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SOME APPLICATIONS OF ASYMPTOTICALLY EQUIVALENCE OF A DOUBLE SEQUENCE OF SETS IN VARIOUS ASPECTS

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ABSTRACT. Savaş [*Generalized asymptotically I-lacunary equivalent of order α for sequences of sets*, Filomat **31** (2017), 1507–1514] studied generalized asymptotically I-lacunary equivalent of order α for a sequence of sets. This article is completely based on a double sequence of sets by way of n-normed spaces. We firstly contrived an Orlicz extension of asymptotically Wijsman equivalence and asymptotically Wijsman lacunary equivalence. By using the hitherto defined concept, we further elongate these notions to asymptotically Orlicz-Wijsman statistical as well as lacunary statistical equivalence. Finally, we explain the concept of ideal extension of order α and present some inclusion relations.

1. Introduction

An attractive theory of 2-normed spaces was introduced and studied by Gähler in [**9**]. In 1989, it was further extended to n-normed spaces by Misiak [**15**]. Since then these spaces were studied by Gunawan [**10**]. In [**11**], Gunawan and Mashadi gave an interesting observation that $(n-1)$-norm is originated from the n-norm.

Definition 1.1. Let X be a real vector space of dimension $d \geq n \geq 2$ and $n \in \mathbb{N}$. We said that real valued function $\|\cdot, \ldots, \cdot\|$ on X^n is an n-norm on X if we have:

1. $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent in X,
2. $\|x_1, x_2, \ldots, x_n\|$ is invariant under permutation,
3. $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \, \|x_1, x_2, \ldots, x_n\|$ for any $\alpha \in \mathbb{R}$,
4. $\|x + x', x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|x', x_2, \ldots, x_n\|$

and the duos $(X, \|\cdot, \ldots, \cdot\|)$ is called a n-normed space over the field $\mathbb{R}$.

Example. The remarkable example of n-normed space is l_∞, well found with

$$\|x_1, x_2, \ldots, x_n\|_\infty = \sup_{j_1, j_2, \ldots, j_n \in \mathbb{N}} |\det(x_{nj_n})|$$

for $x_i = (x_{i1}, x_{i2}, \ldots x_{ij_n}) \in l_\infty$.

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A sequence (x_k) is said to converge to some $L \in X$ in $(X, \|\cdot, \dots, \cdot\|)$ if

$$\lim_{k\to\infty} \|x_k - L, z_1, \dots, z_{n-1}\| = 0 \qquad \text{for every } z_1, \dots, z_{n-1} \in X.$$

A sequence (x_k) is said to be Cauchy in $(X, \|\cdot, \dots, \cdot\|)$ if

$$\lim_{k,p\to\infty} \|x_k - x_p, z_1, \dots, z_{n-1}\| = 0 \qquad \text{for every } z_1, \dots, z_{n-1} \in X.$$

We call X to be complete via n-norm if every Cauchy sequence in X converges to some $L \in X$. Also, recall that complete n-normed space is an n-Banach space.

An Orlicz function M is a function, which is continuous, non-decreasing, and convex on $[0, +\infty)$ with $M(0) = 0$, $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$.

In 1978, Lindenstrauss and Tzafriri [**13**] introduced a new sequence space known as Orlicz sequence space by using the idea of Orlicz function which is defined as

$$\ell_M = \Big\{ x \in \omega : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty \text{ for some } \rho > 0 \Big\}$$

and the norm is defined as

$$\|x\| = \inf \Big\{ \rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1 \Big\}.$$

Using the above norm, we conclude that the space ℓ_M is a Banach space. Further, Orlicz sequence spaces inspected and studied by many prominent authors (see [**16, 23, 25, 26**]).

Freedman et al. [**7**] originated the lacunary strongly convergent sequence space as

$$N_\theta = \Big\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0 \text{ for some } \ L \Big\}.$$

Here, $\theta = (k_r)$ is a lacunary sequence and $k_0 = 0$, $k_r - k_{r-1} \to \infty$ as $r \to \infty$, an increasing sequence of non-negative integers. By $I_r = (k_{r-1}, k_r]$, we symbolize the intervals resoluted by θ. We engrave $h_r = k_r - k_{r-1}$ and $q_r = \frac{k_r}{k_{r-1}}$.

Fast [**6**] familiarised the perception of statistical convergence in 1951.

A sequence $x = (x_k)$ is said to be statistically convergent to a number λ if for each $\varepsilon > 0$, the set $K(\varepsilon) = \{k \leq n : |x_k - \lambda| \geq \varepsilon\}$ has zero asymptotic density, i.e.,

$$\lim_{n\to\infty} \frac{1}{n} |K(\varepsilon)| = 0,$$

we write $S - \lim x = \lambda$.

In 1993, Fridy and Orhan [**8**] introduced a new concept of lacunary statistical convergence. A sequence $x = (x_k)$ of real numbers is said to be lacunary statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{r} \frac{1}{h_r} \big| \big\{ k \in I_r : |x_k - L| \geq \varepsilon \big\} \big| = 0,$$

where $|A|$ denotes the cardinality of $A \subset \mathbb{N}$.

During the course of the paper, $\mathbb{N}$ signifies the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The model of convergence of sequences of numbers has been stretched by abundant authors (see, [**17, 34**]). In 2012, the impression of convergence of sequences to statistical convergence was drawn-out by Nuray and Rhoades [**17**] which also contributed some basic hypotheses. In the same year, Ulusu and Nuray [**31**] introduced the new concept of Wijsman lacunary statistical a convergence of sequence of sets and proved its relations with Wijsman statistical convergence. The association between the ideas like Wijsman statistical convergence, Hausdorff statistical convergence, and Wijsman statistical Cauchy double sequences of sets were deeply probed by Nuray et al. [**18**].

Marouf [**14**] accessibled the characterisation for asymptotically equivalent and asymptotic regular matrices. These theories were protracted by Patterson [**20**] into an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. Later, in 2006 Patterson and Savaş [**22**] extended these definitions to lacunary sequences. Well ahead, the concepts of Wijsman asymptotically equivalence, Wijsman asymptotically statistically equivalence, Wijsman asymptotically lacunary equivalence, and Wijsman asymptotically lacunary statistical equivalence for sequences of sets were also studied. Further, Nuray et al. [**19**] introduced the analog result for a double sequence of sets. In 1993, Marouf [**14**] presented definitions for asymptotically equivalent sequences and asymptotic regular matrices.

Nonnegative sequences $x = \{x_k\}$ and $y = \{y_k\}$ are said to be asymptotically equivalent if $\lim_k \frac{x_k}{y_k} = 1$, it is denoted by $x \sim y$.

Many additional applications of asymptotically statistical equivalent and more investigations in this course can be found in ([**18, 19**]).

Consider a metric space (X, ϱ) for $x \in X$ and $0 \neq A \subseteq X$, the distance from x to A is defined by

$$d(x, A) = \inf_{a \in A} \varrho(x, a).$$

In the whole progression of the paper, we consider $\theta = (k_r)$ to be a lacunary sequence and A, A_k to be any non-empty closed subsets of X. We say that the sequence $\{A_k\}$ is Wijsman convergent to A if

$$d(x, A) = \lim_{k \to \infty} d(x, A_k)$$

for all $x \in X$, written as $W - \lim A_k = A$.

All the vital concepts and theories (background) which is skeleton of this paper are detailed below.

In 2012, Nuray and Rhoades [**17**] outlined the theory of Wijsman convergence and Wijsman statistical convergence of a sequence of sets, and discussed its relationship with other convergence. Later, Ulusu and Nuray [**31**] introduced the concept of Wijsman lacunary statistical convergence. Quite recently, Nuray et al. [**19**] contributed an extension on asymptotically lacunary statistical equivalent set sequences and examined some relations between Wijsman lacunary statistically convergence and Wijsman strongly lacunary statistically convergence on a double sequence of sets.

Firstly in this paper, we explore the above Wijsman statistical convergence in another direction, two new type of convergence called Orlicz-Wijsman statistical convergence and Orlicz-Wijsman lacunary statistical convergence via n-normed space over single sequence space (here we take $\mathcal{M} = \{M_k\}$ to be a sequence of Orlicz functions and A_k, B_k to be non empty closed subset of X) which is investigated as follows.

The sequence $\{A_k\}$ is Orlicz Wijsman statistical convergent to A over n-normed spaces if for each $\varepsilon > 0$ and $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n}\Big|\Big\{k \le n : M_k\Big(\Big\|\frac{d(x,A_k) - d(x,A)}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big) \ge \varepsilon \text{ for some } \rho > 0\Big\}\Big| = 0.$$

Additionally, the sequence $\{A_k\}$ is said to be an Orlicz-Wijsman lacunary statistical convergent to A over n-normed spaces if for all $\varepsilon > 0$ and $x \in X$,

$$\lim_{r} \frac{1}{h_r}\Big|\Big\{k \le I_r : M_k\Big(\Big\|\frac{d(x,A_k) - d(x,A)}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big) \ge \varepsilon \text{ for some } \rho > 0\Big\}\Big| = 0.$$

If $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for all $x \in X$, then the sequences $\{A_k\}$ and $\{B_k\}$ are strongly asymptotically Orlicz-Wijsman lacunary statistical equivalent (Wijsman sense) of multiple L over n-normed spaces if for every $\varepsilon > 0$ and each $x \in X$,

$$\lim_{r} \frac{1}{h_r}\Big|\Big\{k \le I_r : M_k\Big(\Big\|\frac{d(x,A_k)}{d(x,A)} - L, z_1, \dots, z_{n-1}\Big\|\Big) \ge \varepsilon \text{ for some } \rho > 0\Big\}\Big| = 0.$$

It is denoted by $A_k \overset{MWS_\theta^L}{\sim} B_k$.

By the convergence of a double sequence $x = (x_{kj})$ of real numbers to $L \in \mathbb{R}$, we mean the convergence in the Pringsheim sense, i.e., it has Pringsheim limit L (denoted by $P - \lim_{k,j\to\infty} x_{kj} = L$) provided that for given $\varepsilon > 0$, there occurres $n \in N$ such that $|x_{kl} - L| < \varepsilon$ whenever $k, l > n$ (see [**24**]). Recently, Dündar and Pancaroğlu Akın introduced Wijsman regularly ideal convergence of double sequences of sets in [**3**]. A lot of research has been made in this field, for details one may refer to ([**4, 5, 21, 30, 32, 33, 35**]). Throughout the paper, $\{A_{kj}\}$ denotes the double sequence of non-empty closed subsets in n-normed space X.

The double sequence $\{A_{kj}\}$ is Wijsman convergent to A for all $x \in X$ if

$$P - \lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A).$$

The double sequence $\{A_{kj}\}$ is Orlicz-Wijsman statistical convergent over n-normed spaces to A if for each $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn}\Big|\Big\{k \le m, j \le n : M_{kj}\Big(\Big\|\frac{d(x,A_{kj}) - d(x,A)}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big) \ge \varepsilon \text{ for some } \rho > 0\Big\}\Big| = 0.$$

The double sequence $\theta = \{(k_r, j_u)\}$ is called the double lacunary sequence if there exist two increasing sequences of integers such that

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \to \infty \qquad \text{as } r \to \infty,$$

and

$$j_0 = 0, \quad \overline{h}_u = j_u - j_{u-1} \to \infty \qquad \text{as } u \to \infty.$$

In this sequel, we use the following notations

$$k_{ru} = k_r j_u, \qquad h_{ru} = h_r \overline{h}_u,$$
$$I_{ru} = \{(k, j) : k_{r-1} < k < k_r \text{ and } j_{u-1} < j < j_u\},$$
$$q_r = \frac{k_r}{k_{r-1}}, \text{ and } q_u = \frac{j_u}{j_{u-1}}.$$

The double sequence $\{A_{kj}\}$ is said to be Orlicz-Wijsman lacunary statistical convergent to A via n-normed spaces if

$$P - \lim_{r,u\to\infty} \frac{1}{h_r \overline{h}_u} \Big| \Big\{ (k, j) \in I_{ru} : M_{kj} \Big(\Big\| \frac{d(x, A_{kj}) - d(x, A)}{\rho}, z_1, \ldots, z_{n-1} \Big\| \Big) \geq \varepsilon \text{ for some } \rho > 0 \Big\} \Big| = 0$$

for all $\varepsilon > 0$, $x \in X$. $\theta = \{(k_r, j_u)\}$ represents a double lacunary sequence. In this situation, we write $st - \lim_{M_{W_\theta}} A_{kj} = A$.

$\{A_{kj}\}$ is said to be Orlicz-Wijsman strongly lacunary convergent to A via $n-$normed spaces if for all $x \in X$,

$$P - \lim_{r,u\to\infty} \frac{1}{h_r \overline{h}_u} \sum_{k=k_{r-1}+1}^{k_r} \sum_{j=j_{u-1}+1}^{j_u} \Big| \Big\{ M_{kj} \Big(\Big\| \frac{d(x, A_{kj}) - d(x, A)}{\rho}, z_1, \ldots, z_{n-1} \Big\| \Big) \geq \varepsilon \text{ for some } \rho > 0 \Big\} \Big| = 0.$$

2. Orlicz-Wijsman asymptotically equivalent

This section is intended to the study of new theories regarding Orlicz-Wijsman asymptotically equivalence and also tries to potray some definitions and results.

Definition 2.1. Let $M = \{M_{kj}\}$ be a double sequence of Orlicz function, we delineate $d(x; A_{kj},\ B_{kj})$ as follows:

$$d(x; A_{kj},\ B_{kj}) = \begin{cases} \dfrac{d(x, A_{kj})}{d(x, B_{kj})}, & x \notin A_{kj} \cup B_{kj}, \\ L, & x \in A_{kj} \cup B_{kj}. \end{cases}$$

Definition 2.2. The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Orlicz-Wijsman asymptotically equivalent of multiple L if for all $x \in X$,

$$P - \lim_{k,j\to\infty} M_{kj} \Big(\Big\| \frac{d(x; A_{kj},\ B_{kj})}{\rho}, z_1, \ldots, z_{n-1} \Big\| \Big) = L,$$

it is denoted by $A_{kj} \overset{M^2W_2L}{\sim} B_{kj}$.

Definition 2.3. The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Orlicz-Wijsman asymptotically $\mathcal{C}$-equivalent of multiple L if for all $x \in X$,

$$P - \lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj})}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) = L,$$

it is denoted by $A_{kj} \overset{M^2W_2C^L}{\sim} B_{kj}$.

Definition 2.4. The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Orlicz-Wijsman strongly asymptotically $\mathcal{C}$-equivalent of multiple L if for all $x \in X$,

$$P - \lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) = 0,$$

it is denoted by $A_{kj} \overset{[M^2W_2C^L]}{\sim} B_{kj}$.

Definition 2.5. The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Orlicz-Wijsman lacunary asymptotically equivalent of multiple L if for all $x \in X$,

$$P - \lim_{r,u\to\infty} \frac{1}{h_r\bar{h}_u} \sum_{k,j\in I_{ru}} M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj})}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) = L,$$

it is denoted by $A_{kj} \overset{M^2W_2N_\theta^L}{\sim} B_{kj}$.

Definition 2.6. The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Orlicz-Wijsman strongly lacunary asymptotically equivalent of multiple L if for all $x \in X$,

$$P - \lim_{r,u\to\infty} \frac{1}{h_r\bar{h}_u} \sum_{k,j\in I_{ru}} M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) = 0,$$

it is denoted by $A_{kj} \overset{[M^2W_2N_\theta^L]}{\sim} B_{kj}$.

The following theorem is constructed on the basis of the above defined concepts.

Theorem 2.7. *If*

$$1 < \liminf_r q_r \le \limsup_r q_r < \infty \qquad \textit{and} \qquad 1 < \liminf_u q_u \le \limsup_u q_u < \infty,$$

then $A_{kj} \overset{[M^2W_2C^L]}{\sim} B_{kj}$ *if and only if* $A_{kj} \overset{[M^2W_2N_\theta^L]}{\sim} B_{kj}$.

Proof. Suppose $\liminf_r q_r > 1$ and $\liminf_u q_u > 1$, then there occurs $\lambda, \mu > 0$ such that $q_r \ge 1 + \lambda$ and $q_u \ge 1 + \mu$ for every $r, u \ge 1$, which additionally implies that

$$\frac{k_r j_u}{h_r\bar{h}_u} \le \frac{(1+\lambda)(1+\mu)}{\lambda\mu}.$$

Let $A_{kj} \overset{[M^2W_2C^L]}{\sim} B_{kj}$. We can write

$$\begin{aligned}
&\frac{1}{h_r\bar{h}_u}\sum_{k,j\in I_{ru}} M_{kj}\Big(\Big\|\frac{d(x;A_{kj},\ B_{kj})-L}{\rho},z_1,\dots,z_{n-1}\Big\|\Big)\\
&=\frac{1}{h_r\bar{h}_u}\sum_{i,s=1,1}^{k_r,j_u} M_{is}\Big(\Big\|\frac{d(x;A_{is},\ B_{is})-L}{\rho},z_1,\dots,z_{n-1}\Big\|\Big)\\
&\quad-\frac{1}{h_r\bar{h}_u}\sum_{i,s=1,1}^{k_{r-1},j_{u-1}} M_{is}\Big(\Big\|\frac{d(x;A_{is},\ B_{is})-L}{\rho},z_1,\dots,z_{n-1}\Big\|\Big)\\
&=\frac{k_rj_u}{h_r\bar{h}_u}\Bigg(\frac{1}{k_rj_u}\sum_{i,s=1,1}^{k_r,j_u} M_{is}\Big(\Big\|\frac{d(x;A_{is},\ B_{is})-L}{\rho},z_1,\dots,z_{n-1}\Big\|\Big)\Bigg)\\
&\quad-\frac{k_{r-1}j_{u-1}}{h_r\bar{h}_u}\Bigg(\frac{1}{k_{r-1}j_{u-1}}\sum_{i,s=1,1}^{k_{r-1},j_{u-1}} M_{is}\Big(\Big\|\frac{d(x;A_{is},\ B_{is})-L}{\rho},z_1,\dots,z_{n-1}\Big\|\Big)\Bigg).
\end{aligned}$$

Since $A_{kj} \overset{[M^2W_2C^L]}{\sim} B_{kj}$, both terms

$$\frac{1}{k_rj_u}\sum_{i,s=1,1}^{k_r,j_u} M_{is}\Big(\Big\|\frac{d(x;A_{is},\ B_{is})-L}{\rho},z_1,\dots,z_{n-1}\Big\|\Big)$$

and

$$\frac{1}{k_{r-1}j_{u-1}}\sum_{i,s=1,1}^{k_{r-1},j_{u-1}} M_{is}\Big(\Big\|\frac{d(x;A_{is},\ B_{is})-L}{\rho},z_1,\dots,z_{n-1}\Big\|\Big)$$

converge to 0, and it follows that

$$\frac{1}{h_r\bar{h}_u}\sum_{k,j\in I_{ru}} M_{kj}\Big(\Big\|\frac{d(x;A_{kj},\ B_{kj})-L}{\rho},z_1,\dots,z_{n-1}\Big\|\Big)\to 0.$$

Thus, $A_{kj} \overset{[M^2W_2N_\theta^L]}{\sim} B_{kj}$.

Now, we assume that $\limsup_r q_r > 1$ and $\limsup_u q_u > 1$, then there exist $C, D > 0$ such that $q_r < C$ and $q_u < D$ for all r, u. Let $A_{kj} \overset{[M^2W_2N_\theta^L]}{\sim} B_{kj}$ and $\varepsilon > 0$. Then, we can find $R, U > 0$ and $K > 0$ such that

$$\sup_{i\ge R, s\ge U} \kappa_{is} < \varepsilon \quad \text{and} \quad \kappa_{is} < K \qquad \text{for all } i, s = 1, 2, \dots,$$

where

$$\kappa_{ru} = \frac{1}{h_r\bar{h}_u}\sum_{I_{ru}} M_{kj}\Big(\Big\|\frac{d(x;A_{kj},\ B_{kj})-L}{\rho},z_1,\dots,z_{n-1}\Big\|\Big).$$

If we take t, v some integers with conditions like $k_{r-1} < t \leq k_r$ and $j_{u-1} < v \leq j_u$, where $r > R$ and $u > U$, then we can engrave

$$\begin{aligned}
&\frac{1}{tv}\sum_{i,s=1,1}^{t,v} M_{is}\Big(\Big\|d(x; A_{is},\ B_{is}) - L, z_1, \ldots, z_{n-1}\Big\|\Big)\\
&\leq \frac{1}{k_{r-1}j_{u-1}}\sum_{i,s=1,1}^{k_r,j_u} M_{is}\Big(\Big\|\frac{d(x; A_{is},\ B_{is}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\\
&= \frac{1}{k_{r-1}j_{u-1}}\Big(\sum_{I_{11}} M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\\
&\quad + \sum_{I_{12}} M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\\
&\quad + \sum_{I_{21}} M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\\
&\quad + \sum_{I_{22}} M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\\
&\quad + \cdots + \sum_{I_{ru}} M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\|\Big)\Big)\\
&\leq \frac{k_1 j_1}{k_{r-1}j_{u-1}}\kappa_{11} + \frac{k_1(j_2 - j_1)}{k_{r-1}j_{u-1}}\kappa_{12} + \frac{(k_2 - k1)j_1}{k_{r-1}j_{u-1}}\kappa_{21} + \frac{(k_2 - k1)(j_2 - j_1)}{k_{r-1}j_{u-1}}\kappa_{22}\\
&\quad + \cdots + \frac{(k_R - K_{R-1})(j_U - J_{U-1})}{k_{r-1}j_{u-1}}\kappa_{RU} + \cdots + \frac{(k_r - K_{r-1})(j_u - J_{u-1})}{k_{r-1}j_{u-1}}\kappa_{ru}\\
&\leq \Big(\sup_{i,s\geq 1,1}\kappa_{is}\Big)\frac{k_R j_U}{k_{r-1}j_{u-1}} + \Big(\sup_{i\geq R, s\geq U}\kappa_{is}\Big)\frac{(k_r - k_R)(j_u - j_U)}{k_{r-1}j_{u-1}}\\
&\leq K\frac{k_R j_U}{k_{r-1}j_{u-1}} + \varepsilon CD.
\end{aligned}$$

Since $k_{r-1}, j_{u-1} \to \infty$ as $t, v \to \infty$, it follows that

$$\frac{1}{tv}\sum_{i,s=1,1}^{t,v} M_{is}\Big(\Big\|d(x; A_{is},\ B_{is}) - L, z_1, \ldots, z_{n-1}\Big\|\Big) \to 0,$$

and hence $A_{kj} \overset{[M^2 W_2 C^L]}{\sim} B_{kj}$. This completes the proof. □

3. Orlicz-Wijsman asymptotically statistical and Lacunary statistical equivalent

In the present section, we scrutinize statistical convergence as well as lacunary statistical convergence on Orlicz-Wijsman asymptotically equivalent. For this, let $\mathcal{M} = \{M_{kj}\}$ be a double sequence of Orlicz function.

Definition 3.1. The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are said to be an Orlicz-Wijsman asymptotically statistical equivalent of multiple L if for every $\varepsilon > 0$ and $x \in X$,

$$P - \lim_{m,n\to\infty} \frac{1}{mn}\Big|\Big\{k \le m, j \le n : M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big) \ge \varepsilon\Big\}\Big| = 0.$$

It is denoted by $A_{kj} \overset{M^2W_2S^L}{\sim} B_{kj}$.

Definition 3.2. The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are said to be an Orlicz-Wijsman asymptotically lacunary statistical equivalent of multiple L if for every $\varepsilon > 0$ and $x \in X$,

$$P - \lim_{r,u\to\infty} \frac{1}{h_r\bar{h}_u}\Big|\Big\{(k,j) \in I_{ru} : M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big) \ge \varepsilon\Big\}\Big| = 0,$$

it is denoted by $A_{kj} \overset{M^2W_2S^L_\theta}{\sim} B_{kj}$.

Theorem 3.3. *$A_{kj} \overset{[M^2W_2N^L_\theta]}{\sim} B_{kj}$ implies $A_{kj} \overset{M^2W_2S^L_\theta}{\sim} B_{kj}$.*

Proof. Let $\varepsilon > 0$ and $A_{kj} \overset{[M^2W_2N^L_\theta]}{\sim} B_{kj}$. Then, we can write

$$\begin{aligned}
&\sum_{k,j\in I_{ru}} M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big) \\
&= \sum_{k,j\in I_{ru}} M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big) \\
&\quad + \sum_{k,j\in I_{ru}} M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big) \\
&\ge \varepsilon\Big|\Big\{(k,j) \in I_{ru} : M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big) \ge \varepsilon\Big\}\Big|
\end{aligned}$$

which yields the result. □

Theorem 3.4. *If $A_{kj} \overset{M^2W_2S^L_\theta}{\sim} B_{kj}$, where $d(x; A_{kj}) = O(d(x; B_{kj}))$, then*

$$A_{kj} \overset{[M^2W_2N^L_\theta]}{\sim} B_{kj}.$$

Proof. Consider $A_{kj} \overset{M^2W_2S^L_\theta}{\sim} B_{kj}$, where $d(x; A_{kj}) = O(d(x; B_{kj}))$. Then, we take up

$$M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \dots, z_{n-1}\Big\|\Big) \le V \qquad \text{for each } x \in X \text{ and all } k, j.$$

Since $\varepsilon > 0$, we have

$$\begin{aligned}
&\frac{1}{h_r\bar{h}_u}\sum_{(k,j)\in I_{ru}} M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\\
&= \frac{1}{h_r\bar{h}_u}\sum_{k,j\in I_{ru}} M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\\
&\quad + \frac{1}{h_r\bar{h}_u}\sum_{k,j\in I_{ru}} M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\\
&\geq \frac{V}{h_r\bar{h}_u}\big|\big\{(k,j)\in I_{ru} : M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) \geq \varepsilon\big\}\big| + \varepsilon,
\end{aligned}$$

which yields the result. □

Theorem 3.5. *If* $1 < \liminf_r q_r,\ \ 1 < \liminf_u q_u$, *then*

$$A_{kj} \overset{M^2W_2S^L}{\sim} B_{kj} \quad \textit{implies} \quad A_{kj} \overset{M^2W_2S^L_\theta}{\sim} B_{kj}.$$

Proof. We undertake that $\liminf_r q_r > 1$ and $\liminf_u q_u > 1$, then there occur $\lambda, \mu > 0$ such that $q_r \geq 1 + \lambda$ and $q_u \geq 1 + \mu$ for all $r, u \geq 1$, and result into

$$\frac{k_r j_u}{h_r\bar{h}_u} \leq \frac{(1+\lambda)(1+\mu)}{\lambda\mu}.$$

If $A_{kj} \overset{M^2W_2S^L}{\sim} B_{kj}$, then for each $\varepsilon > 0$, suitably large r, u, and for each $x \in X$, we have

$$\begin{aligned}
&\frac{1}{k_r j_u}\Big|\Big\{k \leq k_r, j \leq j_u : M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big)\Big\}\Big|\\
&\geq \frac{1}{k_r j_u}\Big|\Big\{(k,j)\in I_{ru} : M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) \geq \varepsilon\Big\}\Big|\\
&\geq \frac{(1+\lambda)(1+\mu)}{\lambda\mu}\Bigg(\frac{1}{h_r\bar{h}_u}\Big|\Big\{(k,j)\in I_{ru} :\\
&\qquad\qquad M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) \geq \varepsilon\Big\}\Big|\Bigg),
\end{aligned}$$

this completes the proof. □

Theorem 3.6. *If* $\limsup_r q_r < \infty,\ \ \liminf_u q_u < \infty$, *then*

$$A_{kj} \overset{M^2W_2S^L_\theta}{\sim} B_{kj} \quad \textit{implies} \quad A_{kj} \overset{M^2W_2S^L}{\sim} B_{kj}.$$

Proof. Since it is given that $\limsup_r q_r < \infty, \liminf_u q_u < \infty$, then there occur $C, D > 0$ such that $q_r < C$ and $q_u < D$ for all r, u. Let $A_{kj} \overset{M^2W_2S^L_\theta}{\sim} B_{kj}$ and $\varepsilon > 0$. There exists $R > 0$ such that for every $r, s \geq R$,

$$A_{r,s} = \frac{1}{h_r\bar{h}_u}\Big|\Big\{(k,j)\in I_{ru} : M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) \geq \varepsilon\Big\}\Big| < \varepsilon.$$

$H > 0$ such that $A_{r,s} < H$ for all $r, s = 1, 2, \ldots$. Suppose m, n are any integers sustaining $k_{r-1} < m \leq k_r$ and $j_{u-1} < n \leq j_u$, where $r, s > R$. Then,

$$\begin{aligned}
&\frac{1}{mn}\Big|\Big\{k \leq m, j \leq n : M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) \geq \varepsilon\Big\}\Big| \\
&\leq \frac{1}{k_{r-1}j_{u-1}}\Big|\Big\{k \leq k_r, j \leq j_u : M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) \geq \varepsilon\Big\}\Big| \\
&= \frac{1}{k_{r-1}j_{u-1}}\Big|\Big\{(k,j) \in I_{11} : M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) \geq \varepsilon\Big\}\Big| \\
&\quad + \frac{1}{k_{r-1}j_{u-1}}\Big|\Big\{(k,j) \in I_{21} : M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) \geq \varepsilon\Big\}\Big| \\
&\quad + \frac{1}{k_{r-1}j_{u-1}}\Big|\Big\{(k,j) \in I_{12} : M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) \geq \varepsilon\Big\}\Big| \\
&\quad + \frac{1}{k_{r-1}j_{u-1}}\Big|\Big\{(k,j) \in I_{22} : M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) \geq \varepsilon\Big\}\Big| \\
&\qquad\qquad\vdots \\
&\quad + \frac{1}{k_{r-1}j_{u-1}}\Big|\Big\{(k,j) \in I_{ru} : M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) \geq \varepsilon\Big\}\Big| \\
&= \frac{k_1 j_1}{k_{r-1}j_{u-1}k_1 j_1}\Big|\Big\{(k,j) \in I_{11} : M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) \geq \varepsilon\Big\}\Big| \\
&\quad + \frac{(k_2 - k_1)j_1}{k_{r-1}j_{u-1}(k_2 - k_1)j_1}\Big|\Big\{(k,j) \in I_{21} : \\
&\qquad\qquad M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) \geq \varepsilon\Big\}\Big| \\
&\quad + \frac{k_1(j_2 - j_1)}{k_{r-1}j_{u-1}k_1(j_2 - j_1)}\Big|\Big\{(k,j) \in I_{12} : \\
&\qquad\qquad M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) \geq \varepsilon\Big\}\Big| \\
&\quad + \frac{(k_2 - k_1)(j_2 - j_1)}{k_{r-1}j_{u-1}(k_2 - k_1)(j_2 - j_1)}\Big|\Big\{(k,j) \in I_{22} : \\
&\qquad\qquad M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) \geq \varepsilon\Big\}\Big| \\
&\qquad\qquad\vdots \\
&\quad + \frac{(k_R - k_{R-1})(j_R - j_{R-1})}{k_{r-1}j_{u-1}(k_R - k_{R-1})(j_R - j_{R-1})}\Big|\Big\{(k,j) \in I_{RR} : \\
&\qquad\qquad M_{kj}\Big(\Big\|\frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) \geq \varepsilon\Big\}\Big| \\
&\qquad\qquad\vdots
\end{aligned}$$

$$\vdots$$

$$+\frac{(k_r-k_{r-1})(j_r-j_{r-1})}{k_{r-1}j_{u-1}(k_r-k_{r-1})(j_r-j_{r-1})}\Big|\Big\{(k,j)\in I_{rr}: M_{kj}\Big(\Big\|\frac{d(x;A_{kj},\ B_{kj})-L}{\rho},z_1,\ldots,z_{n-1}\Big\|\Big)\geq\varepsilon\Big\}\Big|$$

$$=\frac{k_1j_1}{k_{r-1}j_{u-1}}A_{11}+\frac{(k_2-k_1)j_1}{k_{r-1}j_{u-1}}A_{21}+\frac{(k_2-k_1)j_1}{k_{r-1}j_{u-1}}A_{12}+\frac{(k_2-k_1)(j_2-j_1)}{k_{r-1}j_{u-1}}A_{22}$$

$$\vdots$$

$$+\frac{(k_R-k_{R-1})(j_R-j_{R-1})}{k_{r-1}j_{u-1}}A_{RR}+\cdots+\frac{(k_r-k_{r-1})(j_r-j_{r-1})}{k_{r-1}j_{u-1}}A_{rr}$$

$$\leq\Big\{\sum_{r,s\geq 1}A_{rs}\Big\}\frac{k_Rj_R}{k_{r-1}j_{u-1}}+\Big\{\sum_{r,s\geq R}A_{rs}\Big\}\frac{(k_r-k_R)(j_r-j_R)}{k_{r-1}j_{u-1}}$$

$$\leq H\cdot\frac{k_Rj_R}{k_{r-1}j_{u-1}}+\varepsilon\cdot C\cdot D.$$

This completes the proof. □

By merging Theorem 3.5 and Theorem 3.6, we get the following theorem.

Theorem 3.7. *If* $1<\liminf_r q_r\leq\limsup_r q_r<\infty$ *and* $1<\liminf_u q_u\leq\limsup_u q_u<\infty$, *then* $A_{kj}\overset{M^2W_2S^L_\theta}{\sim}B_{kj}$ *if and only if* $A_{kj}\overset{M^2W_2S^L}{\sim}B_{kj}$, *where* θ *is a double lacunary sequence.*

4. Its ideal extention of order α

In this section of our paper, we use the term ideal. Here we explain the concept of ideal extension narrated in the previous section to number α. Firstly we recall the background of ideal and number α (order α) as follows:

- (i) A kinfolk of sets $I\subseteq 2^{\mathbb{N}}$, where $\mathbb{N}$ is set of natural number is termed to be an ideal if $C,D\in I\Rightarrow C\cup D\in I$, and $C\in I,\ D\subseteq C\Rightarrow D\in I$.
- (ii) A non empty clan of sets $\pounds(I)\subseteq 2^{\mathbb{N}}$ is termed to be filter on $\mathbb{N}$ if and only if $\phi\notin\pounds(I)$.
  - (a) $C\cap D\in\pounds(I)$ for $C,D\in\pounds(I)$.
  - (b) For each $C\in\pounds(I)$ and $C\subseteq D$, we have $D\in\pounds(I)$.
- (iii) A non trivial ideal $I\subseteq 2^{\mathbb{N}}$ is termed to be admissible if $\{\{x\}:x\in\mathbb{N}\}\subseteq I$.
- (iv) The condition for which a non-trivial ideal is maximal if there does not occur any non trivial ideal $O\neq I$ containing I as a subset.
- (v) (IDEAL and FILTER) For each ideal I, there exists a filter $\pounds(I)$ corresponding to I, i.e., $\pounds(I)=\{P\subseteq\mathbb{N}:P^c\in I\}$, where $P^c=\mathbb{N}\setminus P$.
- (vi) (ADMISSIBLE IDEAL and PROPER IDEAL) An admissible ideal I is derived from proper ideal if $\{n\}\in I$ for each $n\in\mathbb{N}$. All over I stands for a proper admissible ideal of $\mathbb{N}$.

Furthermore, Kostyrko et al. [**12**] presented a very interesting generalization of statistical convergence called I-convergence using the notion of ideals of $\mathbb{N}$ with many interesting consequences.

Definition 4.1. Let $I \subset 2^{\mathbb{N}}$ be an admissible ideal in $\mathbb{N}$. Then, the double sequence (x_{kj}) of elements of $\mathbb{R}$ is said to be I convergence to $L \in \mathbb{R}$ if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k, j \in \mathbb{N} : |x_{kj} - L| \geq \varepsilon\} \in I$.

In 2014, the directly above said convergence was further extended to acquaint with the concepts of I-statistical convergence and I-lacunary statistical convergence by Das et al. [**2**]. In the study of statistical convergence, a new trend was originated where the opinion of statistical convergence of order α, $0 < \alpha \leq 1$, was introduced by only replacing n by n^{α} in the denominator and in the definition of statistical convergence by Çolak [**1**].

In 2015, by Savaş [**27**], the scheme of Wijsman I-statistical convergent of order α was introduced in the same article in which he made an attempt to define lacunary convergence for a sequence of sets. Later in 2017, Savaş [**29**] made a new perception of Wijsman asymptotically I-lacunary statistical convergent of order α.

The present section is dedicated to one more application of asymptotically equivalence which is different from the previous sections that means, the Orlicz ideal extension above defined class of order α for a double sequence of sets over n-normed space. Throughout this section, we consider (X, ϱ) as a metric space and $\mathcal{M} = \{M_{kj}\}$ to be an Orlicz function.

Definition 4.2. A double sequence of sets $\{A_{kj}\}$ is said to be an Orlicz-Wijsman I-statistical convergent of order α to A if

$$\Big\{(m, n) \in \mathbb{N} : \frac{1}{m^{\alpha} n^{\alpha}} \Big| \Big\{ M_{kj} \Big(\Big\| \frac{d(x; A_{kj}) - d(x; A)}{\rho}, z_1, \ldots, z_{n-1} \Big\| \Big) \geq \varepsilon \Big\} \Big| \geq \delta \Big\} \in I$$

for all $\varepsilon > 0$, $\delta > 0$, and $x \in X$. Here we write $A_{kj} \to A(S_2(M^2 I_W^2)^{\alpha})$. It is denoted by $S_2(M^2 I_W^2)^{\alpha}$.

Definition 4.3. For $d(x, A_{kj}) > 0$ and $d(x, B_{kj}) > 0$, the double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Orlicz-Wijsman asymptotically I-statistical equivalent to multiple L of order $\alpha, 0 < \alpha \leq 1$, if

$$\Big\{(m, n) \in \mathbb{N} : \frac{1}{m^{\alpha} n^{\alpha}} \Big| \Big\{ M_{kj} \Big(\Big\| \frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1} \Big\| \Big) \geq \varepsilon, \Big\} \Big| \geq \delta \Big\} \in I$$

for all $\varepsilon > 0, \delta > 0$, and $x \in X$. Here we write $A_{kj} \to B_{kj}(S_2^L(M^2 I_W^2)^{\alpha})$. It is denoted by $S_2^L(M^2 I_W^2)^{\alpha}$.

Furthermore, let $S_2^L(M^2 I_W^2)^{\alpha}$ denote the set of $\{A_{kj}\}$ and $\{B_{kj}\}$ such that $A_{kj} \overset{S_2^L(M^2 I_W^2)^{\alpha}}{\sim} B_{kj}$.

For instance, if we take $M_{kj} = \{1, 1. \ldots, 1\}$, I is an admissible ideal, like $I = \{G \subseteq \mathbb{N} : G \text{ is finite subset}\}$, and $\alpha = 1$, then $S_2^L(M^2 I_W^2)^{\alpha}$ coincides with $M^2 W_2 S^L$.

Definition 4.4. $\{A_{kj}\}$ is Orlicz-Wijsman I-lacunary statistical convergent of order $\alpha, 0 < \alpha \leq 1$, to A if for each $\varepsilon > 0, \delta > 0$ and for each $x \in X$,

$$\Big\{(r,u) \in \mathbb{N} : \frac{1}{h_r \bar{h}_u}\Big|\Big\{(k,j) \in I_{ru} : M_{kj}\Big(\Big\|\frac{d(x;A_{kj}) - d(x;A)}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) \geq \varepsilon\Big\}\Big| \geq \delta\Big\} \in I.$$

Here we write $A_{kj} \to A({}_\theta S_2^L(M^2 I_W^2)^\alpha)$. It will be denoted by ${}_\theta S_2(M^2 I_W^2)^\alpha$ for a double sequence of set.

Definition 4.5. For $d(x, A_{kj}) > 0$, and $d(x, B_{kj}) > 0$, the sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Orlicz-Wijsman asymptotically I-lacunary statistical equivalent to multiple L of order $\alpha, 0 < \alpha \leq 1$, if for all $\varepsilon > 0, \delta > 0$, and for $x \in X$,

$$\Big\{(r,u) \in \mathbb{N} : \frac{1}{h_r \bar{h}_u}\Big|\Big\{M_{kj}\Big(\Big\|\frac{d(x;A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) \geq \varepsilon,\Big\}\Big| \geq \delta\Big\} \in I.$$

Here we write $A_{kj} \to B_{kj}({}_\theta S_2^L(M^2 I_W^2)^\alpha)$. It is denoted by ${}_\theta S_2^L(M^2 I_W^2)^\alpha$.

Furthermore, let ${}_\theta S_2^L(M^2 I_W^2)^\alpha$ denote the set of $\{A_{kj}\}$ and $\{B_{kj}\}$ such that $A_{kj} \overset{{}_\theta S_2^L(M^2 I_W^2)^\alpha}{\sim} B_{kj}$.

Now we examine some inclusion relations.

Theorem 4.6. *Prove that* $S_2^L(M^2 I_W^2)^\alpha \subset S_2^L(M^2 I_W^2)^\gamma$ *for* $0 < \alpha \leq \gamma \leq 1$.

Proof. Let $0 < \alpha \leq \gamma \leq 1$, by the definition we have

$$\frac{1}{m^\gamma n^\gamma}\Big|\Big\{k \leq m, j \leq n : M_{kj}\Big(\Big\|\frac{d(x;A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) \geq \varepsilon\Big\}\Big|$$
$$\leq \frac{1}{m^\alpha n^\alpha}\Big|\Big\{k \leq m, j \leq n : M_{kj}\Big(\Big\|\frac{d(x;A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) \geq \varepsilon\Big\}\Big|.$$

So for any $\delta > 0$, we have

$$\Big\{(m,n) \in \mathbb{N} : \frac{1}{m^\gamma n^\gamma}\Big|\Big\{k \leq m, j \leq n : M_{kj}\Big(\Big\|\frac{d(x;A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) \geq \varepsilon\Big\}\Big| \geq \delta\Big\}$$
$$\leq \Big\{(m,n) \in \mathbb{N} : \frac{1}{m^\alpha n^\alpha}\Big|\Big\{k \leq m, j \leq n : M_{kj}\Big(\Big\|\frac{d(x;A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1}\Big\|\Big) \geq \varepsilon\Big\}\Big| \geq \delta\Big\}.$$

Since set on the right hand side belongs to the ideal I, then obviously the set on the left hand side also belongs to I. This shows that $S_2^L(M^2 I_W^2)^\alpha \subset S_2^L(M^2 I_W^2)^\gamma$. □

Theorem 4.7. *Let* $0 < \alpha \leq \gamma \leq 1$, *then*

(i) ${}_\theta S_2^L(M^2 I_W^2)^\alpha \subset {}_\theta S_2^L(M^2 I_W^2)^\gamma$.

(ii) *In particular,* ${}_\theta S_2^L(M^2 I_W^2)^\alpha \subset {}_\theta S_2^L(M^2 I_W^2)$.

Proof. Proof is same as done in Theorem 4.6. □

Definition 4.8. The sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are strongly Orlicz-Wijsman asymptotically I-lacunary statistical equivalent to multiple L of order $\alpha, 0 < \alpha \leq 1$, if for all $\varepsilon > 0$, $\delta > 0$, and for $x \in X$,

$$\Big\{(r,u) \in \mathbb{N} : \frac{1}{h_r \bar{h}_u} \sum_{k,j \in I_{ru}} \Big| M_{kj}\Big(\Big\| \frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1} \Big\|\Big)\Big| \geq \varepsilon \Big\} \in I.$$

Here, we write $A_{kj} \to\ B_{kj}({}_{\theta}N_2^L(M^2 I_W^2)^{\alpha})$. It is denoted by ${}_{\theta}N_2^L(M^2 I_W^2)^{\alpha}$.

Furthermore, let ${}_{\theta}N_2^L(M^2 I_W^2)^{\alpha}$ denote the set of $\{A_{kj}\}$ and $\{B_{kj}\}$ such that $A_{kj} \overset{{}_{\theta}N_2^L(M^2 I_W^2)^{\alpha}}{\sim} B_{kj}$.

Theorem 4.9. *Let θ be a lacunary sequence. If $A_{kj} \overset{{}_{\theta}N_2^L(M^2 I_W^2)^{\alpha}}{\sim} B_{kj}$, then $A_{kj} \overset{{}_{\theta}S_2^L(M^2 I_W^2)^{\alpha}}{\sim} B_{kj}$.*

Proof. Let $\varepsilon > 0$ and $A_{kj} \overset{{}_{\theta}N_2^L(M^2 I_W^2)^{\alpha}}{\sim} B_{kj}$ such that

$$\begin{aligned}
&\sum_{k,j \in I_{ru}} \Big| M_{kj}\Big(\Big\| \frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1} \Big\|\Big)\Big| \\
&\geq \sum_{\substack{k,j \in I_{ru} \\ \left| M_{kj}\left(\left\| \frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1} \right\|\right)\right| \geq \varepsilon}} \Big| M_{kj}\Big(\Big\| \frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1} \Big\|\Big)\Big| \\
&\geq \varepsilon \Big| (k,j) \in I_{ru} : M_{kj}\Big(\Big\| \frac{d; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1} \Big\|\Big) \geq \varepsilon \Big|,
\end{aligned}$$

and so

$$\begin{aligned}
&\frac{1}{\varepsilon} \frac{1}{h_r \bar{h}_u} \sum_{k,j \in I_{ru}} \Big| M_{kj}\Big(\Big\| \frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1} \Big\|\Big)\Big| \\
&\quad \geq \frac{1}{h_r \bar{h}_u} \Big| (k,j) \in I_{ru} : M_{kj}\Big(\Big\| \frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1} \Big\|\Big) \geq \varepsilon \Big|.
\end{aligned}$$

Then, for any $\delta > 0$

$$\begin{aligned}
&\Big\{(r,u) \in \mathbb{N} : \frac{1}{h_r \bar{h}_u} \Big| \Big\{ M_{kj}\Big(\Big\| \frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1} \Big\|\Big) \geq \varepsilon, \Big\} \Big| \geq \delta \Big\} \\
&\subseteq \Big\{(r,u) \in \mathbb{N} : \frac{1}{h_r \bar{h}_u} \sum_{k,j \in I_{ru}} \Big| M_{kj}\Big(\Big\| \frac{d(x; A_{kj},\ B_{kj}) - L}{\rho}, z_1, \ldots, z_{n-1} \Big\|\Big)\Big| \geq \varepsilon \cdot \delta \Big\} \in I.
\end{aligned}$$

This proves the result. □

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