Acta Math. Univ. Comenianae
Vol. XC, 4 (2021), pp. 353–376

NEIGHBORHOOD-PRIME LABELING OF SNAKE GRAPHS

S. K. PATEL and A. N. KANSAGARA

Abstract. We study neighborhood-prime labeling in the context of snake graphs of the types C_k^m and $C_{k,q}^m$. In particular, we prove that the snake graphs of the type C_k^m are neighborhood-prime if and only if either $k \not\equiv 2 \pmod 4$ or $m \not\equiv 1 \pmod 4$. Further, we also show that $C_{k,2}^m$ and $C_{k,3}^m$ are neighborhood-prime for all $m \geq 2$.

1. Introduction

All graphs considered in this paper are simple, finite and undirected. The vertex and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. $|V(G)|$ and $|E(G)|$ denote the cardinality of these sets and in general $|S|$ denotes the cardinality of any given set S. We use the notation $N_G(v)$ to denote the set of vertices in a graph G which are adjacent to the vertex v. This set is known as the *neighborhood of* v and when the context of the graph is clear, it is simply denoted by $N(v)$.

Definition 1.1. A bijection $f\colon V(G) \to \{1, 2, \dots, n\}$ is said to be a *prime labeling* of a graph G with n vertices if $\gcd(f(u), f(v)) = 1$, whenever u and v are adjacent vertices of G. A graph that admits a prime labeling is called a prime graph.

The neighborhood-prime labeling of a graph G is a variant of prime labeling introduced by Patel and Shrimali [**6**] where they require greatest common divisor (gcd) of the labels of all the vertices in the neighborhood $N(v)$ to be 1. More precisely, any bijective function $f\colon V(G) \to \{1, 2, \dots, |V(G)|\}$ is said to be a *neighborhood-prime labeling* on G if $\gcd(f(N(v))) := \gcd\{f(u) : u \in N(v)\} = 1$ for every vertex $v \in V(G)$ whose degree is at least 2. A graph that admits neighborhood-prime labeling is called a *neighborhood-prime graph.*

Note that a prime graph may not be neighborhood-prime and a neighborhood-prime graph may not be prime. For instance, the complete graph K_4 is neighborhood-prime and not prime whereas the cycle C_6 is prime and not neighborhood-prime. However, Lemma 2.1 and Theorem 3.2 in the following sections provide interesting links between these two concepts. Prime labeling has been extensively

Received May 29, 2020; revised July 13, 2021.
2020 *Mathematics Subject Classification.* Primary 05C78.
Key words and phrases. Prime labeling; neighborhood-prime labeling; snake graphs.

studied since its introduction about forty years ago but neighborhood-prime labeling has been introduced very recently and so a lot remains to explore in this direction. Patel and Shrimali in their initial research ([**6, 7, 8**]) on neighborhood-prime labeling have shown that paths, complete graphs, wheels, helms, flowers, cartesian and tensor products of two paths are neighborhood-prime. They have shown that the cycle C_k is neighborhood-prime if and only if $k \not\equiv 2 \pmod 4$ and have further characterized neighborhood-prime graphs amongst the class of union of two cycle graphs. Cloys and Fox [**3**] have shown that certain classes of trees like caterpillars, spiders, firecrackers, trees without degree 2 vertices are neighborhood-prime and finally conjectured that all trees are neighborhood-prime. This is in line with the famous conjecture that all trees are prime which has remain unsolved for almost forty years. However, based on the partial result that all trees of sufficiently large orders are prime [**5**]; Asplund et al. [**1**] have shown that all trees of sufficiently large orders are neighborhood-prime. In the same paper based on Hamiltonicity of the graph, they have also shown that the generalized Petersen graphs and grid graphs are neighborhood-prime. A brief summary of results related to prime labeling, neighborhood-prime labeling and some of the variants of these two labelings is available in the dynamic survey of graph labeling by J. Gallian [**4**]. Now, we discuss the results of the present paper.

This paper deals with neighborhood-prime labeling of snake graphs of the type C_k^m and $C_{k,q}^m$ whose definitions and other terminology are explained in Section 2 and Section 3, respectively. The initial part of the present work is motivated by some of the partial results by Cloys and Fox [**3**] about neighborhood-prime labeling of snake graphs $S_{k,n}$ (same as C_k^{n-1} in our notation). In Section 2, we give a full answer on neighborhood-prime labeling of these snake graphs. The remaining part of our paper is motivated by a paper by A. Bigham et al. [**2**] where they introduce general snake graphs $C_{k,q}^m$ and study prime labeling of these newly introduced snake graphs. In the concluding section of the same paper, they have posed a question about neighborhood-prime labeling in the context of general snake graphs. In Section 3, we have come up with some positive results in response to their query. In particular, we show that the general snake graphs $C_{k,2}^m$ and $C_{k,3}^m$ are neighborhood-prime for $m \geq 2$.

All theorems are supported with appropriate examples and figures for a better understanding of their proofs. The following elementary number theory result is used while proving certain theorems and so it is stated over here in the form of a Lemma.

Lemma 1.1. *Let* $\{m_i\}_{i\geq 0}$ *be a sequence of integers defined by* $m_i := a + id$, *where* $a, d \in \mathbb{Z}$. *If* $\gcd(a, d) = 1$ *then* m_i *and* m_{i+1} *are relatively prime for all* i.

Proof. Let q be a positive integer such that $q|m_i$ (i.e., q divides m_i) and $q|m_{i+1}$. Then q divides $m_{i+1} - m_i = d$ and consequently $q|id$. Now $q|id$, $q|a + id$ and so $q|a$. Thus q divides both a and d. But $\gcd(a, d) = 1$, and hence $q = 1$. $\square$

2. Neighborhood-prime labeling of snake graphs C_k^m

We introduce the snake graph (also known as k-polygon snake) of [**3**] with a different notation C_k^m over here. The reason for this is that it can also be visulalized as $C_{k,1}^m$ which is a special case of the general snake graphs $C_{k,q}^m$ that, we study in Section 3.

A *snake graph* C_k^m is obtained by considering a path P_{m+1} on the vertices $u_1, u_2, \ldots, u_{m+1}$ where each edge u_iu_{i+1} is replaced by a cycle of length $k \geq 3$, one of whose edge is u_iu_{i+1}. Thus C_k^m is a graph with $m(k-1)+1$ vertices and mk edges and it is the same as the snake graph $S_{k,m+1}$ of [**3**]. In order to understand the neighborhood-prime labeling on C_k^m, we shall understand it as a path on $n = m(k-1)+1$ vertices $v_1, v_2, \ldots, v_n$ with additional m edges as $v_{(j-1)(k-1)+1}v_{j(k-1)+1}$, where $j = 1, 2, \ldots, m$. See for instance graph of C_7^4 in Figure 2.

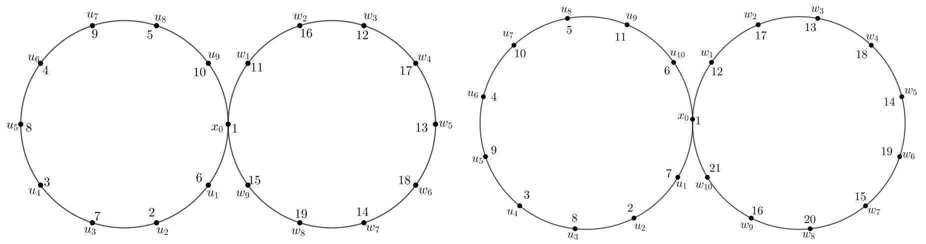
Now before, we establish our results on neighborhood-prime labeling of snake graphs C_k^m, let us review a known result about it.

Theorem 2.1 ([**3**])**.** *The k-polygonal snake $S_{k,n}$ (i.e., C_k^{n-1}) has a neighborhood-prime labeling for the following cases where $k \geq 6$ and $n \geq 3$:*

- $k \equiv 1 \pmod 4$ *and* $n = 2^l + 1$ *for* $l \geq 1$
- $k \equiv 0 \pmod 4$ *and* $n = 2^l$ *for* $l \geq 2$
- $k \equiv 0 \pmod 4$ *and* $n = 2^l + 1$ *for* $l \geq 1$
- $k = 2^l + 2$ *for* $l \geq 2$ *and* $n \equiv 3 \pmod 4$
- k *even and* $n = 3$
- $k = 2^l + 3$ *for* $l \geq 2$ *and* n *even.*

We see that the above theorem due to [**3**] discusses a very limited and special cases only. In this paper, we prove that the snake graph C_k^m is neighborhood-prime if and only if either $k \not\equiv 2 \pmod 4$ or $m \not\equiv 1 \pmod 4$.

Note that if $m = 1$, then C_k^m is just a cycle of length k which is known to be neighborhood-prime iff $k \not\equiv 2 \pmod 4$. See for instance [**6**]. Further, C_k^2 is just a one point union (or fusion) of two cycles of length k which can be easily shown as neighborhood-prime. For instance, refer to Figure 1a and 1b, where, we have illustrated this case for even as well as odd cycles. Therefore, while proving the results about neighborhood-prime labeling of the snake graphs, we always consider $m \geq 3$. A common approach of showing a graph is neighborhood-prime is to define a bijection $f\colon V(G) \to \{1, 2, \ldots, |V(G)|\}$ and verify that $\gcd(f(N(v))) := \gcd\{f(u) : u \in N(v)\} = 1$ for every vertex v whose degree is at least 2. This verification is trivial or obvious whenever $f(N(v))$ contains two or more consecutive integers. So throughout the paper, we shall use the terminology that $\gcd(f(N(v)))$ *is trivially* 1 to mean that $f(N(v))$ contains two or more consecutive integers. Proving negative results about neighborhood-prime labeling is usually much more difficult and same happens here also. So, we begin with positive results.



(a) One point union of cycles of length 10 **(b)** One point union of cycles of length 11

Figure 1. Neighborhood-prime labeling of one point union of two cycles

Theorem 2.2. *For $m \geq 3$, the snake graph C_k^m is neighborhood-prime if k is odd.*

Proof. Let $G = C_k^m$ be a snake graph with $V(G) = \{v_i : 1 \leq i \leq m(k-1)+1\}$ and $E(G) = \{v_i v_{i+1}, v_{(j-1)(k-1)+1} v_{j(k-1)+1} : 1 \leq i \leq m(k-1), 1 \leq j \leq m\}$. Then $|V(G)| = m(k-1)+1$ and $|E(G)| = mk$.

Define $f \colon V(G) \to \{1, 2, \ldots, |V(G)|\}$ by

$$f(v_{2i}) = i, \qquad 1 \leq i \leq \frac{(m-1)(k-1)}{2},$$

$$f(v_{2i-1}) = \frac{(m-1)(k-1)}{2} + i, \qquad 1 \leq i \leq \frac{m(k-1)}{2} + 1,$$

$$f(v_{2i}) = \frac{m(k-1)}{2} + 1 + i; \qquad \frac{(m-1)(k-1)}{2} + 1 \leq i \leq \frac{m(k-1)}{2}.$$

Observe that

$$\begin{aligned} \gcd(f(N(v_{m(k-1)+1}))) &= \gcd(f(v_{m(k-1)}), f(v_{(m-1)(k-1)+1})) \\ &= \gcd(m(k-1)+1, (m-1)(k-1)+1) = 1 \end{aligned}$$

due to Lemma 1.1. Also $f(v_2) = 1$, so that $\gcd(f(N(v_1))) = \gcd(f(v_2), f(v_k)) = 1$. Further, for any $v \neq v_1, v_{m(k-1)+1}$; $N(v)$ contains either 2 or 4 vertices for which $\gcd(f(N(v)))$ is trivially 1. Hence f is a neighborhood-prime labeling on G. $\square$

Example 2.1. Neighborhood-prime labeling of snake graph C_7^4 is as shown in Figure 2.

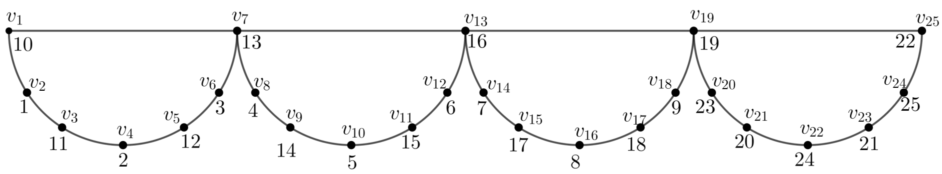


Figure 2. Neighborhood-prime labeling of snake graph C_7^4

Theorem 2.3. *For $m \geq 3$, the snake graph C_k^m is neighborhood-prime if:*

1. $k \equiv 0 \pmod 4$.
2. $k \equiv 2 \pmod 4$ *and m is even.*
3. $k \equiv 2 \pmod 4$ *and* $m \equiv 3 \pmod 4$.

Proof. Let $G = C_k^m$ be a snake graph with $V(G)$ and $E(G)$ as in the previous theorem.

1. $k \equiv 0 \pmod 4$

If m is even, define $f\colon V(G) \to \{1, 2, \ldots, |V(G)|\}$ by

$$\begin{aligned} f(v_{2i}) &= i+1, && 1 \leq i \leq \frac{(m-1)(k-1)-1}{2}, \\ &= 1, && i = \frac{(m-1)(k-1)-1}{2} + 1, \\ &= i, && \frac{(m-1)(k-1)-1}{2} + 2 \leq i \leq \frac{m(k-1)}{2}, \\ f(v_{2i-1}) &= \frac{m(k-1)}{2} + i, && 1 \leq i \leq \frac{m(k-1)}{2} + 1. \end{aligned}$$

If m is odd, define f by

$$\begin{aligned} f(v_{2i}) &= i+1, && 1 \leq i \leq \frac{m(k-1)+1}{2}, \\ f(v_{2i-1}) &= \frac{m(k-1)+1}{2} + 1 + i, && 1 \leq i \leq \frac{(m-1)(k-1)}{2}, \\ &= 1, && i = \frac{(m-1)(k-1)}{2} + 1, \\ &= \frac{m(k-1)+1}{2} + i, && \frac{(m-1)(k-1)}{2} + 2 \leq i \leq \frac{m(k-1)+1}{2}. \end{aligned}$$

The key points about verifying that f is a neighborhood-prime labeling are discussed below:
Since $k \equiv 0 \pmod 4$; $\frac{k}{2} + 1$ is odd and so

$$\gcd(f(N(v_1))) = \gcd(f(v_2), f(v_k)) = \gcd\left(2, \frac{k}{2} + 1\right) = 1.$$

Also

$$\begin{aligned} \gcd(f(N(v_{m(k-1)+1}))) &= \gcd(f(N(v_{(m-1)(k-1)}))) \\ &= \gcd(f(N(v_{(m-1)(k-1)+2}))) = 1 \end{aligned}$$

because all these vertices have $v_{(m-1)(k-1)+1}$ as one of their neighbors and

$$f(v_{(m-1)(k-1)+1}) = 1.$$

For $v \neq v_1, v_{m(k-1)+1}, v_{(m-1)(k-1)}, v_{(m-1)(k-1)+2}$; $\gcd(f(N(v))$ is trivially one and so, we are through. Neighborhood-prime labeling of snake graphs C_8^4 and C_8^3 are illustrated in Figure 3 and Figure 4, respectively.

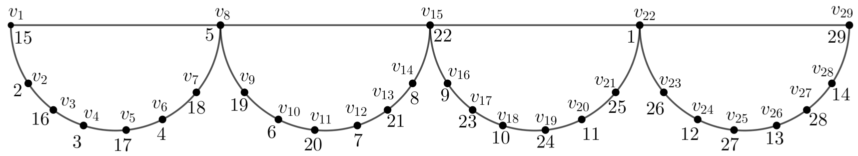


Figure 3. Neighborhood-prime labeling of snake graph C_8^4

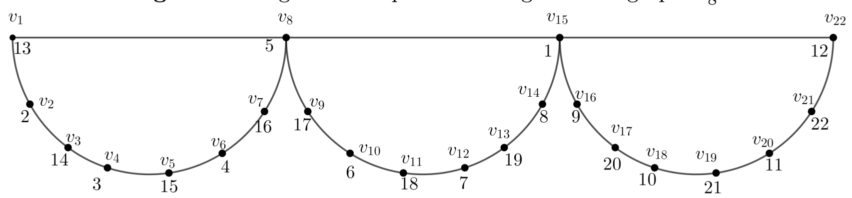


Figure 4. Neighborhood-prime labeling of snake graph C_8^3

2. $k \equiv 2 \pmod 4$ and m is even.
Define $f\colon V(G) \to \{1, 2, \ldots, |V(G)|\}$ by

$$
\begin{aligned}
f(v_{2i}) &= 4, && i = 1,\\
&= 3, && i = 2,\\
&= i + 2, && 3 \le i \le \frac{m(k-1)-2}{2},\\
&= 1, && i = \frac{m(k-1)}{2},\\
f(v_{2i-1}) &= \frac{m(k-1)-2}{2} + 2 + i, && 1 \le i \le \frac{m(k-1)}{2},\\
&= 2, && i = \frac{m(k-1)}{2} + 1.
\end{aligned}
$$

Since $k \equiv 2 \pmod 4$, $\frac{k}{2} + 2$ is odd and so $\gcd(f(N(v_1))) = \gcd(f(v_2), f(v_k)) = \gcd\left(4, \frac{k}{2}+2\right) = 1$. Moreover, $\gcd(f(N(v_5))) = \gcd(f(v_4), f(v_6)) = \gcd(3, 5) = 1$. Also $f(v_{m(k-1)}) = 1$, and so

$$\gcd(f(N(v_{m(k-1)+1}))) = \gcd(f(N(v_{m(k-1)-1}))) = 1.$$

Further as m is even,

$$
\begin{aligned}
\gcd(f(N(v_{m(k-1)}))) &= \gcd\left(f(v_{m(k-1)-1}), (f(v_{m(k-1)+1})\right)\\
&= \gcd(m(k-1)+1, 2) = 1.
\end{aligned}
$$

Finally, for any $v \ne v_1, v_5, v_{m(k-1)-1}, v_{m(k-1)}, v_{m(k-1)+1}$; $\gcd(f(N(v))$ is trivially one.

Neighborhood-prime labeling of snake graph C_6^4 is shown in Figure 5.

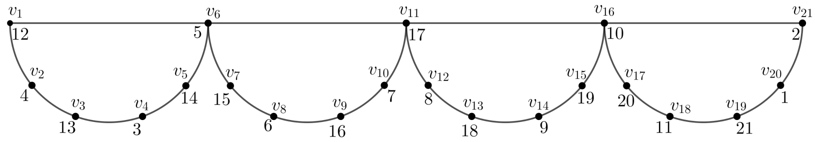


Figure 5. Neighborhood-prime labeling of snake graph C_6^4

Note that if m is odd, then the labeling f is no longer neighborhood-prime since $\gcd(f(N(v_{m(k-1)})))$ is equal to 2 now. So, we have to think differently when m is odd.

3. $k \equiv 2 \pmod 4$ and $m \equiv 3 \pmod 4$.

Here, we consider the following two cases on k.

<u>*Case 1.*</u> $\frac{k}{2} \not\equiv 1 \pmod 3$.

Define $f\colon V(G) \to \{1, 2, \ldots, |V(G)|\}$ as

$$
\begin{aligned}
f(v_{2i}) &= i + 2, && 1 \le i \le \frac{m(k-1)+1}{2},\\
f(v_{2i-1}) &= m(k-1) + 2 - i, && 1 \le i \le \frac{m(k-1)+1}{2} - 2,\\
&= 2, && i = \frac{m(k-1)+1}{2} - 1,\\
&= 1, && i = \frac{m(k-1)+1}{2}.
\end{aligned}
$$

We see that $\gcd(f(N(v_1))) = \gcd(f(v_2), f(v_k)) = \gcd\left(3, \frac{k}{2} + 2\right) = 1$ as $\frac{k}{2} + 2 \not\equiv 0 \pmod 3$. Since $f(v_{m(k-1)}) = 1$, we have

$$\gcd(f(N(v_{m(k-1)+1}))) = \gcd(f(N(v_{m(k-1)-1}))) = 1.$$

Also

$$
\begin{aligned}
\gcd(f(N(v_{m(k-1)-3}))) &= \gcd(f(v_{m(k-1)-2}), f(v_{m(k-1)-4}))\\
&= \gcd\left(2, \frac{m(k-1)+1}{2} + 3\right) = 1,
\end{aligned}
$$

because $\frac{m(k-1)+1}{2} + 3$ is odd due to the assumption that $k \equiv 2 \pmod 4$ and $m \equiv 3 \pmod 4$. For the remaining vertices v; $\gcd(f(N(v))$ is trivially one. Neighborhood-prime labeling of snake graph C_{10}^3 is shown in Figure 6.

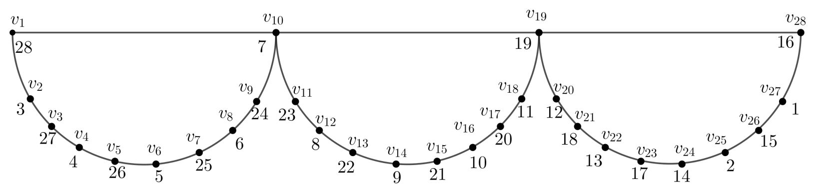


Figure 6. Neighborhood-prime labeling of snake graph C_{10}^3

Case 2. $\frac{k}{2} \equiv 1 \pmod 3$.
Define $f\colon V(G) \to \{1, 2, \ldots, |V(G)|\}$ as

$$
\begin{aligned}
f(v_{2i}) &= i+2 && 1 \le i \le \frac{k}{2} - 1,\\
&= i+4, && \frac{k}{2} \le i \le \frac{m(k-1)+1}{2},\\
f(v_{2i-1}) &= m(k-1)+2-i, && 1 \le i \le \frac{m(k-1)+1}{2} - 4,\\
&= 2, && i = \frac{m(k-1)+1}{2} - 3,\\
&= \frac{k}{2}+2, && i = \frac{m(k-1)+1}{2} - 2,\\
&= \frac{k}{2}+3, && i = \frac{m(k-1)+1}{2} - 1,\\
&= 1, && i = \frac{m(k-1)+1}{2}.
\end{aligned}
$$

First, we note that $\frac{k}{2} + 2 \equiv 0 \pmod 3$ and so $\frac{k}{2} + 4 \not\equiv 0 \pmod 3$. Hence $\gcd(f(N(v_1))) = \gcd(f(v_2), f(v_k)) = \gcd\left(3, \frac{k}{2}+4\right) = 1$ and further,

$$
\begin{aligned}
\gcd(f(N(v_{k-1}))) &= \gcd(f(v_{k-2}), f(v_k)) = \gcd\left(\frac{k}{2}+1, \frac{k}{2}+4\right)\\
&= \gcd\left(3, \frac{k}{2}+4\right) = 1.
\end{aligned}
$$

Now $f(v_{m(k-1)}) = 1$ and $v_{m(k-1)}$ is a neighbor of both $v_{m(k-1)+1}$ and $v_{m(k-1)-1}$ and so

$$\gcd(f(N(v_{m(k-1)+1}))) = \gcd(f(N(v_{m(k-1)-1}))) = 1.$$

Also

$$\gcd(f(N(v_{m(k-1)-5}))) = \gcd(f(v_{m(k-1)-4}), f(v_{m(k-1)-6})) = \gcd\left(\frac{k}{2}+2, 2\right) = 1$$

and

$$
\begin{aligned}
\gcd(f(N(v_{m(k-1)-7}))) &= \gcd(f(v_{m(k-1)-6}), f(v_{m(k-1)-8}))\\
&= \gcd\left(2, \frac{m(k-1)+1}{2}+5\right) = 1,
\end{aligned}
$$

because both $\frac{k}{2}+2$ and $\frac{m(k-1)+1}{2}+5$ are odd due to the assumption that $k \equiv 2 \pmod 4$ and $m \equiv 3 \pmod 4$. Finally, if

$$v \ne v_1, v_{k-1}, v_{m(k-1)-7}, v_{m(k-1)-5}, v_{m(k-1)-1}, v_{m(k-1)+1},$$

then $\gcd(f(N(v)))$ is trivially 1.

Neighborhood-prime labeling of snake graph C_{14}^3 is shown in Figure 7. $\square$

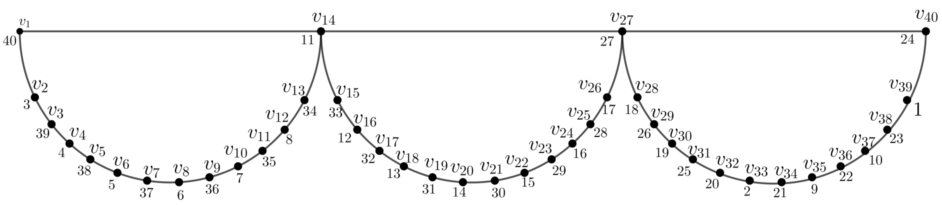


Figure 7. Neighborhood-prime labeling of snake graph C_{14}^3

We now proceed to proving that C_k^m is not neighborhood-prime if $k \equiv 2 \pmod 4$ and $m \equiv 1 \pmod 4$. Note that if $m = 1$, then C_k^m is a cycle of length $k \equiv 2 \pmod 4$ which is not neighborhood-prime as shown in [**6**]. But the proof of C_k^1 extends no further. Therefore, we first show that C_k^5 is not neighborhood-prime and then explain, essentially how the same argument can be used while proving the result for all higher values of m. For this, we introduce a concept of neighbor graph $H(G)$ of a given graph G, and prove a lemma regarding the connection between the neighborhood-prime labeling of G and prime labeling of its neighbor graph $H(G)$.

Definition 2.1. The neighbor graph $H(G)$ of a given graph G is a graph whose vertex set is same as the vertex set of the graph G and in which two vertices u and v are adjacent if and only if there exists a vertex w in G such that $N_G(w) = \{u, v\}$.

Note that if k is even then the neighbor graph of the cycle C_k is given by the union of two cycles of lengths $k/2$ and if k is odd then it is C_k itself. Also note that if a graph G does not contain any vertex of degree 2, then its neighbor graph does not have an edge. Consequently, the neighbor graph of the complete graph K_4 is $\overline{K_4}$ (i.e, complement graph of K_4).

Lemma 2.1. *Every neighborhood-prime labeling defined on a graph G is a prime labeling on its neighbor graph $H(G)$.*

Proof. Let f be any neighborhood-prime labeling defined on the graph G. Consider an arbitrary edge uv in graph $H(G)$. Then there exists a vertex w in G such that $N_G(w) = \{u, v\}$. But f is a neighborhood-prime labeling on G and hence $\gcd(f(N_G(w))) = \gcd\{f(u), f(v)\} = 1$. Thus, we have shown that $f(u)$ and $f(v)$ are relatively prime whenever uv is an edge in $H(G)$ and hence f is a prime labeling on $H(G)$. $\square$

Theorem 2.4. *The snake graph C_k^5 is not neighborhood-prime if $k \equiv 2 \pmod 4$.*

Proof. Let $G = C_k^5$ with $V(G) = \{v_i : 1 \le i \le 5(k-1)+1\}$ and

$$E(G) = \{v_i v_{i+1}, v_1 v_k, v_k v_{2k-1}, v_{2k-1} v_{3k-2}, v_{3k-2} v_{4k-3}, v_{4k-3} v_{5k-4} \\ : 1 \le i \le 5(k-1)\}.$$

Thus $|V(G)| = 5(k-1)+1 = 5k-4$ and $|E(G)| = 5(k-1)+5 = 5k$. Our first observation here is about the neighbor graph $H(G)$ (which, we shall denote by H

throughout the proof for the ease of notation) of the graph G whose edge set is

$$E(H) = \{v_{i-1}v_{i+1}, v_2v_k, v_{4k-3}v_{5k-5} \\ : \text{ where } 2 \leq i \leq 5k-5 \text{ and } i \neq k, 2k-1, 3k-2, 4k-3\}.$$

Observe that in the absence of the edges v_2v_k and $v_{4k-3}v_{5k-5}$, H is simply the (disjoint) union of six different paths each of which consists of vertices with either all even suffixes or odd suffixes. Based on this observation, we now consider the two subgraphs of H induced by the set of vertices with even suffixes and odd suffixes and, we denote these two induced subgraphs of H by H_1 and H_2, respectively. Thus

$$E(H_1) = \{v_{i-1}v_{i+1}, v_2v_k : \\ \text{where } i \text{ is odd only, } 2 \leq i \leq 5k-5 \text{ and } i \neq 2k-1, 4k-3\}$$

and

$$E(H_2) = \{v_{i-1}v_{i+1}, v_{4k-3}v_{5k-5} : \\ \text{where } i \text{ is even only, } 2 \leq i \leq 5k-5 \text{ and } i \neq k, 3k-2\}.$$

The neighbor graph H of the snake graph C_{10}^5 is shown in Figure 8.

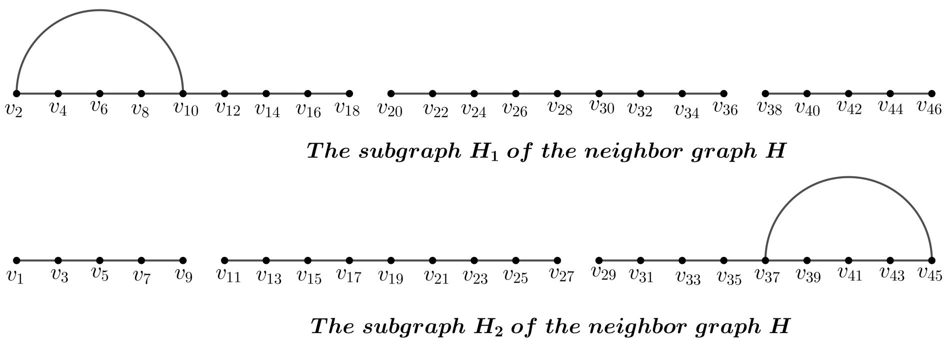


Figure 8. The neighbor graph H of the snake graph C_{10}^5

Obviously

$$|V(H_1)| = |V(H_2)| = \frac{|V(H)|}{2} = \frac{|V(G)|}{2} = \frac{5k-4}{2}$$

and moreover the graphs H_1 and H_2 are isomorphic. A very important observation here is that the number $\frac{5k-4}{2}$ is odd and this is because $k \equiv 2 \pmod 4$. Our proof is by contradiction. So assume that there exists a neighborhood-prime labeling of G which, we denote by f. By Lemma 2.1, f is a prime labeling on its neighbor graph H with $5k-4$ vertices where exactly $\frac{5k-4}{2}$ vertices must be labeled with even integers. For $i = 1, 2$; let N_i denote the number of vertices in H_i which are labeled with even integers. Since, we know that $N_1 + N_2 = \frac{5k-4}{2}$ and that $\frac{5k-4}{2}$ is odd, one of the numbers N_i is at least $\frac{1}{2}\left(\frac{5k-4}{2}+1\right) = \frac{5k-2}{4}$. Since H_1 and H_2 are isomorphic, without loss of generality, we may assume that $N_1 \geq \frac{5k-2}{4}$. Now the graph H_1 consists of three components; two of which are paths and the remaining one is a path after the removal of the additional edge v_2v_k. So the assignment of $\frac{5k-2}{4}$ even integers (through the prime labeling f) to the vertices of H_1 is extremely tight as, we shall see now. Recall that any collection of vertices which are labeled with even integers under a prime labeling is always an independent set. We denote the three components of H_1 by H_1^1, H_1^2 and H_1^3, where

$$E(H_1^1) = \{v_{i-1}v_{i+1}, v_2v_k : \text{ where } i \text{ is odd and } 3 \le i \le 2k-3\},$$
$$E(H_1^2) = \{v_{i-1}v_{i+1} : \text{ where } i \text{ is odd and } 2k+1 \le i \le 4k-5\},$$
$$E(H_1^3) = \{v_{i-1}v_{i+1} : \text{ where } i \text{ is odd and } 4k-1 \le i \le 5k-5\}.$$

Now as per the description of the component H_1^1 given above; it may be verified that its independence number is $\frac{k}{2} - 1$ (in the absence of the edge v_2v_k it is $\frac{k}{2}$ though) and so at the most $\frac{k}{2} - 1$ vertices in H_1^1 can only be labeled with even integers under f. Hence at least

$$\Big(\frac{5k-2}{4}\Big) - \Big(\frac{k}{2} - 1\Big) = \frac{3k+2}{4} \tag{2.1}$$

number of vertices from $H_1^2 \cup H_1^3$ must be labeled with even integers. But H_1^2 and H_1^3 are paths of even lengths $k-2$ and $\frac{k}{2} - 1$ (or, we may say paths on odd number of vertices $k-1$ and $\frac{k}{2}$), respectively. So the independence number of $H_1^2 \cup H_1^3$ is

$$\frac{1}{2}\big((k-1)+1\big) + \frac{1}{2}\Big(\frac{k}{2} + 1\Big) = \frac{3k+2}{4}. \tag{2.2}$$

Since the two numbers in (2.1) and (2.2) turn out to be the same; the only possible way of labeling at least $\frac{3k+2}{4}$ number of vertices of $H_1^2 \cup H_1^3$ with even integers is to start with the very first vertex and then label every alternate vertex in both the paths by the even integers. Accordingly, (looking at the edge sets of H_1^2 and H_1^3 described above) the vertices in the following union set

$$\big\{v_{2k}, v_{2k+4}, v_{2k+8}, \dots, v_{3k-2}, \dots, v_{4k-8}, v_{4k-4}\big\} \cup \big\{v_{4k-2}, v_{4k-6}, \dots, v_{5k-8}, v_{5k-4}\big\}$$

must all be labeled with even integers. (The presence of v_{3k-2} in the first set is justified due to the assumption that $k \equiv 2 \pmod 4$). But if this is true then all the four vertices of $N_G(v_{4k-3})$ namely $v_{3k-2}, v_{4k-4}, v_{4k-2}$ and v_{5k-4} get even labels and hence

$$\gcd(f(N_G(v_{4k-3}))) \ge 2.$$

But this contradicts our assumption that f is a neighborhood-prime labeling on G. □

Remark 2.1. In Theorem 2.4, we prove that if $k \equiv 2 \pmod 4$ then C_k^m is not neighborhood-prime when $m = 5$. The following key points are useful in extending the proof to the general case $m \equiv 1 \pmod 4$ $(m > 1)$.

1. $\frac{m(k-1)+1}{2}$ is an odd number which is also the cardinality of the vertex sets of the subgraphs H_1 and H_2 in the general case. The number $\frac{m(k-1)+1}{2}$ being odd, once again H_1 (without loss of generality) must be labeled with at least $\frac{m(k-1)+3}{4}$ even integers.

2. In the general case, H_1 consists of $\frac{m+1}{2}$ components out of which $\frac{m-1}{2}$ are paths and the remaining one is a path after the removal of the edge v_2v_k. Moreover out of the $\frac{m-1}{2}$ components which are paths; $\frac{m-3}{2}$ are of (even) lengths $k-2$ and the remaining one is of (even) length $\frac{k}{2}-1$.
3. The component of H_1 with the additional edge v_2v_k can accommodate at most $\frac{k}{2}-1$ vertices with even labels and so the remaining $\frac{m-1}{2}$ components must accommodate at least
$$\Big(\frac{m(k-1)+3}{4}\Big)-\Big(\frac{k}{2}-1\Big)=\frac{(m-2)k-m+7}{4}$$
vertices with even labels. While doing so, once again, we end up with a situation where one of the vertices from the set $\{v_{4k-3}, v_{6k-5}, v_{8k-7}, \ldots, v_{(m-1)k-(m-2)}\}$ has all its four neighbors being labeled with even integers and so, we arrive at a contradiction.

The results of Section 2 can now be summarised and given as:

Theorem 2.5. *The snake graph C_k^m is neighborhood-prime if and only if either* $k \not\equiv 2 \pmod 4$ *or* $m \not\equiv 1 \pmod 4$.

3. Neighborhood-prime labeling of general snake graphs $C_{k,q}^m$

The general snake graphs $C_{k,q}^m$ were first introduced by Bigham et al. [**2**] wherein they also defined terms like spine and belly of such snake graphs. In the following paragraph, we explain this definition and terminology but simultaneously ask the reader to refer to Figures 9a, 9b and 9c. The parameters k, q and m associated with the graph $C_{k,q}^m$ are positive integers satisfying the conditions $k \geq 3$ and $q \leq \frac{k}{2}$.

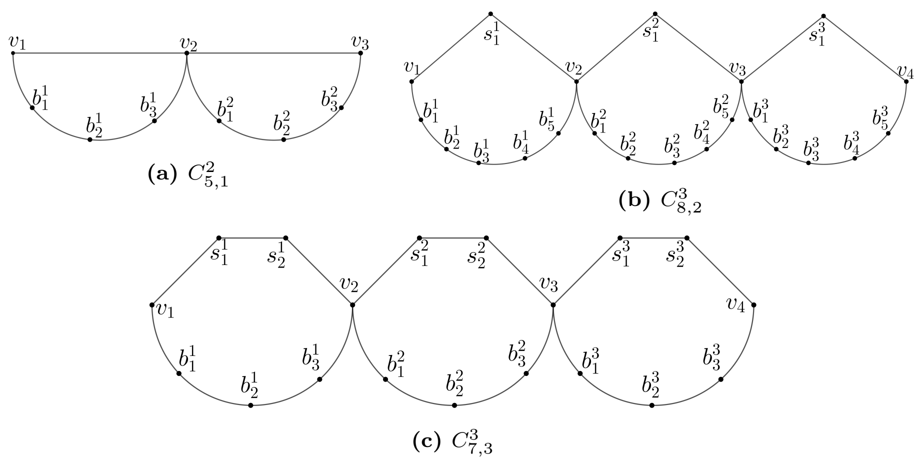


Figure 9. A few examples of snake graphs $C_{k,q}^m$

In order to define $C^m_{k,q}$, first, we consider $m+1$ vertices $v_1, v_2, \dots, v_{m+1}$ arranged from left to right. Then the *general snake graph* $C^m_{k,q}$ is a graph with $m(k-1)+1$ vertices and mk edges which is obtained by completing a cycle $C^{(i)}_k$ of length k between every pair of points v_i and v_{i+1} (where $1 \leq i \leq m$) in such a way that the lengths of the shorter and the longer paths (along the cycle $C^{(i)}_k$) joining v_i and v_{i+1} are q and $k-q$, respectively. Further, the newly introduced $q-1$ vertices along this shorter path joining v_i and v_{i+1} are known as *spine vertices* denoted by s^i_l, $1 \leq l \leq q-1$; where i refers to the cycle $C^{(i)}_k$ which the spine vertex belongs and l refers to the distance between s^i_l and v_i. Similarly, the $k-q-1$ newly introduced vertices along the longer path joining v_i and v_{i+1} are known as *belly vertices* denoted by b^i_j, $1 \leq j \leq k-q-1$; where i refers to the cycle $C^{(i)}_k$ which the belly vertex belongs and j refers to the distance between b^i_j and v_i. The vertices other than the spine and belly vertices are precisely the vertices v_i which are all known as *vertebrae.* The shortest path of length qm from v_1 to v_{m+1} is called the *spine.* The path of $(k-q)m$ edges from v_1 to v_{m+1} that does not contain any vertices on the spine is called the *belly* of the snake graph. Figures 9a, 9b and 9c show various examples of the general snake graphs labeled with spine and belly vertices and the vertebrae.

Note that if $q = 1$, then the graph $C^m_{k,q}$ is same as the snake graph C^m_k studied in Section 2 but if $q > 1$ and $m > 2$, then these two graphs are non-isomorphic. Bigham et al. [**2**] introduced a concept of cyclic snake labeling related to snake graphs $C^m_{k,q}$ and analysed the cases under which the cyclic snake labeling (or its slight modification) results into prime labeling. Accordingly, they managed to show that $C^m_{k,1}$ is prime for all k and m and that $C^m_{k,2}$ is prime whenever k is odd and m is less than or equal to the least prime factor of $k-2$. For further results in this direction refer [**2**]. Our focus will be on proving that $C^m_{k,2}$ and $C^m_{k,3}$ are neighborhood-prime. Note that if $m = 1$, then $C^m_{k,q}$ is just a cycle of length k and if $m = 2$, then $C^m_{k,q}$ is just a one point union (or fusion) of two cycles of length k and these two situations have already been discussed in Section 2. Therefore, while proving the results about neighborhood-prime labeling of the snake graphs $C^m_{k,q}$, we consider $m \geq 3$.

3.1. The case $q = 2$

Theorem 3.1. *The snake graph $C^m_{k,2}$ is neighborhood-prime for all $m \geq 3$.*

Proof. Let $G = C^m_{k,2}$ be a snake graph with $V(G) = \{v_j, v_{m+1}, s^j_1, b^j_i : 1 \leq j \leq m, 1 \leq i \leq k-3\}$ and $E(G) = \{v_j b^j_1, b^j_i b^j_{i+1}, b^j_{k-3} v_{j+1}, v_j s^j_1, s^j_1 v_{j+1} : 1 \leq j \leq m, 1 \leq i \leq k-4\}$. Then $|V(G)| = m(k-1)+1$ and $|E(G)| = mk$. We need to define the required labeling $f \colon V(G) \to \{1, 2, \dots, |V(G)|\}$ separately for odd and even k.

<u>*Case 1:*</u> k is odd.
For $j = 1, 2, \dots, m$ and $i = 1, 2, \dots, \frac{k-3}{2}$;, we define f by

$$f(v_j) = (j-1)(k-1)+1,$$

$$f(b^j_{2i}) = (j-1)(k-1) + 1 + i,$$
$$f(s^j_1) = (j-1)(k-1) + \frac{k-1}{2} + 1,$$
$$f(b^j_{2i-1}) = (j-1)(k-1) + \frac{k-1}{2} + 1 + i.$$

Note that $\gcd(f(N(s^j_1))) = \gcd(f(v_j), f(v_{j+1})) = \gcd((j-1)(k-1)+1, j(k-1)+1) = 1$ due to Lemma 1.1.

Case 2: k is even.

For $j = 1, 2, \ldots, m$; $i = 1, 2, \ldots, \frac{k-2}{2}$ and $l = 1, 2, \ldots, \frac{k-4}{2}$; here, we define f by

$$f(b^j_{2i-1}) = \begin{cases} (j-1)\Big(\dfrac{k-2}{2}\Big) + i, & \text{if } j = 1, 2, \ldots, m-1, \\ \dfrac{(m-1)(k-2)}{2} + \dfrac{m(k-2)}{2} + 1 + i, & \text{if } j = m, \end{cases}$$
$$f(v_j) = \frac{(m-1)(k-2)}{2} + (j-1)\Big(\frac{k-2}{2}\Big) + 1,$$
$$f(b^j_{2l}) = \frac{(m-1)(k-2)}{2} + (j-1)\Big(\frac{k-2}{2}\Big) + 1 + l,$$
$$f(v_{m+1}) = \frac{(m-1)(k-2)}{2} + m\Big(\frac{k-2}{2}\Big) + 1,$$
$$f(s^j_1) = \frac{(m-1)(k-2)}{2} + (m+1)\Big(\frac{k-2}{2}\Big) + 2 + (m-j).$$

Using Lemma 1.1 $\big($with $a = \frac{(m-1)(k-2)}{2} + 1$ and $d = \frac{(k-2)}{2}\big)$, we claim that

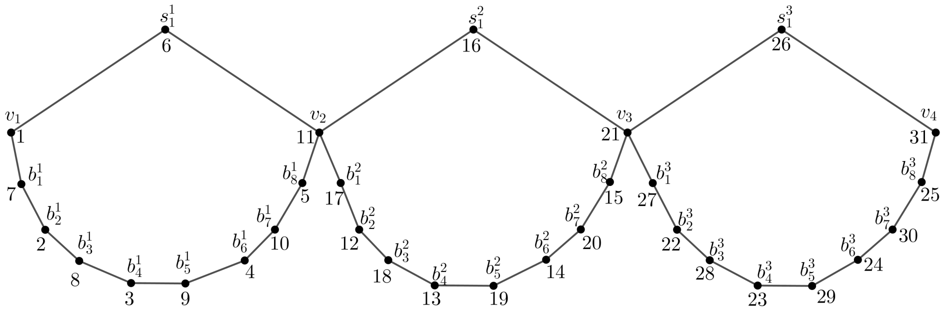
$$\gcd(f(N(s^j_1))) = \gcd(f(v_j), f(v_{j+1})) = 1.$$

Also when k is even, $f(b^1_1) = 1$ gives $\gcd(f(N(v_1))) = \gcd(f(b^1_1), f(s^1_1)) = 1$. Finally, for $v \neq s^j_1$ when k is odd and for $v \neq s^j_1, v_1$ when k is even, it may be verified that $\gcd(f(N(v)))$ is trivially 1. Thus $G = C^m_{k,2}$ is neighborhood-prime. □

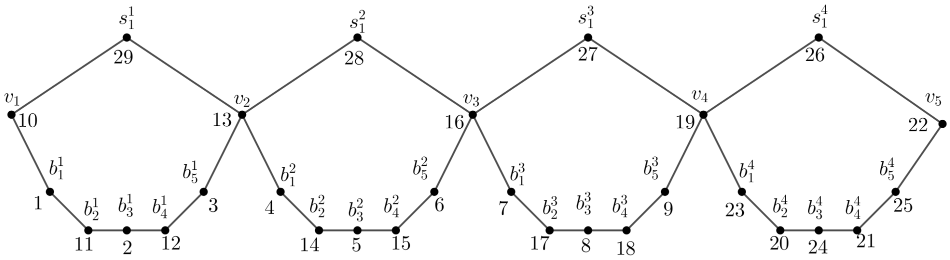
Example 3.1. Neighborhood-prime labeling of the snake graphs $C^3_{11,2}$ and $C^4_{8,2}$ are shown in Figure 10.

3.2. The case $q = 3$

Asplund et al. [**1**] define *the set of neighborhood graphs* of a given graph G as the set of all possible graphs H such that $V(H) = V(G)$, and for each $v \in V(G)$ with $\deg_G(v) \geq 2$, there exists exactly one edge $uw \in E(H)$ where $u, w \in N_G(v)$. We alert the reader about the noticeable difference between the concept of neighbor graph introduced in Section 2 and the concept of neighborhood graphs introduced over here. A major difference is that given a graph G, its neighbor graph is unique whereas its neighborhood graphs can be more than one. We saw in Section 2 that neighbor graphs are used to give a necessary condition for the existence of neighborhood-prime labeling of a given graph G, whereas the following theorem



(a) $C^3_{11,2}$



(b) $C^4_{8,2}$

Figure 10. Neighborhood-prime labeling of the snake graphs $C^3_{11,2}$ and $C^4_{8,2}$

due to Asplund et al. [**1**] uses neighborhood graphs to give a sufficient condition for the same.

Theorem 3.2 ([**1**]). *For a graph G, if there exists its neighborhood graph H which is prime, then G is neighborhood-prime.*

Proof. Assume $f\colon V(H) \to \{1, 2, \ldots, |V(H)|\}$ is a prime labeling of H. Let $v \in V(G)$ with $\deg_G(v) \geq 2$, and suppose u, w are the neighbors of v for which uw is an edge in H. Since f is a prime labeling of H, we have $\gcd\{f(u), f(w)\} = 1$, and thus $\gcd\{f(N_G(v))\} = 1$ as well because $\{f(u), f(w)\} \subseteq f(N_G(v))$. $\square$

We shall show that $C^m_{k,3}$ is neighborhood-prime by showing that one of its neigborhood graph is prime. For this, we introduce a new graph in this paper denoted by $P^+_{n,r}$, which is associated with the path P_n on n vertices. We shall prove an important theorem related to a labeling of $P^+_{n,r}$ and this theorem will form an important base for proving our main theorem.

Let P_n be a path on n vertices $v_1, v_2, \ldots, v_n$ and $\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$ be an independent set of r vertices of the path for some $1 \leq r \leq \lceil \frac{n}{2} \rceil$. Then $P^+_{n,r}$ *is defined as* a graph obtained from the path P_n after attaching exactly 2 pendent edges to each of the r vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$. Thus, $P^+_{n,r}$ is a graph on $n+2r$ vertices with $n+2r-1$ edges. If $n > 1$, then the choice of the independent set $\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$ is not unique and so every such choice results into a graph $P^+_{n,r}$. For instance, if

$n = 4$, then, we have seven graphs of the type $P^+_{4,r}$ with respect to seven different choices of the independent sets namely $\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_1, v_3\}, \{v_1, v_4\}$ and $\{v_2, v_4\}$. One of the graphs $P^+_{n,r}$ for $n = 14$ and $r = 5$ can be seen in Figure 11.

So far, we have defined $P^+_{n,r}$ for $1 \leq r \leq \lceil \frac{n}{2} \rceil$. If, we use the notation $P^+_{n,0}$ for the path P_n, then in future, we can use the notation $P^+_{n,r}$ for all $0 \leq r \leq \lceil \frac{n}{2} \rceil$. This is just for an added convenience. We know that for any positive integer k, the vertices of a path P_n can be labeled with n consecutive integers $k, k+1, \ldots, k+n-1$ in such a way that any two adjacent vertices have relatively prime labels. We shall now see that a similar type of labeling is possible in case of the graph $P^+_{n,r}$ also.

Lemma 3.1. *Let P_3 be a path on vertices v_1, v_2, v_3 and $P^+_{3,1}$ be the graph obtained from the path P_3 after attaching two pendent edges $v_2^1 v_2$ and $v_2^2 v_2$ to the vertex v_2 of P_3. Then for any arbitrary positive integer k, the graph $P^+_{3,1}$ can be labeled with 5 consecutive integers $k, k+1, \ldots, k+4$ such that the labels of any two adjacent vertices in $P^+_{3,1}$ are relatively prime and moreover v_1 and v_3 are assigned the labels k and $k+4$, respectively.*

Proof. First, we fix the labels k and $k+4$ for the vertices v_1 and v_3, respectively. Further, if k is odd then assign $k+2$ to v_2. Now assigning $k+1$ and $k+3$ to v_2^1 and v_2^2 randomly completes our requirement.

If k is even then $k+1$ and $k+3$ both are odd and further, both of them cannot be a multiple of 3. So whichever is not a multiple of 3, assign that label to v_2 and the remaining one along with $k+2$ can be assigned to the vertices v_2^1 and v_2^2 randomly. It may be verified that this assignment meets our requirement. $\square$

We ask the reader to verify that the method discussed in the proof of Lemma 3.1 can easily be adopted and extended to prove the following two theorems. As a matter of help, see Figure 11.

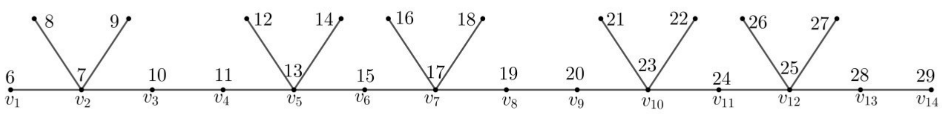


Figure 11. A labeling of $P^+_{14,5}$

Theorem 3.3. *Consider a graph $P^+_{n,r}$ associated with the path P_n on n vertices $v_1, v_2, \ldots, v_n$ for some $0 \leq r \leq \lceil \frac{n}{2} \rceil$ and in which v_1 and v_n are free from pendent edges. Then for any arbitrary positive integer k, the graph $P^+_{n,r}$ can be labeled with $n+2r$ consecutive integers $k, k+1, \ldots, k+n+2r-1$ such that the labels of any two adjacent vertices in $P^+_{n,r}$ are relatively prime and moreover v_1 and v_n are assigned the labels k and $k+n+2r-1$, respectively.*

Theorem 3.4. *Consider a graph $P^+_{n,r}$ associated with the path P_n on n vertices $v_1, v_2, \ldots, v_n$ for some $1 \leq r \leq \lceil \frac{n}{2} \rceil$ and in which v_1 has two pendent edges. Then for any arbitrary positive integer k, the graph $P^+_{n,r}$ can be labeled with $n+2r$*

consecutive integers $k, k+1, \ldots, k+n+2r-1$ *such that the labels of any two adjacent vertices in* $P^+_{n,r}$ *are relatively prime and* v_n *is assigned the label* k. *Moreover, if* $k+2n-1$ *is neither even nor a multiple of* 3, *then such a labeling is possible with an additional condition that* v_1 *is assigned* $k+2n-1$.

With this, we are now ready to prove our main theorem on neighborbood-prime labeling of $C^m_{k,3}$.

Theorem 3.5. *The graph* $C^m_{k,3}$ *is neighborhood-prime for all* $m \geq 3$.

Proof. The definition of $C^m_{k,q}$ suggests that if $q=3$, then $k \geq 6$ and so this is assumed throughout the proof. Let $G = C^m_{k,3}$ with vertex set

$$V(G) = \{v_j, v_{m+1}, s^j_1, s^j_2, b^j_i : 1 \leq j \leq m, 1 \leq i \leq k-4\}$$

and

$$E(G) = \{v_j b^j_1, b^j_i b^j_{i+1}, b^j_{k-4} v_{j+1}, v_j s^j_1, s^j_1 s^j_2, s^j_2 v_{j+1} : 1 \leq j \leq m, 1 \leq i \leq k-5\}.$$

Then $|V(G)| = m(k-1)+1$ and $|E(G)| = mk$. Consider a neighborhood graph H of G with $V(H) = V(G)$ and whose edges are given by

$$\begin{aligned} E(H) = \{&uw : u, w \in N_G(v) \text{ and where } v \in V(G) \text{ is such that } \deg_G(v) = 2\} \\ &\cup \{b^j_{k-4} b^{j+1}_1 : 1 \leq j \leq m-1\}. \end{aligned} \tag{3.1}$$

Due to Theorem 3.2, it is enough to show that H is a prime graph. We do this by taking two cases.

<u>*Case 1:*</u> k is even.

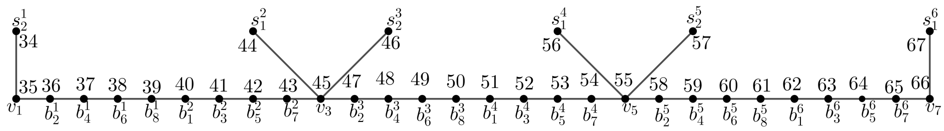
Subcase (i): m is even.

In this case notice that $E(H) = E(H_1) \cup E(H_2)$, where

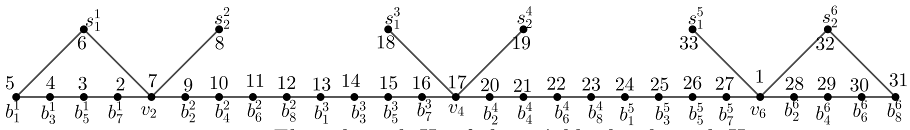
$$\begin{aligned} E(H_1) = \Big\{&s^1_2 v_1,\ v_{2j-1} b^{2j-1}_2,\ b^{2j-1}_{2i} b^{2j-1}_{2i+2},\ b^{2j-1}_{k-4} b^{2j}_1,\ b^{2j}_{2i-1} b^{2j}_{2i+1},\ s^{2j}_1 v_{2j+1}, \\ &b^{2j}_{k-5} v_{2j+1},\ v_{2l+1} s^{2l+1}_2 : 1 \leq i \leq \frac{k-6}{2}, 1 \leq j \leq \frac{m}{2}, 1 \leq l \leq \frac{m-2}{2}\Big\} \\ E(H_2) = \Big\{&b^1_1 s^1_1,\ b^{2j-1}_{2i-1} b^{2j-1}_{2i+1},\ v_{2j} b^{2j-1}_{k-5},\ s^{2j-1}_1 v_{2j},\ v_{2j} s^{2j}_2,\ v_{2j} b^{2j}_2,\ b^{2l}_{k-4} b^{2l+1}_1, \\ &b^{2j}_{2i} b^{2j}_{2i+2}, s^m_2 b^m_{k-4} : 1 \leq i \leq \frac{k-6}{2}, 1 \leq j \leq \frac{m}{2}, 1 \leq l \leq \frac{m-2}{2}\Big\}. \end{aligned}$$

Note that if $k=6$, then i runs over an empty range which means that the edges defined in terms of i will be absent from both edge sets. This understanding shall prevail throughout the proof where such instances occur. Neighborhood graph H of $C^6_{12,3}$ can be observed in Figure 12.

One can verify that the neighborhood graph H of G is actually a (disjoint) union of two graphs H_1 and H_2, where $|V(H_1)| = \frac{m(k-1)}{2}+1$, $|V(H_2)| = \frac{m(k-1)}{2}$ and moreover, H_1 is of the type $P^+_{\frac{m(k-3)+6}{2}, \frac{m-2}{2}}$. We keep in mind that H_2 is only marginally different from H_1 in the sense that after the removal of edges $b^1_1 s^1_1$ and $b^m_{k-4} s^m_2$; H_2 is of the type $P^+_{\frac{m(k-3)}{2}, \frac{m}{2}}$. In view of Theorem 3.3 it is possible to label the vertices of H_1 with the help of $\frac{m(k-1)}{2}+1$ consecutive integers starting



The subgraph H_1 of the neighborhood graph H



The subgraph H_2 of the neighborhood graph H

Figure 12. Neighborhood graph H of $C^6_{12,3}$ with a prime labeling

with $\frac{m(k-1)}{2}+1$ and ending with $m(k-1)+1$, such that any two adjacent vertices of H_1 get relatively prime labels. Thus, we are done if, we show that the graph H_2 can be labeled with first $\frac{m(k-1)}{2}$ positive integers in such a way that any two adjacent vertices of H_2 get relatively prime labels. For this, we consider the three subgraphs of H_2 say H_2^1, H_2^2 and H_2^3 which are induced (respectively) by the three disjoint sets of vertices $S_1 = \{v_2, b_1^1, b_3^1, \ldots, b_{k-7}^1, b_{k-5}^1, s_1^1, s_2^2\}$, $S_2 = \{v_m, b_2^m, b_4^m, \ldots, b_{k-6}^m, b_{k-4}^m, s_1^{m-1}, s_2^m\}$ and $S_3 = V(H_2) - S$, where $S = S_1 \bigcup S_2$. Note that H_2^1, H_2^2 and H_2^3 can also be understood as the three connected components of the graph H_2 obtained on removal of the two edges $v_2 b_2^2$ and $b_{k-5}^{m-1} v_m$. Define the labels of vertices of the set $S_1 \cup S_2$ by

$$
\begin{aligned}
f(b_{2i-1}^1) &= \frac{k}{2} - i,\ 1 \le i \le \frac{k-4}{2},\\
f(s_1^1) &= \frac{k}{2},\\
f(s_2^2) &= \begin{cases} \dfrac{k}{2}+1, & \text{if } \dfrac{k}{2} \text{ is odd;}\\ \dfrac{k}{2}+2, & \text{if } \dfrac{k}{2} \text{ is even,}\end{cases}\\
f(v_2) &= \begin{cases} \dfrac{k}{2}+2, & \text{if } \dfrac{k}{2} \text{ is odd;}\\ \dfrac{k}{2}+1, & \text{if } \dfrac{k}{2} \text{ is even,}\end{cases}\\
f(b_{2i}^m) &= \frac{m(k-1)}{2} - \frac{k}{2} + i, \qquad 1 \le i \le \frac{k-4}{2},\\
f(s_2^m) &= \frac{m(k-1)}{2} - 1,\\
f(s_1^{m-1}) &= \frac{m(k-1)}{2},\\
f(v_m) &= 1.
\end{aligned}
$$

We have assigned the labels to the vertices of sets S_1 and S_2 with the help of two sets of integers

$$\Big\{2, 3, \ldots, \frac{k}{2}+1, \frac{k}{2}+2\Big\}$$
$$\cup \Big\{1, \frac{m(k-1)}{2} - \frac{k}{2}+1, \frac{m(k-1)}{2} - \frac{k}{2}+2, \ldots, \frac{m(k-1)}{2} - 1, \frac{m(k-1)}{2}\Big\}.$$

The intermediate set of consecutive integers $\Big\{\frac{k}{2}+3, \frac{k}{2}+4, \ldots, \frac{m(k-1)}{2} - \frac{k}{2}\Big\}$ can be used to label all the vertices of the set S_3 as per the following rule. We observe that the subgraph H_2^3 of H_2 which is induced by the vertices of the set S_3 is $P^+_{n,r}$ (where $n = (m-2)(\frac{k-4}{2}) + \frac{m}{2} - 2$ and $r = \frac{m}{2} - 2$) with end vertices as b_2^2 and b_{k-5}^{m-1}. So for $u \in S_3$, we may define $f(u)$ as per Theorem 3.3 so that $f(b_2^2) = \frac{k}{2}+3$ and $f(b_{k-5}^{m-1}) = \frac{m(k-1)}{2} - \frac{k}{2}$ and any two adjacent vertices in S_3 get relatively prime labels. Keeping in mind that $f(v_2)$ and $f(b_2^2)$ are always relatively prime (being consecutive or consecutive odd integers) and that $f(b_{k-5}^{m-1})$ and $f(v_m)$ are relatively prime (as $f(v_m) = 1$); it is not difficult to verify that f as defined above is a prime labeling on H_2 and so we are done in the case m is even.

Subcase (ii): m is odd.
In this case notice that $E(H) = E(H_1) \cup E(H_2)$, where

$$E(H_1) = \Big\{s_2^1 v_1, v_{2j-1}b_2^{2j-1}, b_{2i}^{2j-1}b_{2i+2}^{2j-1}, b_{k-4}^{2l-1}b_1^{2l},\ b_{2i-1}^{2l}b_{2i+1}^{2l},\ b_{k-5}^{2l}v_{2l+1},\ s_1^{2l}v_{2l+1},$$
$$v_{2l+1}s_2^{2l+1}, s_2^m b_{k-4}^m : 1 \le i \le \frac{k-6}{2}, 1 \le j \le \frac{m+1}{2}, 1 \le l \le \frac{m-1}{2}\Big\}$$
$$E(H_2) = \Big\{b_1^1 s_1^1,\ b_{2i-1}^{2j-1}b_{2i+1}^{2j-1},\ b_{k-5}^{2j-1}v_{2j},\ s_1^{2j-1}v_{2j},\ v_{2l}s_2^{2l},\ v_{2l}b_2^{2l},\ b_{2i}^{2l}b_{2i+2}^{2l},$$
$$b_{k-4}^{2l}b_1^{2l+1} : 1 \le i \le \frac{k-6}{2}, 1 \le j \le \frac{m+1}{2}, 1 \le l \le \frac{m-1}{2}\Big\}.$$

Neighborhood graph H of $C^5_{14,3}$ is as in Figure 13.

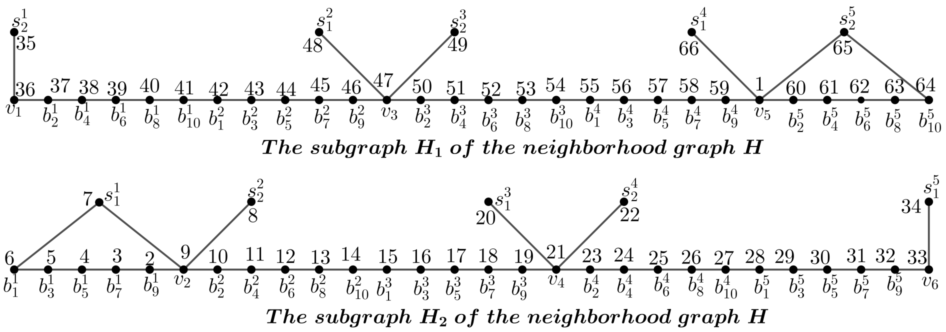


Figure 13. Neighborhood graph H of $C^5_{14,3}$ with a prime labeling

The graph H is once again a (disjoint) union of two graphs H_1 and H_2 with $|V(H_1)| = |V(H_2)| = \frac{m(k-1)+1}{2}$. In fact, here H_1 and H_2 are isomorphic graphs. Moreover, if the edges $b^m_{k-4}s^m_2$ and $b^1_1s^1_1$ are removed from the graphs H_1 and H_2, respectively then both these graphs are of type $P^+_{n,r}$ with $n = \frac{m(k-3)+3}{2}$ and $r = \frac{m-1}{2}$. We describe a method to label the vertices of H_1 and H_2 by considering the two induced subgraphs of each of these two graphs and conclude that this method results into prime labeling of H.

First consider the two subgraphs of H_2 say H^1_2 and H^2_2 which are induced by the two disjoint sets of vertices $S^1_2 = \{v_2, b^1_1, b^1_3, \ldots, b^1_{k-7}, b^1_{k-5}, s^1_1, s^2_2\}$ and $S^2_2 = V(H_2) - S^1_2$, respectively. H^1_2 and H^2_2 can be understood as the two connected components of H_2 obtained after the removal of the edge $v_2b^2_2$.

We label the vertices of H^1_2 using the integers $2, 3, \ldots, \frac{k}{2} + 2$ as follows:

$$\begin{aligned}
f(b^1_{2i-1}) &= \frac{k}{2} - i; \quad 1 \le i \le \frac{k-4}{2},\\
f(s^1_1) &= \frac{k}{2},\\
f(s^2_2) &= \begin{cases} \frac{k}{2} + 1, & \text{if } \dfrac{k}{2} \text{ is odd;}\\ \frac{k}{2} + 2, & \text{if } \dfrac{k}{2} \text{ is even,}\end{cases}\\
f(v_2) &= \begin{cases} \dfrac{k}{2} + 2, & \text{if } \dfrac{k}{2} \text{ is odd;}\\ \dfrac{k}{2} + 1, & \text{if } \dfrac{k}{2} \text{ is even.}\end{cases}
\end{aligned}$$

It is easily seen that any two adjacent vertices of H^1_2 have relatively prime labels under this f. Now, H^2_2 is of the type $P^+_{n,r}$ (with $n = \frac{m(k-3)-k+5}{2}$ and $r = \frac{m-3}{2}$) with end vertices as b^2_2 and s^m_1. Hence the vertices of H^2_2 are labeled under f using the integers $\frac{k}{2}+3, \frac{k}{2}+4, \ldots, \frac{m(k-1)+1}{2} + 1$ as per Theorem 3.3. Consequently any two adjacent vertices of H^2_2 have relatively prime labels and moreover $f(b^2_2) = \frac{k}{2}+3$.

Now consider the two subgraphs of H_1 say H^1_1 and H^2_1 which are induced by two disjoint sets of vertices $S^1_1 = \{v_m, b^m_2, b^m_4, \ldots, b^m_{k-6}, b^m_{k-4}, s^{m-1}_1, s^m_2\}$ and $S^2_1 = V(H_1) - S^1_1$, respectively. H^1_1 and H^2_1 can be understood as the two connected components of H_1 after the removal of the edge $b^{m-1}_{k-5}v_m$. We label the vertices of H^1_1 using the integer 1 along with consecutive integers $m(k-1) - \frac{k}{2} + 2$, $m(k-1) - \frac{k}{2} + 3 \ldots, m(k-1) + 1$ as follows:

$$\begin{aligned}
f(b^m_{2i}) &= m(k-1) - \frac{k}{2} + 1 + i; \quad 1 \le i \le \frac{k-4}{2},\\
f(s^m_2) &= m(k-1),\\
f(s^{m-1}_1) &= m(k-1) + 1,\\
f(v_m) &= 1.
\end{aligned}$$

Note that any two adjacent vertices of H_1^1 have relatively prime labels under f. Now, H_1^2 is of the type $P_{n,r}^{+}$ (with $n = \frac{m(k-3)-k+5}{2}$ and $r = \frac{m-3}{2}$) with end vertices as s_2^1 and b_{k-5}^{m-1}. Hence the vertices of H_1^2 are labeled under f using the consecutive integers $\frac{m(k-1)+1}{2}+2, \frac{m(k-1)+1}{2}+3, \dots, m(k-1)-\frac{k}{2}+1$ as per Theorem 3.3. As a result any two adjacent vertices of H_1^2 have relatively prime labels and moreover $f(b_{k-5}^{m-1}) = m(k-1)-\frac{k}{2}+1$. Finally, $f(u)$ is now defined for all vertices of the graph $H_1 \bigcup H_2 = H$. In view of the above arguments and the fact that $\gcd(f(v_2), f(b_2^2)) = 1 = \gcd(f(v_m), f(b_{k-5}^{m-1}))$, we conclude that f is a prime labeling on H.

Case 2: k is odd.

We shall assume that $k \geq 9$ and later comment on the case $k = 7$ (this is the minimum value for odd k).

If k is odd, then the edge set of the neighborhood graph H of G as described in (3.1) is given by

$$E(H) = \big\{v_j b_2^j, b_{2i}^j b_{2i+2}^j,\ b_{k-5}^j v_{j+1},\ s_1^j v_{j+1},\ s_2^j v_j,\ s_1^1 b_1^1,\ s_2^m b_{k-4}^m, b_{2l-1}^j b_{2l+1}^j,$$
$$b_{k-4}^r b_1^{r+1} : 1 \leq i \leq \frac{k-7}{2}, 1 \leq j \leq m, 1 \leq l \leq \frac{k-5}{2}, 1 \leq r \leq m-1\big\}.$$

Neighborhood graph H of $C_{9,3}^6$ is as in Figure 14.

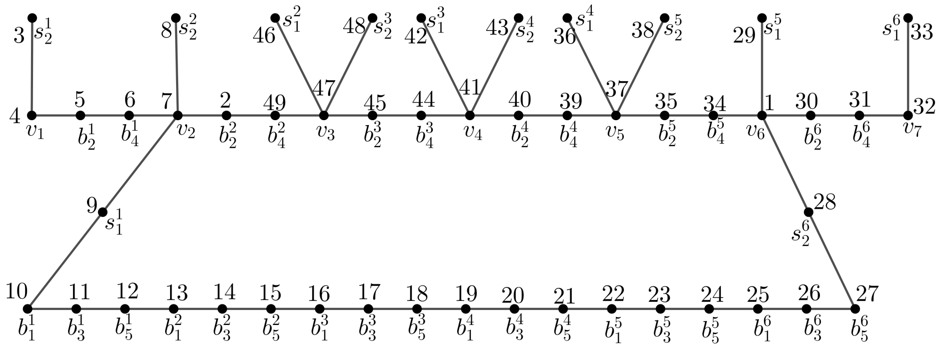


Figure 14. Neighborhood graph H of $C_{9,3}^6$ with a prime labeling

We split the vertex set $V(H)$ into sets S_1 and S_2 as

$$S_1 = \big\{b_{2i}^q, b_{2l-1}^j, v_1, v_2, v_m, v_{m+1}, s_1^1, s_2^1, s_2^2, s_1^{m-1}, s_1^m, s_2^m : 1 \leq i \leq \frac{k-5}{2},$$
$$1 \leq l \leq \frac{k-3}{2}, 1 \leq j \leq m \text{ and } q = 1, m\big\} \cup \{b_2^2\}$$

and

$$S_2 = V(H) - S_1 = \big\{b_{2i}^j, v_r, s_1^2, s_2^{m-1} s_p^q : 1 \leq i \leq \frac{k-5}{2}, 3 \leq r \leq m-1,$$
$$2 \leq j \leq m-1, 3 \leq q \leq m-2 \text{ and } p = 1, 2\big\} - \{b_2^2\}.$$

The reason for this type of division is to see that the subgraph induced by S_2 is of the type $P^+_{n,r}$ (for some n and r) and so assigning labels to the vertices of S_2 shall be easy. We assign the labels to the vertices of S_1 as follows:

$$f(v_m) = 1, \qquad f(b^2_2) = 2,$$

$$f(s^1_2) = 3, \qquad f(v_1) = 4,$$

$$f(b^1_{2i}) = 4 + i, \qquad 1 \le i \le \frac{k-5}{2},$$

$$f(v_2) = \begin{cases} \dfrac{k+5}{2}, & \text{if } \dfrac{k-1}{2} \text{ is even;} \\ \dfrac{k+7}{2}, & \text{if } \dfrac{k-1}{2} \text{ is odd.} \end{cases}$$

$$f(s^2_2) = \begin{cases} \dfrac{k+7}{2}, & \text{if } \dfrac{k-1}{2} \text{ is even;} \\ \dfrac{k+5}{2}, & \text{if } \dfrac{k-1}{2} \text{ is odd.} \end{cases}$$

$$f(s^1_1) = \frac{k+9}{2},$$

$$f(b^j_{2l-1}) = \frac{k+9}{2} + (j-1)\Big(\frac{k-3}{2}\Big) + l, \qquad 1 \le l \le \frac{k-3}{2} \quad 1 \le j \le m,$$

$$f(s^m_2) = f(b^m_{k-4}) + 1,$$

$$f(s^{m-1}_1) = f(s^m_2) + 1,$$

$$f(b^m_{2i}) = f(s^{m-1}_1) + i, \qquad 1 \le i \le \frac{k-5}{2},$$

$$f(v_{m+1}) = f(b^m_{k-5}) + 1,$$

$$f(s^m_1) = f(v_{m+1}) + 1 = \frac{m(k-3)}{2} + k + 6.$$

Clearly, if u and w are any two adjacent vertices from the set S_1 which are different from v_m and b^2_2, then the gcd of their labels is always 1 because they are either consecutive or consecutive odd integers. Also if u or w is v_m then the same is true as $f(v_m) = 1$. Finally, if $u = b^2_2$, then (within the set S_1) it is adjacent to only v_2, which is assigned an odd label whereas $f(b^2_2) = 2$ and so in this way any two arbitrary vertices in S_1 get relatively prime labels and further, these labels come from the set $\Big\{1, 2, \ldots, \frac{m(k-3)}{2} + k + 6\Big\}$. Now the subgraph of H which is induced by the vertices from the set S_2 is of the type $P^+_{n,r}$ (where $n = \frac{m(k-3)}{2} - k + 1$ and $r = m - 3$) with end point vertices as b^2_{k-5} and b^{m-1}_{k-5}. So, we label the vertices of $P^+_{n,r}$ with consecutive integers $\frac{m(k-3)}{2} + k + 7, \frac{m(k-3)}{2} + k + 8, \ldots, m(k-1) + 1$ as per Theorem 3.3 (but in the direction b^{m-1}_{k-5} to b^2_{k-5}) so that $f(b^{m-1}_{k-5}) = \frac{m(k-3)}{2} +$

$k+7$ and $f(b^2_{k-5}) = m(k-1)+1$. Thus, so far, we have shown that any two adjacent vertices within the two subgraphs induced by the set of vertices S_1 and S_2 have relatively prime labels under f and moreover, the labels come from the set $\{1, 2, \ldots, m(k-1)+1\}$. Hence, in order to conclude that f is a prime labeling on H, we just have to verify that if a vertex of S_1 is adjacent to a vertex of S_2 in the graph H, then they have relatively prime labels. But there are only two such pairs of vertices. One is $u_1 = b^2_2, w_1 = b^2_{k-5}$ and the other is $u_2 = v_m, w_2 = b^{m-1}_{k-5}$. But $f(v_m) = 1$, $f(w_2) = \frac{m(k-3)}{2} + k + 7$, $f(b^2_2) = 2$ and $f(b^2_{k-5}) = m(k-1)+1$ is an odd integer (as k is odd) and hence

$$\gcd(f(u_1), f(w_1)) = 1 = \gcd(f(u_2), f(w_2)).$$

So f is a prime labeling on H for $k \geq 9$.
If $k = 7$ and further $m = 3$, then one has to observe that the set S_2 is an empty set now and so, we need to label the vertices of S_1 only. But this is done precisely as in the case $k \geq 9$ and, we get through. Finally, if $k = 7$ and $m > 3$, then the edge set $E(H)$ misses out the edges of the form $b^j_{2i}b^j_{2i+2}$ and consequently, the vertex b^2_2 is adjacent to v_3. As a result, the graph induced by S_2 is of the type $P^+_{n,r}$ (where $n = 2(m-3)$ and $r = m-3$), with end point vertices as v_3 and b^{m-1}_2. Once again, label the vertices of S_1 as before and the vertices of the graph $P^+_{n,r}$ as per Theorem 3.4 with the help of consecutive integers $2m+14, 2m+15, \ldots, 6m, 6m+1$ so that $f(v_3) = 6m+1$ and $f(b^{m-1}_2) = 2m+14$. Using the fact that $\gcd(f(b^2_2), f(v_3)) = \gcd(2, 6m+1) = 1$, it may be verified that f defines a prime labeling on H in this case also and so, we are done. ☐

4. Conclusion

We have shown that C^m_k is neighborhood-prime iff either $k \not\equiv 2 \pmod 4$ or $m \not\equiv 1 \pmod 4$. Further, it is shown that the general snake graphs $C^m_{k,q}$ are neighborhood-prime for $q = 2, 3$. Investigating similar results for higher values of q seems to be very challenging and a good scope for future work in this direction.

References

1. Asplund J., Fox N. B. and Hamm A., *New persectives of neighborhood-prime labelings on graphs*, `arXiv:1804.02473v1`.
2. Bigham A., Donovan E., Pack J., Turley J. and Wiglesworth L., *Prime labelings of snake graphs*, PUMP J. Undergrad. Res. **2** (2019), 131–149.
3. Cloys M. and Fox N. B., *Neighborhood-prime labelings of trees and other classes of graphs*, `arXiv:1801.01802`.
4. Gallian J., *A dynamic survey of graph labeling*, Electron. J. Comb. (2019), `https://doi.org/10.37236/27`.
5. Haxell P., Pikhurko O. and Taraz A., *Primality of trees*, Combinatorics **2** (2011), 481–500.
6. Patel S. K. and Shrimali N. P., *Neighborhood-prime labeling*, Int. J. Math. Soft Computing **5**(2) (2015), 135–143.

7. Patel S. K. and Shrimali N. P., *Neighborhood-prime labeling of some union graphs*, Int. J. Math. Soft Computing **6**(1) (2016), 39–47.

8. Patel S. K. and Shrimali N. P., *Neighborhood-prime labeling of some product graphs*, Algebra Discret. Math. **25**(1) (2018), 118–129.

S. K. Patel, Department of Mathematics, Government Engineering College, Bhuj-370 001, India,
e-mail: skpatel27@yahoo.com

A. N. Kansagara, Department of Mathematics, Gujarat University, Ahmedabad-380 009, India,
e-mail: ankurk@dkvjamnagar.edu.in