NUMERICAL SOLUTION OF A STOCHASTIC CONTROL PROBLEM OF OPTION PRICING FOR A LIQUIDITY SWITCHING MARKET

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Abstract. We consider the problem of European option pricing in a market which experiences instances of liquidity and illiquidity. The investor’s problem of utility maximisation whose solution is characterised by a semilinear coupled Hamilton Jacobi Bellman (HJB) equation is analysed numerically. A discrete maximum principle is derived and conditions on the time step which ensure the preservation of positivity of the solution are deduced. We analyse the numerical scheme and present results on the order of convergence and error in the maximum norm.

1. Introduction

Recent financial crises have shown that assumptions of a frictionless market where investors can trade without restrictions on the amount of stocks and no delays in executing trades and that all stocks are liquid, do not reflect the behaviour of real markets. In this paper, we consider the liquidity crisis which is characterised by low trade volumes of stocks and abrupt and huge changes in prices. We consider the problem of option pricing in the event of that the market experiences liquidity shocks. Under these circumstances, the market is incomplete due to the restrictions on stock holdings which investors can have contrary to the classical Black-Scholes framework.

We assume a regime-switching Markov model of market liquidity with two states; liquid state and the illiquid state. In the liquid state the market is frictionless and investors can trade arbitrary amounts of stocks without affecting stock prices.

In order to price derivatives in an incomplete market, we will use a utility maximisation method which takes into account risk preferences of investors. For this approach, we can get indifference prices depending on the risk appetite of investors. Other approaches involve pricing under equivalent martingale measure...
for which the prices of derivatives are found as expectations, e.g., [1]. In the utility maximisation framework, the prices of derivatives are characterised by the Hamilton-Jacobi-Bellman equations which are usually non-linear parabolic partial differential equations. In regime-switching Markov model, the Hamilton-Jacobi-Bellman equations results in coupled system of non-linear parabolic differential equations.

Our contribution is derivation of a discrete Maximum principle and conditions on the time step so as to ensure the preservation of positivity of solutions. We extend Ludkovski [1] by analysing the numerical solutions.

The paper is organised as follows. The first section introduces the market model and in the second section, we deduce the PDE system whose solutions are the buyer’s indifference prices for the investor’s utility maximisation problem. The prices corresponding to the two optimal value functions satisfy a semi-linear parabolic system. The third section focuses on the numerical method for the parabolic system. Numerical experiments are presented in the next section. Conclusions are given in the fifth section.

2. Market model

The market model we describe here was suggested by [1]. We consider a market with two assets: a cash account with risky free rate $r$ and a risky asset which is driven by a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $W_t$ is a Wiener process defined on a probability space $(\Omega, F, \mathbb{P})$. We consider the Merton problem where an investor can invest a proportion $\pi_t$ of their wealth $X_t$ in the risky and $1 - \pi_t$ in the risk-free asset. The dynamics of the wealth are given by

$$dX_t = \mu \pi_t X_t dt + \sigma \pi_t X_t dW_t.$$

In order to model the Markov modulated market, we assume the market $M_t$ has two states, liquid (0) and illiquid (1), and can be modeled as a Markov chain with state space $E = \{0, 1\}$. A matrix associated with $M_t$ is the generator matrix

$$G = \begin{pmatrix}
-\nu_{01} & \nu_{01} \\
\nu_{10} & -\nu_{10}
\end{pmatrix},$$

where $\nu_{01}$ and $\nu_{10}$ are transition intensities from state 0 to 1 and vice versa, respectively. In the liquid state, the market is frictionless and trading is done continuously. We assume that in the illiquid state, trading of the stock is not allowed, the stock price does not change and so is the wealth, $dS_t = dX_t = 0$. This assumption is in line with closure of the stock exchanges in the event of liquidity crises [2]. We note that other assumptions of an illiquid market are possible, e.g., severe changes in the stock price [2], but in this paper, we assume the stock price is fixed during the crisis.

We consider an extension of the Merton model where the investor maximises their utility not only from wealth of investing in stock but also from holding a
long position on European contingent claims. We consider the case where the investment horizon coincides with the maturity of the claims. For simplicity, we assume the claims have a maturity $T$ and risk-free rate $r = 0$ which imply that

$$dS_t = \sigma S_t d\tilde{W}_t$$

under an equivalent martingale measure $Q$ (see, e.g., [13]).

As we have mentioned, in order to price derivatives, an equivalent martingale measure should be found under with prices of the stock is a martingale. By using Girsanov theorem (see, e.g., [13]) such a measure that $Q$ can be found as shown in [9].

The minimal entropy martingale measure (MEMM), $Q^E$ is such that

$$Q^E := \arg \min H_T(Q|P), \quad H_T(Q|P) := \begin{cases} E^Q \left( \log \frac{dQ}{dP} | G_T \right), & Q \ll P, \\ +\infty, & \text{otherwise} \end{cases}$$

(see, e.g., [7]). Here the minimum taken over the class of equivalent martingale measures, $E^Q$ is the expectation under $Q$ and $G_T = \sigma(S_t, M_t : t \leq T)$ is the sigma algebra generated by the stock price and Markov chain $M_t$.

In order to obtain the risk neutral transition intensities, we introduce the following ODE system $F'(t) = (D - G)F(t)$ with $F(t) \equiv (F_0(t), F_1(t))$ and $F(T) = [1, 1]^T$, where

$$D = \begin{pmatrix} \mu^2/2\sigma^2 & 0 \\ 0 & 0 \end{pmatrix}.$$ 

The solutions $F_0(t), F_1(t)$ are explicitly given by

$$F_0(t) = c_1 e^{-\lambda_1(T-t)} + c_2 e^{-\lambda_2(T-t)},$$

$$F_1(t) = \frac{1}{\nu_{01}} \left\{ c_1(d_0 + \nu_{01} - \lambda_1) e^{-\lambda_1(T-t)} + c_2(d_0 + \nu_{01} - \lambda_2) e^{-\lambda_2(T-t)} \right\},$$

where

$$d_0 := D(1,1) = \frac{\mu^2}{2\sigma^2}, \quad \lambda_{1,2} = \frac{d_0 + \nu_{01} + \nu_{10} \pm \sqrt{(d_0 + \nu_{01} + \nu_{10})^2 - 4d_0\nu_{10}}}{2},$$

$$c_1 = \frac{\lambda_2 - d_0}{\lambda_2 - \lambda_1}, \quad c_2 = \frac{\lambda_1 - d_0}{\lambda_1 - \lambda_2}, \quad t \in [0,T].$$

3. Utility maximisation and indifference pricing

In this section, we solve the problem of option pricing by utility maximisation. Following [1], we assume the investor’s utility is described by an exponential utility function $u(x) = -e^{-\gamma x}$, where $\gamma > 0$ is the coefficient of risk aversion. Similar results on pricing problem can be found using power, logarithmic and other utility functions [13].

The investor seeks to maximise utility of both terminal wealth and option payoff at $T$. Letting $h(S_T)$ denote payoff function, terminal utility is given by
The indifference price is the price that the investor would have the same optimal wealth only in the two states. In this case, which is the classical Merton problem where the investor seeks to maximise the solution of the ODE system

\[
\begin{align*}
U^i_t &+ \sup_{(\pi t) \in A} \mathbb{E}^P_{t,X,S,i} \left[ -e^{-\gamma(X_T + h(S_T))} \right] = 0, \\
V^i_t &+ \nu_0(\bar{U}^i - \bar{U}^0) + \pi \mu X \bar{U}^0_t + \frac{1}{2} \sigma^2 \pi^2 X^2 \bar{U}^0_x + \pi \sigma^2 SX \bar{U}^0_x = 0,
\end{align*}
\]

(see Øksendal [11], [5], Fleming and Soner [6]). Following [1], let the value functions \( \bar{U}^i(t, X, S) \) with \( (t, X, S) \in [0,T] \times R \times R^+ \) for a holder of an option with payoff \( h(S_T) \) at terminal time \( T \) be given by

\[
\bar{U}^i(t, X, S) = -e^{-\gamma X - \gamma R^i(t,S)},
\]

where \( R^i(t,S) \) characterises the price of the options in the two states. Simplifying (5), we deduce that \( R^i(t,S) \) are the unique viscosity solutions of the coupled nonlinear system

\[
\begin{align*}
R^0_t + \frac{1}{2} \sigma^2 S^2 R^0_{SS} - \frac{\mu S}{\gamma} e^{-\gamma(R^i-R^0)} + \frac{(d_0 + \nu_0)}{\gamma} &= 0, \\
R^1_t - \frac{\mu S}{\gamma} e^{-\gamma(R^1-R^0)} + \frac{\nu_0}{\gamma} = 0
\end{align*}
\]

with the terminal conditions \( R^i(T,S) = h(S), \quad i = 0,1 \).

In particular, if \( h(S) \equiv 0 \), then optimal investment problem becomes

\[
\bar{V}^i(t, X) := \sup_{(\tau t) \in A} \mathbb{E}^P_{t,X} \left[ -e^{-\gamma X_T} \right], \quad i = 0,1,
\]

which is the classical Merton problem where the investor seeks to maximise terminal wealth only in the two states. In this case, \( R^i(t,S) \) is independent of the stocks prices, i.e., \( R^i(t,S) = R^i(t) \) and the optimal solution is given by

\[
\bar{V}^i(t, X) = -e^{-\gamma X - \gamma R^i(t)}.
\]

Let \( F_i(t) = -e^{-\gamma R^i(t)} \), then substituting that into (6) indicates that \( F_i(t) \) is the solution of the ODE system \( F'(t) = (D - G)F(t) \). In summary, the values functions \( \bar{U}^i(t, X, S) \) and \( \bar{V}^i(t, X) \) give optimal utility when the investor holds or does not hold options, respectively.

We consider indifference pricing [3], a notion associated with utility pricing. The indifference price is the price that the investor would have the same optimal
utility level between holding and not holding options. The buyer’s indifference prices $p$ (for initial state 0) and $q$ (for initial state 1) are defined by

$$U^0(t, X - p, S) = \tilde{V}^0(t, X), \quad \tilde{U}^1(t, X - q, S) = \tilde{V}^1(t, X).$$

Since $\tilde{V}^i(t, X) = -e^{-\gamma X} F_i(t)$, $\tilde{U}^0(t, X - p, S) = -e^{-(X-p)-\gamma R^0(t,S)}$ and $\tilde{U}^1(t, X - q, S) = -e^{-(X-q)-\gamma R^1(t,S)}$, the definition of the indifference prices implies

$$R^0(t, S) - p(t, S) = -\gamma^{-1} \ln F_0(t), \quad R^1(t, S) - q(t, S) = -\gamma^{-1} \ln F_1(t).$$

Substituting the above equalities into (6), we conclude that the option buyer’s indifference prices $p$ and $q$ satisfy the semilinear elliptic PDE system

$$p_t + \frac{1}{2} \sigma^2 S^2 p_{SS} = \frac{\nu_{10}}{\gamma} F_1 e^{-\gamma(q-p)} + \frac{(d_0 + \nu_{10})}{\gamma} - \frac{1}{\gamma} F_0', \quad q_t = \frac{\nu_{10}}{\gamma} F_0 e^{-\gamma(p-q)} + \frac{\nu_{01}}{\gamma} - \frac{1}{\gamma} F_1' = 0,$$

with the terminal conditions $p(T, S) = h(S) = q(T, S)$, $i = 0, 1$. The most important property of system (9) is the non-negativity of the solution $(p(t, S), q(t, S))$, see [8].

**Theorem 3.1 (Positivity).** Let $h(S)$ be bounded from below (or above) by a constant, i.e., $h(S) \geq h_*$ (resp., $h(S) \leq h^*$) and $(p(t, S), q(t, S))$ be a classical solution of the terminal value problem (9). Then $p(t, S) \geq h_*$ and $q(t, S) \geq h_*$ ($p(t, S) \leq h^*$ and $q(t, S) \leq h^*$).

4. NUMERICAL SOLUTION

In this section, we propose a numerical method to approximate the option buyer’s indifference prices $p$ and $q$. We will solve the system of PDEs using the finite difference method (see, e.g., Smith [15], Strikwerda [14], etc). We refer the reader to Duffy [4] and Tavella and Randall [16] on the use of the method in financial applications. We remark that although we can solve (9), for $p$ and $q$, we will solve (6) instead of (9) since it is simpler. We can then recover $p$ and $q$ through the relationships (8).

By making the substitution $\tau = T - t$, the PDE (6) becomes

$$\begin{cases}
\gamma(R^0 - \frac{1}{2} \sigma^2 S^2 R^0_{SS}) = -\nu_{01} e^{-\gamma R^0} e^\gamma R^0 + d_0 + \nu_{01} \\
\gamma R^1 = -\nu_{10} e^{-\gamma R^0} e^\gamma R^1 + \nu_{10}.
\end{cases}$$

Let $r^0 = \gamma R^0$ and $r^1 = \gamma R^1$, then the system becomes

$$\begin{cases}
r^0_t - \frac{1}{2} \sigma^2 S^2 r^0_{SS} = -\nu_{01} e^{-r^0} e^{\gamma r^0} + d_0 + \nu_{01} \\
r^1 = -\nu_{10} e^{-r^0} e^{\gamma r^1} + \nu_{10}
\end{cases}$$

with initial conditions

$$r^0(0, S) = \gamma h(S) = r^1(0, S), \quad h(S) = \max(S - K, 0).$$
The boundary conditions are
\[ r^i(x, 0) = 0 = r^i(x, T), \quad r^0(x, S) \approx S \approx r^1(x, S) \]
as \( S \to \infty \). The computational domain is such that \((x, S) \in [0, T] \times [0, \infty)\), but for the sake of calculations, we seek a solution on \([0, T] \times [0, S_{\text{max}}]\). The last equation can be written as
\[ (e^{-r^i})_t = \nu_
abla e^{-r^i} - \nu_{10} e^{-r^i}. \]

Let \( e^{-r^i} = w \). We make the following approximation \( V^i_j \approx r^0(jk, ih), W^i_j \approx w(jk, ih) \). Using an implicit scheme, the \( r^i\)-PDE can be discretised as follows:
\[
\frac{W_{i}^{j+1} - W_{i}^{j}}{\nu} = \nu W_{i}^{j+1} - \nu_{10} W_{i}^{j+1} \]
\[
W_{i}^{j+1} = \frac{W_{i}^{j}}{1 + k \nu} + \frac{k \nu_{10}}{1 + k \nu_{10}} e^{-V_{i}^{j+1}}. \]

We also apply an implicit scheme for the \( r^0\)-PDE to get
\[
\frac{V_{i}^{j+1} - V_{i}^{j}}{k} - \frac{1}{2} \sigma^2 S_{i}^{2} \frac{V_{i}^{j+1} - 2V_{i}^{j} + V_{i}^{j-1}}{k^2} = -\nu_{01} W_{i}^{j} + \nu_{01} W_{i}^{j+1} e^{-V_{i}^{j+1}} + d_{0} - \nu_{01} = -\frac{\nu_{01} W_{i}^{j}}{1 + k \nu_{10}} e^{-V_{i}^{j+1}} - \frac{k \nu_{01} \nu_{10}}{1 + k \nu_{10}} + d_{0} + \nu_{01}. \]

Due to the nonlinearity of \( e^{V_{i}^{j+1}} \), we use Taylor’s expansion about \((j, i)\) to linearize, i.e.,
\[ e^{V_{i}^{j+1}} \approx e^{V_{i}^{j}} + e^{V_{i}^{j}} (V_{i}^{j+1} - V_{i}^{j}) \]
as a first order approximation. The \( r^0\)-PDE becomes
\[
V_{i}^{j+1} - V_{i}^{j} = -\frac{1}{2} \sigma^2 S_{i}^{2} \frac{k}{h^2} (V_{i}^{j+1} + 2V_{i}^{j+1} + V_{i}^{j-1}) - k \nu_{01} W_{i}^{j} \frac{V_{i}^{j}}{1 + k \nu_{10}} e^{V_{i}^{j}} (1 + V_{i}^{j+1} - V_{i}^{j}) - \frac{k^2 \nu_{01} \nu_{10}}{1 + k \nu_{10}} + k(d_{0} + \nu_{01}). \]

Let
\[
a_{i} = \frac{1}{2} \sigma^2 S_{i}^{2}, \quad b_{i} = 1 + \sigma^2 S_{i}^{2} + \frac{k \nu_{01} W_{i}^{j}}{1 + k \nu_{10}} e^{V_{i}^{j}} \quad c_{i} = 1 - \frac{1}{2} \sigma^2 S_{i}^{2}, \]
where \( \alpha = k/h^2 \) is the parabolic ratio. The \( r^0\)-PDE can be written as
\[
-c_{i} V_{i}^{j+1} + b_{i} V_{i}^{j+1} - a_{i} V_{i+1}^{j+1} = V_{i}^{j} - \frac{k \nu_{01} W_{i}^{j}}{1 + k \nu_{10}} e^{V_{i}^{j}} (1 - V_{i}^{j}) - \frac{k^2 \nu_{01} \nu_{10}}{1 + k \nu_{10}} + k(d_{0} + \nu_{01}). \]

Define
\[ D^{k,h} = \{ (\tau_{j}, S_{i}) : \tau_{j} = jk, S_{i} = ih, j = 0, 1, \ldots, N, i = 0, 1, \ldots, N_x \}. \]

Let \( B = (a_{1}, a_{2}) \times (b_{1}, b_{2}) \), such that \( B \subset D^{k,h} \) and assume the corners of \( B \) are points of the mesh \( D^{k,h} \), i.e.,
\[ a_1 = m_1 h, a_2 = m_2 h, b_1 = n_1 k, b_2 = n_2 k \]
for some non-negative integers \( m_1, m_2, n_1, n_2 \). Let \( B' \) be the parabolic boundary of \( B \), i.e., \( B' = \overline{B} - B \).

We remark that \( B' \) is used to establish a maximum principle, see, e.g., [10]. The idea being to derive a maximum principle on a sub-domain of \( D^{k,h} \) and then conclude that it holds for the whole domain.

**Lemma 4.1.** If the following time step restriction

\[
(15) \quad k \leq \frac{V_i^j}{\nu_0 W_i^j e^{V_i^j} (1 - V_i^j) - (d_0 + \nu_0) - \nu_0 V_i^j}
\]

holds, then

\[
(16) \quad V_i^j - \frac{k \nu_0 W_i^j}{1 + k \nu_10} e^{V_i^j} (1 - V_i^j) - \frac{k^2 \nu_0 \nu_10}{1 + k \nu_10} + k(d_0 + \nu_0) \geq 0.
\]

**Proof.** The proof follows by noting that if (16) holds, then rearranging gives (15). \( \square \)

**Theorem 4.2** (Discrete Maximum Principle). Suppose \( V_i^j \geq 0 \) for all \((t_j, S_i) \in \partial B' \cap D^{k,h}\) and \( k \) satisfies (15). If

\[
- c_i V_i^{j+1} + b_i V_i^j - a_i V_i^{j+1} - (V_i^j - \frac{k \nu_0 W_i^j}{1 + k \nu_10} e^{V_i^j} (1 - V_i^j) - \frac{k^2 \nu_0 \nu_10}{1 + k \nu_10} + k(d_0 + \nu_0)) \geq 0
\]

for all \((t_j, S_i) \in B \cap D^{k,h}\), then \( V_i^j \geq 0 \) for all \((t_j, S_i) \in \mathcal{B} \cap D^{k,h}\).

**Proof.** Let

\[
V^j = (V_{m_1+1}^j, \ldots, V_{m_2-1}^j)^T
\]

for \( j, n_1 + 1 \leq j \leq n_2 - 1 \). From the hypothesis, i.e., \( V_i^j \geq 0 \) for all \((t_j, S_i) \in \partial B' \cap D^{k,h}\), we deduce that \( V_{m_1}^j \geq 0, V_{m_2}^j \geq 0 \) for \( j \) such that \( n_1 \leq j \leq n_2 \) and we also have \( V^{n_1} \geq 0 \). The proof proceeds by induction on \( j \).

The condition (17) can be written as

\[
AV_i^{j+1} - (V_i^j - \frac{k \nu_0 W_i^j}{1 + k \nu_10} e^{V_i^j} (1 - V_i^j) - \frac{k^2 \nu_0 \nu_10}{1 + k \nu_10} + k(d_0 + \nu_0)) - b_i^{j+1} \geq 0,
\]

where \( A \) is the tridiagonal matrix and

\[
b_i^{j+1} = (c_{m_1+1} V_{m_1+1}^j, 0, \ldots, 0, a_{m_2+1} V_{m_2+1}^j)^T.
\]

Taking \( V_i^j \geq 0 \) as the inductive hypothesis, it follows that

\[
AV_i^{j+1} \geq (V_i^j - \frac{k \nu_0 W_i^j}{1 + k \nu_10} e^{V_i^j} (1 - V_i^j) - \frac{k^2 \nu_0 \nu_10}{1 + k \nu_10} + k(d_0 + \nu_0)) + b_i^{j+1} \geq 0,
\]

since \( b_i^{j+1} \geq 0 \) and \( k \) satisfies (15).

\( A \) is an \( M \)-matrix and reducibly strictly diagonally dominant so \( A^{-1} \) exists and \( A^{-1} > 0 \), see Varga [17]. Therefore,

\[
V_i^{j+1} \geq A^{-1} (V_i^j - \frac{k \nu_0 W_i^j}{1 + k \nu_10} e^{V_i^j} (1 - V_i^j) - \frac{k^2 \nu_0 \nu_10}{1 + k \nu_10} + k(d_0 + \nu_0) + b_i^{j+1}) \geq 0.
\]
This completes the induction step.

5. Numerical results

In this section, we present some results based on the numerical scheme that we developed. The error is computed using the uniform norm as

$$E^{N_x,V} = \|V^*(\tau, S) - V^{N_x}(\tau, S)\|_\infty,$$

where $V^*(\tau, S)$ is the solution of the finest mesh with $N_x = N^*_x = 2048$. Table 1 shows the error values for a market characterised by the following parameters [1]: $\mu = 0.06, \sigma = 0.3, \nu_{01} = 1, \nu_{10} = 12, K = 10, T = 1$ and $\gamma = 1$. The order of convergence is calculated using

$$D^{N_t,V} = |V^{N_t}(T, S_0) - V^{2N_t}(T, S_0)|, \quad \rho^{N_t,V} = \log_2 \frac{D^{N_t,V}}{D^{2N_t,V}}$$

where $V^{N_t}(T, S_0)$ is the value of the option for a given spot price $S_0$ (see, e.g., [12]). Table 2 shows that the scheme is approximately of first order.

Table 1. Maximum error $E^{N_x,V}$ and $E^{N_x,W}$ for discrete solution.

<table>
<thead>
<tr>
<th>$N_x$</th>
<th>$E^{N_x,V}$</th>
<th>$E^{N_x,W}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>1.84-02</td>
<td>2.94-02</td>
</tr>
<tr>
<td>64</td>
<td>1.45-02</td>
<td>1.62-02</td>
</tr>
<tr>
<td>128</td>
<td>1.06-02</td>
<td>8.30-03</td>
</tr>
<tr>
<td>256</td>
<td>7.20-03</td>
<td>4.00-03</td>
</tr>
<tr>
<td>512</td>
<td>4.30-03</td>
<td>1.70-03</td>
</tr>
<tr>
<td>1024</td>
<td>2.00-03</td>
<td>5.80-04</td>
</tr>
</tbody>
</table>

Table 2. Order of convergence for at the money ($S_0 = 2, K = 2, S_{\text{min}} = 0, S_{\text{max}} = 5, k = h/2$).

<table>
<thead>
<tr>
<th>$N_t$</th>
<th>$V$ Value</th>
<th>Difference $\rho^{N_t,V}$</th>
<th>$W$ Value</th>
<th>Difference $\rho^{N_t,W}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>0.246669</td>
<td>0.235165</td>
<td>7.70e-04</td>
<td>7.52e-04</td>
</tr>
<tr>
<td>60</td>
<td>0.247438</td>
<td>3.11e-04</td>
<td>2.48 (1.31)</td>
<td>3.01e-04</td>
</tr>
<tr>
<td>120</td>
<td>0.247749</td>
<td>2.14e-04</td>
<td>2.25 (1.17)</td>
<td>1.31e-04</td>
</tr>
<tr>
<td>240</td>
<td>0.247887</td>
<td>6.50e-05</td>
<td>2.12 (1.09)</td>
<td>1.30e-04</td>
</tr>
<tr>
<td>480</td>
<td>0.247952</td>
<td>3.10e-05</td>
<td>2.10 (1.07)</td>
<td>2.10 e-05</td>
</tr>
<tr>
<td>960</td>
<td>0.247983</td>
<td>2.90 e-05</td>
<td>2.10 (1.07)</td>
<td></td>
</tr>
</tbody>
</table>

Table 3 shows the indifference buyer’s price for different number of options that the investor has in his portfolio. Positive numbers show the buyer’s price and negative ones the writer’s prices and these are given by $p_t$. The indifference price decreases with number of contracts due to increased risk of having more options. As expected, option prices increase with spot price due to moneyness. As also expected, Figure 1 shows that the option value increases with time to maturity.
Table 3. Indifference price per option for buyer and seller.

<table>
<thead>
<tr>
<th>Number of options</th>
<th>$S_0=8$</th>
<th>$S_0=10$</th>
<th>$S_0=12$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>0.3006</td>
<td>1.1011</td>
<td>2.4682</td>
</tr>
<tr>
<td>4</td>
<td>0.3161</td>
<td>1.1326</td>
<td>2.4922</td>
</tr>
<tr>
<td>1</td>
<td>0.3226</td>
<td>1.1447</td>
<td>2.5021</td>
</tr>
<tr>
<td>$-1$</td>
<td>0.3256</td>
<td>1.1502</td>
<td>2.5066</td>
</tr>
<tr>
<td>$-4$</td>
<td>0.3290</td>
<td>1.1562</td>
<td>2.5117</td>
</tr>
<tr>
<td>$-8$</td>
<td>0.3323</td>
<td>1.1619</td>
<td>2.5166</td>
</tr>
</tbody>
</table>

Figure 1. Comparing European option values at issue and maturity in the liquid and illiquid states.

(a) $r^0$ at $t = 0$ and $t = T$

(b) $r^1$ at $t = 0$ and $t = T$

Figure 2. Solution surfaces.
We considered the option pricing problem in a market that is liquid or illiquid at any given time. We solved an investor’s utility maximisation problem by deriving a system of semilinear parabolic equations. Under certain conditions, we derived a numerical solution for the indifference prices which preserves positivity and obeys a discrete maximum principle. Numerical experiments show that the method is nearly first order accurate. The results also showed a trade-off between size of contract and the price of option which can be explained by the risk involved.

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