [**2**, Chapter 3, Section 7]) because abelian groups can be considered as modules over the ring $\mathbb{Z}$ of integers. By these module results, we have that the finiteness of index of H is not necessary, i.e., if H is a subgroup of a finitely generated free abelian group G, then there is a group basis $g_1, g_2, \ldots, g_n$ of G and $l_1, l_2, \ldots l_k \in \mathbb{N} \setminus \{0\}$ (where $k \leq n$) such that $l_1 g_1, l_2 g_2, \ldots, l_k g_k$ form a group basis of H.

Take a lattice $L \subseteq \mathbb{R}^n$ with basis $v_1, v_2, \ldots, v_k \in L$. Each basis of L is also a group basis. Thus the rank of L equals k. Conversely, each group basis $\alpha_1, \alpha_2, \ldots, \alpha_k$ is linearly independent over $\mathbb{R}$ because $v_1, v_2, \ldots, v_k$ are contained in the least subspace of $\mathbb{R}^n$ generated by $\alpha_1, \alpha_2, \ldots, \alpha_k$. Hence concepts of basis of a lattice and of group basis of a free abelian group are equivalent for lattices. Next, take a subgroup K of L. It is a free group and there is a group basis $\gamma_1, \gamma_2, \ldots, \gamma_k$ of L such that $b_1\gamma_1, b_2\gamma_2, \ldots, b_m\gamma_m$ form a group basis of K for some $b_1, b_2, \ldots, b_m \in \mathbb{N} \setminus \{0\}$ (where m is the rank of K). Then $b_1\gamma_1, b_2\gamma_2, \ldots, b_m\gamma_m$ are linearly independent over $\mathbb{R}$ because $\gamma_1, \gamma_2, \ldots, \gamma_k$ are independent. Hence K is a lattice of dimension m. Finally, take $\beta_1, \beta_2, \ldots, \beta_m \in L$ linearly independent over $\mathbb{Z}$. Then the subgroup J of L generated by these elements is free of rank m, so J is a lattice of dimension m. Hence $\beta_1, \beta_2, \ldots, \beta_m$ are linearly independent over $\mathbb{R}$.

A subset M of $\mathbb{R}^n$ is *convex* (see [**3**]) if $(1-a)v + aw \in M$ for all $v, w \in M$ and $0 \leq a \leq 1$. Next, $N \subseteq \mathbb{R}^n$ is a *cone* (see [**1**]) if $av \in N$ for all $v \in N$ and each $a \in \mathbb{R}_+$, where $\mathbb{R}_+$ is the set of all non-negative real numbers. For example, the set $\mathbb{R}^n_+ = \{(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n\colon\ a_1, a_2, \ldots, a_n \in \mathbb{R}_+\}$ is a convex cone. Vectors of the convex cone $\mathbb{R}^n_+$ are called *non-negative*, a basis consisting of non-negative vectors is also called *non-negative* (see [**1**]).

A set $S \subseteq \mathbb{R}^n$ is said to have the dimension k if the least affine subspace of $\mathbb{R}^n$ containing S has the dimension k. The least affine subspace containing a given cone C is linear (because $\mathbf{0} \in C$). Thus it is a standard observation that a cone C is of dimension k if and only if all (equivalently, at least one) maximal (up to inclusion) subsets of C of linearly independent vectors have k elements.

Some interesting properties of bases of lattices in $\mathbb{R}^n$ are investigated by Cherednik in the paper [**1**]. The first main result (see [**1**, Theorem 1]) shows that each full lattice in $\mathbb{R}^n$ has a non-negative basis. Applying this theorem, Cherednik also proved (see [**1**, Proposition 2 and Corollary 2]) that an arbitrary lattice $L \subseteq \mathbb{R}^n$ has a non-negative basis if and only if L is generated by all its non-negative vectors (of course, " $\Longrightarrow$ " is trivial). Next, these two results are generalized for arbitrary convex cones (see [**1**, Theorem 3 and Proposition 3]). The last main result shows that each full lattice in $\mathbb{R}^n$ has a basis in any translation of every convex cone of dimension n (see [**1**, Theorem 4]).

In this paper, we apply basic facts of linear algebra to present new proofs of Cherednik's results which are simpler and much shorter than the original.

We need the fact proved in [**1**] (see Proposition 1) that each full integer lattice has a non-negative basis. Its proof is not long, so we recall it now. More precisely, applying methods from this proof, we can show the following slightly more general fact.

Lemma 1. *Let $L \subseteq \mathbb{Z}^n$ be an integer lattice. Then there is a basis $v_1, v_2, \ldots, v_k$ of L (where $k \leq n$) such that* $\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{pmatrix}$ *is an upper triangular $k \times n$-matrix and the first non-zero entry in each row is positive.*

Proof. The trivial lattice $\{\mathbf{0}\}$ has the empty basis, so we can assume that L has a non-zero vector. For $i = 1, 2, \ldots, n$, let $\pi_i \colon \mathbb{Z}^n \longrightarrow \mathbb{Z}$ be the projection on the i-th coordinate. Take the least positive integer $i_1 \leq n$ such that $\pi_{i_1}(L) \subseteq \mathbb{Z}$ has non-zero element. Then $\pi_{i_1}(L)$ is a non-trivial subgroup of $\mathbb{Z}$, so there is a positive integer a_1 which generates $\pi_{i_1}(L)$. Take any $v_1 \in L$ such that $\pi_{i_1}(v_1) = a_1$ and the set $K_1 = \{w \in L \colon \pi_{i_1}(w) = 0\}$. Then K_1 is a subgroup (equivalently, a sublattice) of L and $L = \langle v_1 \rangle_L \oplus K_1$. If $K_1 = \{\mathbf{0}\}$, then the proof is complete. If not, then, we can repeat this procedure to K_1, and so on. $\square$

Corollary 2 ([**1**, Proposition 1])**.** *Each full integer lattice $L \subseteq \mathbb{Z}^n$ has a non-negative basis.*

Proof. Take a basis $v_1 = (a_1^1, a_2^1, \ldots, a_{n-1}^1, a_n^1), v_2 = (0, a_2^2, \ldots, a_{n-1}^2, a_n^2), \ldots, v_n = (0, 0, \ldots, 0, a_n^n)$ of L from Lemma 1 (here $k = n$ because L is a full lattice). Since $v_1, v_2, \ldots, v_n$ are linearly independent vectors (over $\mathbb{R}$), the matrix $\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} a_1^1 & a_2^1 & \ldots & a_{n-1}^1 & a_n^1 \\ 0 & a_2^2 & \ldots & a_{n-1}^2 & a_n^2 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & a_n^n \end{pmatrix}$ is non-singular. Thus $\det \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = a_1^1 \cdot a_2^2 \cdot \cdots \cdot a_n^n \neq 0$. This fact and Lemma 1 imply that integers $a_1^1, a_2^2, \ldots, a_n^n$ are positive. Then, for each $i = 1, 2, \ldots, n-1$, there are positive integers $b_{i+1}^i, b_{i+2}^i, \ldots, b_n^i$ such that $w_i = v_i + b_{i+1}^i v_{i+1} + \cdots + b_n^i v_n \in \mathbb{R}_+^n \cap L$. Of course, $w_n = v_n$ is also non-negative. It is easy to see that $v_1, v_2, \ldots, v_n$ are linear combinations of $w_1, w_2, \ldots, w_n$ with integer coefficients. Hence $w_1, w_2, \ldots, w_n$ generate L (over $\mathbb{Z}$). Moreover, they span $\mathbb{R}^n$ (over $\mathbb{R}$) because $v_1, v_2, \ldots, v_n$ span $\mathbb{R}^n$. So $w_1, w_2, \ldots, w_n$ are linearly independent. $\square$

Intersection of a family of convex cones is a convex cone. Thus for each $S \subseteq \mathbb{R}^n$, there is the least (up to inclusion) convex cone denoted by $\mathrm{cone}_{\mathbb{R}^n} S$, containing S. For example, $\mathrm{cone}_{\mathbb{R}^n}\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\} = \mathbb{R}_+^n$, where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ is the standard basis of $\mathbb{R}^n$.

The interior of a set $S \subseteq \mathbb{R}^n$ (i.e., the largest open set contained in S) is denoted by $\mathrm{int} S$.

Lemma 3. (a) *Each convex cone C is closed under non-negative linear combinations, i.e., $a_1 v_1 + a_2 v_2 + \cdots + a_k v_k \in C$ for all $v_1, v_2, \ldots, v_k \in C$ and $a_1, a_2, \ldots, a_k \in \mathbb{R}_+$.*

(b) *For* $v_1, v_2, \dots, v_k \in \mathbb{R}^n$, $\mathrm{cone}_{\mathbb{R}^n}\{v_1, v_2, \dots, v_k\} = \{a_1v_1 + a_2v_2 + \cdots + a_kv_k\colon\ a_1, a_2, \dots, a_k \in \mathbb{R}_+\}$.

(c) *For a basis* $v_1, v_2, \dots v_n$ *of* $\mathbb{R}^n$, $\mathrm{int}\,\mathrm{cone}_{\mathbb{R}^n}\{v_1, v_2, \dots v_n\} = \{a_1v_1 + a_2v_2 + \cdots + a_nv_n\colon\ a_1, a_2, \dots, a_n \in \mathbb{R}_+ \setminus \{0\}\}$. *In particular,* $\mathrm{int}\,\mathrm{cone}_{\mathbb{R}^n}\{v_1, v_2, \dots v_n\}$ *is a non-empty set.*

Proof. (a) is implied by the equality $v + w = 2(\frac{1}{2}v + \frac{1}{2}w)$ for $v, w \in \mathbb{R}^n$. The point (b) is obtained by (a) because $\{a_1v_1 + \cdots + a_nv_n\colon\ a_1, \dots, a_n \in \mathbb{R}_+\}$ is a convex cone. Finally, (c) follows from (b). □

Now, we prove two main lemmas of this paper.

Lemma 4. *For each convex cone* $C \subseteq \mathbb{R}^n$ *of dimension* n*, there is a basis of* $\mathbb{R}^n$ *which belongs to* $C \cap \mathbb{Z}^n$.

Proof. It is sufficient to show that there is a basis $\beta_1, \beta_2, \dots, \beta_n$ of $\mathbb{R}^n$ which belongs to $C \cap \mathbb{Q}^n$ (where $\mathbb{Q}$ is the field of rational numbers) because C is closed under multiplication by positive numbers. Since C has the dimension n, there is a basis $v_1, v_2, \dots, v_n$ of $\mathbb{R}^n$ which belongs to C. Take $C_1 = \mathrm{cone}_{\mathbb{R}^n}\{v_1, v_2, \dots, v_n\} \subseteq C$. Then $\mathrm{int}C_1 \neq \emptyset$ (Lemma 3(c)), so there is $\mathbf{0} \neq \beta_1 \in \mathrm{int}C_1 \cap \mathbb{Q}^n$ because the set $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$. Since $\beta_1 \in \mathrm{int}C_1$, we have that β_1 and any $n-1$-element subset of $\{v_1, v_2, \dots, v_n\}$ form basis of $\mathbb{R}^n$. In particular, β_1 and $v_2, \dots, v_n$ are linearly independent.

Now let $C_2 = \mathrm{cone}_{\mathbb{R}^n}\{\beta_1, v_2, \dots, v_n\} \subseteq C_1$. As above, we can choose $\mathbf{0} \neq \beta_2 \in \mathrm{int}C_2 \cap \mathbb{Q}^n$ and obtain that $\beta_1, \beta_2, v_3, \dots, v_n$ are linearly independent. Repeating this procedure n times, we obtain linearly independent vectors $\beta_1, \beta_2, \dots, \beta_n$ which belong to $C \cap \mathbb{Q}^n$. □

Recall that if an $n \times n$-matrix A is non-singular (i.e., its determinant $\det A$ is not equal to 0), then the linear system of equations $A(x_1, x_2, \dots, x_n)^T = \beta^T$ has an exactly one solution given by Cramer's formulas $x_i = \frac{\det A_i}{\det A}$ for $i = 1, 2, \dots, n$, where the matrix A_i is obtained from A by replacing the i-th column by β^T. Here ^{T} denotes the matrix transposition, in particular, $(x_1, x_2, \dots, x_n)^T$ is a matrix with one column and n rows.

Lemma 5. *Let vectors* $v_1, v_2, \dots, v_n \in \mathbb{Z}^n$ *be linearly independent (over* $\mathbb{R}$*). Then* $\mathbb{Z}^n$ *has a basis which belongs to* $\mathrm{cone}_{\mathbb{R}^n}\{v_1, v_2, \dots v_n\}$.

Proof. Take the matrix $A = \left(v_1^T, v_2^T, \dots, v_n^T\right)$ and $d = \left|\det A\right|$. Then $d \gneqq 0$ and A is invertible because $v_1, v_2, \dots, v_n$ form a basis. Let $\theta, \psi\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be maps such that $\theta(v) = d \cdot v$ for $v \in \mathbb{R}^n$ and $\psi(\varepsilon_i) = v_i$ for $i = 1, 2, \dots, n$. They are linear automorphisms and $M(\psi)_{\mathcal{S}t}^{\mathcal{S}t} = A$, i.e., $\psi(v) = A \cdot v^T$ for $v \in \mathbb{R}^n$, where $M(\psi)_{\mathcal{S}t}^{\mathcal{S}t}$ denotes the matrix of ψ in the standard basis $\mathcal{S}t\colon \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$. Hence $M(\psi^{-1})_{\mathcal{S}t}^{\mathcal{S}t} = A^{-1}$. Let $\varphi = \theta \circ \psi^{-1}$, i.e., $\varphi(v) = d \cdot (A^{-1} \cdot v^T)$ for $v \in \mathbb{R}^n$. Then $\varphi(v_i) = d \cdot \varepsilon_i$ for $i = 1, 2, \dots, n$, so $\varphi(\mathrm{cone}_{\mathbb{R}^n}\{v_1, v_2, \dots v_n\}) = \mathrm{cone}_{\mathbb{R}^n}\{\varepsilon_1, \varepsilon_2, \dots \varepsilon_n\} = \mathbb{R}^n_+$.

Observe that A is the change of basis matrix from the basis $\mathcal{A}\colon\ v_1, v_2, \dots, v_n$ to the standard basis $\mathcal{S}t$, so the inverse A^{-1} is the change of basis matrix from $\mathcal{S}t$ to $\mathcal{A}$. Hence $A^{-1} \cdot v^T$ gives coordinates of v in the basis $\mathcal{A}$ for each $v \in \mathbb{R}^n$. On the other hand, Cramer's formulas implies that if $v \in \mathbb{Z}^n$, then these coordinates are equal to $\frac{b_1}{d}, \frac{b_2}{d}, \dots, \frac{b_n}{d}$ for some $b_1, b_2, \dots, b_n \in \mathbb{Z}$. Thus $\varphi(\mathbb{Z}^n) \subseteq \mathbb{Z}^n$. Next, $\varphi(\mathbb{Z}^n)$ is of dimension n because φ is an automorphism. Hence and by Corollary 2, $\varphi(\mathbb{Z}^n)$ has a basis $w_1, w_2, \dots, w_n \in \mathbb{R}^n_+$. Then $\varphi^{-1}(w_1), \varphi^{-1}(w_2), \dots, \varphi^{-1}(w_n)$ form a basis of $\mathbb{Z}^n$ and belong to $\mathrm{cone}_{\mathbb{R}^n}\{v_1, v_2, \dots v_n\}$. □

Now, we can give new simpler and shorter proofs of main results from [**1**].

Theorem 6 ([**1**, Theorem 3])**.** *A full lattice $L \subseteq \mathbb{R}^n$ has a basis in a convex cone $C \subseteq \mathbb{R}^n$ iff C is of dimension n.*

Proof. The implication “ $\Longrightarrow$ ” is obvious.

“ $\Longleftarrow$ ” : Take a basis of L and the linear automorphism φ of $\mathbb{R}^n$ which transforms this basis on the standard basis. Then $\varphi(L) = \mathbb{Z}^n$ and $\varphi(C)$ is also a convex cone of dimension n. By Lemma 4, there are linearly independent $w_1, w_2, \dots, w_n \in \varphi(C) \cap \mathbb{Z}^n$. Next, by Lemma 5, $\mathbb{Z}^n$ has a basis $v_1, v_2, \dots, v_n \in \mathrm{cone}_{\mathbb{R}^n}\{w_1, w_2, \dots, w_n\} \subseteq \varphi(C)$. Now $\varphi^{-1}(v_1), \varphi^{-1}(v_2), \dots, \varphi^{-1}(v_n) \in C$ form a basis of L. □

$\mathbb{R}^n_+$ is a convex cone, so the following consequence of Theorem 6 holds.

Corollary 7 ([**1**, Theorem 1])**.** *Each full lattice $L \subseteq \mathbb{R}^n$ has a non-negative basis.*

Corollary 8 ([**1**, Proposition 3 and Corollary 4])**.** *A lattice $L \subseteq \mathbb{R}^n$ has a basis in a convex cone $C \subseteq \mathbb{R}^n$ iff $L \cap C$ generates L.*

Proof. The implication “ $\Longrightarrow$ ” is trivial.

“ $\Longleftarrow$ ” : Assume that L has the dimension k, and take the least linear subspace V of $\mathbb{R}^n$ which contains L (equivalently, $L \cap C$). Then V has the dimension k and is also the least linear subspace of $\mathbb{R}^n$ which contains $V \cap C$. Hence $V \cap C$ is a convex cone of dimension k. Now, we can apply Theorem 6 to obtain a basis of L which is contained in $V \cap C \subseteq C$. Formally, we take (similarly as in the proof of Theorem 6) a linear isomorphism $\varphi\colon V \longrightarrow \mathbb{R}^k$ such that $\varphi(L) = \mathbb{Z}^k$. Then $\varphi(V \cap C)$ is a convex cone of dimension k in $\mathbb{R}^k$, so by Theorem 6, there is a basis of $\mathbb{Z}^k$ which belongs to $\varphi(V \cap C)$. The inverse image of this basis under φ is the required basis of L. □

A particular case of the Corollary 8 is the following fact.

Corollary 9 ([**1**, Proposition 2])**.** *A lattice $L \subseteq \mathbb{R}^n$ has a non-negative basis iff $L \cap \mathbb{R}^n_+$ generates L.*

For each $S \subseteq \mathbb{R}^n$ and $v \in \mathbb{R}^n$, the set $v + S = \{v + w\colon\ w \in S\}$ is a translation of S (by the vector v).

At the end of [**1**], Cherednik showed that every full lattice $L \subseteq \mathbb{R}^n$ has a basis in each translation of every convex cone of dimension n. However, his proof has a gap because he applied the false conjecture that a translation of a convex cone is closed under addition. In particular, this result does not hold for $n = 1$ (see the example below the next fact). Moreover, his proof is unnecessary complicated. Now, we present an elementary and very short proof of this result.

Lemma 10. *For all $n \geq 2$, $\mathbb{Z}^n$ has a basis in each translation of $\mathbb{R}^n_+$.*

Proof. Take a vector $v = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ and $a \in \mathbb{Z}$ such that $a \geq \max\{a_1, a_2, \dots, a_n\}$. Of course, $(x_1, x_2, \dots, x_n) \in v + \mathbb{R}^n_+$ iff $x_i \geq a_i$ for $i = 1, 2, \dots, n$. Let $\alpha_1 = (a+1, a, a, \dots, a, a)$, $\alpha_2 = (a, a+1, a, \dots, a, a)$, $\alpha_3 = (a, a, a+1, \dots, a, a), \dots, \alpha_{n-1} = (a, a, a, \dots, a+1, a)$ and $\alpha_n = ((n-1)a+2, (n-1)a+2, (n-1)a+2, \dots, (n-1)a+1)$. Of course, $\alpha_1, \alpha_2, \dots, \alpha_{n-1} \in (v + \mathbb{R}^n_+) \cap \mathbb{Z}^n$. Since $n \geq 2$, we have $(n-1)a + 1 \geq a$, so α_n belongs to $(v + \mathbb{R}^n_+) \cap \mathbb{Z}^n$ too. Next, $(1, 1, \dots, 1) = \alpha_n - (\alpha_1 + \alpha_2 + \dots + \alpha_{n-1})$, $\varepsilon_i = \alpha_i - a(1, 1, \dots, 1)$, for $i = 1, 2, \dots, n-1$, $\varepsilon_n = (1, 1, \dots, 1) - (\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_{n-1})$. Thus $\alpha_1, \alpha_2, \dots, \alpha_n$ generate $\mathbb{Z}^n$ and span $\mathbb{R}^n$, so $\alpha_1, \alpha_2, \dots, \alpha_n$ form a basis of $\mathbb{Z}^n$ which belongs to $v + \mathbb{R}^n_+$. □

Lemma 10 is not true for $n = 1$, because $\mathbb{Z}$ has no generator in $y + \mathbb{R}_+ = \{x \in \mathbb{R} \colon x \geq y\}$ for all $y > 1$. Thus the next result is also not true for $n = 1$.

Theorem 11 ([**1**, Theorem 4])**.** *For all $n \geq 2$, each full lattice $L \subseteq \mathbb{R}^n$ has a basis in each translation of every convex cone C of dimension n.*

Proof. L has a basis $\alpha_1, \alpha_2, \dots, \alpha_n \in L \cap C$ by Theorem 6. Take the linear automorphism φ of $\mathbb{R}^n$ which transforms $\alpha_1, \alpha_2, \dots, \alpha_n$ on the standard basis of $\mathbb{R}^n$. Then $\varphi(L) = \mathbb{Z}^n$ and $\varphi(D) = \mathbb{R}^n_+$, where $D = \operatorname{cone}_{\mathbb{R}^n}\{\alpha_1, \alpha_2, \dots, \alpha_n\}$. Hence and by Lemma 10, for each $w \in \mathbb{R}^n$, there are $v_1, v_2, \dots, v_n \in \varphi(w) + \mathbb{R}^n_+ = \varphi(w) + \varphi(D) = \varphi(w + D)$ which form a basis of $\mathbb{Z}^n$. The inverse images of these vectors form a basis of L which is contained in $w + D \subseteq w + C$, because $D \subseteq C$. □

At the end of the paper, observe that Corollary 8 does not hold for translations of convex cones. Take the lattice $L = \mathbb{Z} \times \mathbb{Z} \times \{0\} \subseteq \mathbb{R}^3$, the convex cone $C = \operatorname{cone}_{\mathbb{R}^3}\{(-1, 0, -1), (0, -1, -1), (0, 0, -1)\}$ and $v = (2, 2, 1)$. Then $(v + C) \cap L = \{(1, 2, 0), (2, 1, 0), (2, 2, 0)\}$. This set generates L, but of course, is not linearly independent. On the other hand, $(1, 0, 0)$ does not belong to the sublattice of L generated by $\{(2, 1, 0), (2, 2, 0)\}$, $(0, 1, 0)$ does not belong to the sublattice of L generated by $\{(1, 2, 0), (2, 2, 0)\}$, and it is easy to see that both these vectors do not belong to the sublattice of L generated by $\{(1, 2, 0), (2, 1, 0)\}$. Thus L has no basis in $v + C$.

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