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MATHEMATICAL FORMALIZATION OF MACROECONOMIC STABILIZATION POLICY IN A HIGH-DIMENSIONAL DYNAMIC KEYNESIAN MODEL WITH PUBLIC DEBT ACCUMULATION

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ABSTRACT. In this paper, we mathematically study the effect of macroeconomic stabilization policy in a high-dimensional dynamic Keynesian model with public debt accumulation. The reduced form of our model is described by a five-dimensional system of nonlinear differential equations. The dynamic effect of the fiscal and monetary policy mix on the macroeconomic stability, instability and cyclical fluctuations are studied both analytically and numerically.

1. INTRODUCTION

In this paper, we study the impact of the fiscal and monetary policy mix on macroeconomic stability by using a variant of the 'high-dimensional dynamic Keynesian model' that was developed by Asada, Chiarella, Flaschel and Franke (2003, 2010).[1] The reduced form of our model consists of a five-dimensional system of nonlinear differential equations. We study the dynamic effect of fiscal and monetary policy mix on macroeconomic stability, instability and cyclical fluctuations both analytically and numerically.

In Section 2, we present the outline of the model that is based on Asada and Ouchi (2013), and describe a reduced form of the system that consists of the five-dimensional system of nonlinear differential equations. In Section 3, we describe the characteristics of the long run equilibrium solution. Section 4 is devoted to the local stability/instability analysis of the long run equilibrium point and the analysis of cyclical fluctuations around the long run equilibrium point. Section 5 provides some numerical simulations that supplement our analytical results. In Section 6, we summarize the main conclusion of this paper. Some complicated calculations and proofs are contained in appendices.

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[1]'High-dimensional' dynamic model simply implies the dynamic model with many (in fact, at least three) variables. 'Dimension' of the system is equal to the number of variables.

2. A System of Fundamental Dynamic Equations

In this paper, we utilize the model that was formulated by Asada and Ouchi (2013). The detailed exposition of the full system of equations is contained in Asada and Ouchi (2013), so that in this paper, we explicitly explain only 8 key equations among 24 equations including some definitional equations.

This model consists of the following system of equations: (a) The equilibrium condition for the goods market (IS equation). (b) The standard Keynesian consumption function. (c) The income tax function. (d) The standard Keynesian investment function, which implies that the private investment rate (i) is a decreasing function of the 'expected real interest rate' ($r - \pi^e$). (e) The equilibrium condition for the money market (LM equation). (f) The budget constraint of the 'consolidated government' that includes the central bank (equation (1) below). (g) A standard version of the 'expectations-augmented wage Phillips curve' (equation (2) below). (h) The dynamic of the rate of employment (equation (3) below). (i) The capital accumulation equation (equation (4) below). (j) A version of Kaldor's (1957) technical progress function, which implies that the rate of technical progress ($\dot{a}/a$) is positively correlated with the rate of capital accumulation ($\dot{K}/K$) (equation (5) below). Following three relationships (k), (l) and (m) in particular are important in this model on fiscal and monetary stabilization policies.

(k) A formalization of the fiscal policy rule that considers both employment and balance of public debt (equation (6) below), where $\bar{e}$ is the 'natural' rate of employment that is consistent with equation (2), and $\bar{b}$ is the target value of the public debt-capital ratio that is set by the government. The parameter θ in equation (6) is the weight of the importance of the employment consideration compared with the debt consideration that is determined by the government, and the parameter α is the measure of the strength of the fiscal policy response to the employment and/or the public debt.

(l) A formalization of the monetary policy rule in spirit of the 'Taylor rule' due to Taylor (1993) (equation (7) below). In this formulation, the 'nonnegative' constraint, which means that the nominal interest rate (r) cannot become negative, is considered. The parameter $\overline{\pi}$ is the target rate of inflation that is set by the central bank. This monetary policy rule is the interest rate rule that is called the 'flexible inflation targeting', which is the mixture of the inflation targeting and the employment targeting. The parameters β_1 and β_2 are the measures of the strength of the central bank's monetary policy responses to the inflation and the employment.

(m) A formalization of the inflation expectation formation by the public (equation (8) below). This is a mixture of the 'forward looking' and the 'backward looking' (or 'adaptive') expectations. The parameter ξ in equation (8) is the weight of the 'forward looking' expectation formation, which can be interpreted as the measure of the 'credibility' of the central bank's inflation targeting. We can consider that the higher the parameter value ξ, the more credible the central bank's announcement concerning the inflation targeting will be. The parameter γ

is the measure of the speed of response to the inflation expectations.

$$(1) \qquad T + \dot{B}/p + \dot{H}/p = G + rB/p$$

$$(2) \qquad \dot{w}/w = \kappa(e - \overline{e}) + \dot{a}/a + \pi^e, \qquad \kappa > 0,\ 0 < e = N/N^s \leqq 1$$

$$(3) \qquad \dot{e}/e = \dot{y}/y + \dot{K}/K - \dot{a}/a - n_s, \qquad n_s = \dot{N}^s/N^s$$

$$(4) \qquad \dot{K} = I + \sigma G, \qquad 0 < \sigma < 1$$

$$(5) \qquad \dot{a}/a = \varepsilon(\dot{K}/K) + \varepsilon_0, \qquad 0 < \varepsilon < 1,\ \varepsilon_0 > 0$$

$$(6) \qquad \dot{g} = \alpha\{\theta(\overline{e} - e) + (1-\theta)(\overline{b} - b)\}, \qquad \alpha > 0,\ 0 < \theta < 1,\ \overline{b} > 0$$

$$(7) \qquad \dot{r} = \begin{cases} \beta_1(\pi - \overline{\pi}) + \beta_2(e - \overline{e}) & \text{if } r > 0,\ \beta_1 > 0,\ \beta_2 > 0 \\ \max[0, \beta_1(\pi - \overline{\pi}) + \beta_2(e - \overline{e})] & \text{if } r = 0,\ \beta_1 > 0,\ \beta_2 > 0 \end{cases}$$

$$(8) \qquad \dot{\pi}^e = \gamma[\xi(\overline{\pi} - \pi^e) + (1-\xi)(\pi - \pi^e)], \qquad \gamma > 0,\ 0 \leqq \xi \leqq 1,\ \overline{\pi} > 0$$

The meanings of the symbols of the endogenous variables in this model are as follows. Y =real national income (real output). C =real private consumption expenditure. I = real private investment expenditure. G =real government expenditure. K = real capital stock. $y = Y/K$ = output-capital ratio, which is proportional to the rate of capacity utilization of capital stock. $c = C/K$ = private consumption-capital ratio. $i = I/K$ = private investment-capital ratio (rate of private investment). $g = G/K$ = government expenditure-capital ratio (rate of government expenditure). T = real income tax. B = nominal public debt. $t = T/K$ = income tax-capital ratio. $b = B/pK$ = public debt-capital ratio. p = price level. r = nominal interest rate of public debt. H = nominal high-powered money (nominal base money). $h = H/pK$ = high-powered money-capital ratio. w = nominal wage rate.$\pi = \dot{p}/p = \dot{w}/w - \dot{a}/a$ = rate of price inflation. π^e = expected rate of price inflation. $r - \pi^e$ = expected real interest rate of public debt, e = rate of employment = 1 − rate of unemployment. N =labor employment. N^s =labor supply, $a = Y/N$ =average labor productivity. Asada and Ouchi (2013) showed that the above system can be transformed into the following five-dimensional system of nonlinear differential equations, which is called 'a system of fundamental dynamic equations' of our model.

$$(9) \quad \begin{aligned}
&\text{(i)} \quad \dot{g} = F_1(b, e) = \alpha\{\theta(\overline{e} - e) + (1-\theta)(\overline{b} - b)\} \\
&\text{(ii)} \quad \dot{r} = F_2(\pi^e, e) = \begin{cases} (\kappa\beta_1 + \beta_2)(e - \overline{e}) + \beta_1(\pi^e - \overline{\pi}) & \text{if } r > 0 \\ \max[0, (\kappa\beta_1 + \beta_2)(e - \overline{e}) + \beta_1(\pi^e - \overline{\pi})] & \text{if } r = 0 \end{cases} \\
&\text{(iii)} \quad \dot{\pi}^e = F_3(\pi^e, e) = \gamma[\xi(\overline{\pi} - \pi^e) + (1-\xi)\kappa(e - \overline{e})] \\
&\text{(iv)} \quad \dot{b} = F_4(g, r, \pi^e, b, e) = [g - \tau y(r, \pi^e, b, g) + t_0 + \{r(1-\tau) - \kappa(e - \overline{e}) \\
&\qquad -\pi^e - i(r - \pi^e) - \sigma g\}b - \{\pi + i(r - \pi^e) + \sigma g\}h(r, \pi^e, b, g) \\
&\qquad -\{\psi'(r)y(r, \pi^e, b, g) + \psi(r)y_r\}F_2(\pi^e, e) \\
&\qquad -\psi(r)\{y_{\pi^e}F_3(\pi^e, e) + y_g F_1(b, e)\}]/\{1 + \psi(r)y_b\} \\
&\text{(v)} \quad \dot{e} = F_5(g, r, \pi^e, b, e) = e[\{y_r F_2(\pi^e, e) + y_{\pi^e}F_3(\pi^e, e) \\
&\qquad +y_b F_4(g, r, \pi^e, b, e) + y_g F_1(b, e)\}/y(r, \pi^e, b, g) \\
&\qquad +(1-\varepsilon)\{i(r - \pi^e) + \sigma g\} - (\varepsilon_0 + n_s)].
\end{aligned}$$

where

$$(10)\quad \begin{aligned} &y(r,\pi^e,b,g) = \frac{1}{1-\delta(1-\tau)}\{\delta(1-\tau)rb + \delta t_0 + c_0 + i(r-\pi^e) + g\},\\ &\qquad 0<\delta<1 \quad 0<\tau<1,\\ &y_r = \partial y/\partial r = \delta(1-\tau)b + \underset{(-)}{i_{r-\pi^e}}\}/\{1-\delta(1-\tau)\},\\ &y_{\pi^e} = \partial y/\partial \pi^e = -\underset{(-)}{i_{r-\pi^e}}/\{1-\delta(1-\tau)\} > 0,\\ &y_b = \partial y/\partial b = \delta(1-\tau)r/\{1-\delta(1-\tau)\} \geqq 0,\\ &y_g = \partial y/\partial g = 1/\{1-\delta(1-\tau)\} > 1 > 0, \end{aligned}$$

$$(11)\quad h(r,\pi^e,b,g) = \psi(r)y(r,\pi^e,b,g), \quad \psi(r) = \phi(r)/m(r), \quad \psi'(r) < 0.$$

The parameters δ and τ in Eq. (10) are the marginal propensity to consume of the economic agents and the marginal income tax rate, respectively.

3. Characteristics of the long run equilibrium solution

In this section, we consider the 'long run equilibrium' solution of the system (9) that satisfies

$$(12)\qquad \dot{g} = \dot{r} = \dot{\pi}^e = \dot{b} = \dot{e} = 0, e = \overline{e}.$$

Substituting these conditions into Eq. (9), we have the following set of conditions for the long run equilibrium values $(g^*, r^*, \pi^{e*}, b^*, e^*)$:

$$(13)\quad \begin{aligned} &\text{(i)} \quad e^* = \overline{e}, \quad b^* = \overline{b}, \quad \pi^{e*} = \pi^* = \overline{\pi}\\ &\text{(ii)} \quad i(r^*-\overline{\pi}) = \frac{\varepsilon_0+n_s}{1-\varepsilon} - \sigma g^*\\ &\text{(iii)} \quad g^* - \tau y(r^*,\overline{\pi},\overline{b},g^*) + t_0 + \Big\{r^*(1-\tau) - \overline{\pi} - \frac{\varepsilon_0+n_s}{1-\varepsilon}\Big\}\overline{b}\\ &\qquad -\Big\{\overline{\pi} + \frac{\varepsilon_0+n_s}{1-\varepsilon}\Big\}h(r^*,\overline{\pi},\overline{b},g^*) = 0 \end{aligned}$$

In this paper, we assume that this system of equations has the unique economically meaningful solution $(g^*, r^*) > (0,0)$.[2] Incidentally, we have $(\dot{K}/K)^* = \frac{\varepsilon_0+n_s}{1-\varepsilon} > 0$, $(\dot{a}/a)^* = \frac{\varepsilon_0+\varepsilon n_s}{1-\varepsilon} > 0$, and $n^* = n_s + (\dot{a}/a)^* = (\dot{K}/K)^* > 0$ at the long run equilibrium point. This means that the equilibrium rate of capital accumulation $(\dot{K}/K)^*$, the equilibrium rate of technical progress $(\dot{a}/a)^*$, and the equilibrium 'natural' growth rate $n^* = n_s + (\dot{a}/a)^*$ are determined endogenously in this model.

4. Analysis of local stability/instability and cyclical fluctuations

Next, we study the local stability/instability of the long run equilibrium point. For this purpose, let us consider the following (5×5) Jacobian matrix of the system (9) that is evaluated *at the equilibrium point*.

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[2]In Section 5, we provide a numerical example in which such an equilibrium point exists.

$$J = \begin{bmatrix} 0 & 0 & 0 & -\alpha(1-\theta) & -\alpha\theta \\ 0 & 0 & \beta_1 & 0 & \kappa\beta_1 + \beta_2 \\ 0 & 0 & -\gamma\xi & 0 & \gamma(1-\xi)\kappa \\ F_{41} & F_{42} & F_{43} & F_{44} & F_{45} \\ F_{51} & F_{52} & F_{53} & F_{54} & F_{55} \end{bmatrix} \tag{14}$$

The detailed expressions of the partial derivatives F_{ij} are given in *Appendix A*.

The characteristic equation of this system becomes

$$\Gamma(\lambda) = |\lambda I - J| = \lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5 = 0, \tag{15}$$

where

$$a_1 = -\mathrm{trace}\, J = \gamma\xi - F_{44} - F_{55}, \tag{16}$$

$$a_j = (-1)^j \quad \text{(sum of all principal j-th order minors of } J) \quad (j = 2, 3, 4), \tag{17}$$

$$a_5 = -\det J. \tag{18}$$

It is worth noting that the Liénard-Chipart expression of the Routh-Hurwitz conditions for stable roots implies that a set of necessary (but not sufficient) conditions for the local stability of the equilibrium point of the system (9) is expressed by

$$a_j > 0 \qquad \text{for all } j \in \{1, 2, \cdots, 5\}. \tag{19}$$[3]

This means that the equilibrium point of this system is locally unstable if we have $a_j < 0$ for at least one of $j \in \{1, 2, \cdots, 5\}$. The following proposition follows from this fact.

Proposition 1 (Instability Proposition)**.** *Suppose that the following set of conditions is satisfied:*

(1) *Fiscal policy parameter θ is close to 0.*

(2) *Fiscal policy parameter α is sufficiently large.*

(3) *Monetary policy parameters β_1 and β_2 are close to 0.*

(4) *Credibility parameter of the central bank's inflation targeting ξ is close to 0.*

Then, the equilibrium point of the system (9) *is locally unstable.*

Proof. See *Appendix B*. □

At this stage of the analysis, let us introduce the following assumption.[4]

Assumption 1.

(i) $\left| i^*_{r-\pi^e} \right| > \delta(1-\tau)\bar{b}$.

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[3]See Gandolfo (2009) Chap. 16. Unlike the so-called 'New Keynesian' dynamic model that is represented by Woodford (2003), all initial conditions of the endogenous variables are assumed to be fixed in this model. This means that (1) the equilibrium point is considered to be locally stable only if all characteristic roots have negative real parts, and (2) it is considered to be locally unstable if at least one characteristic root has positive real part in our model.

[4]As already noted in *Appendix A*, asterisk($*$) means that the values are evaluated *at the equilibrium point.*

(ii) $\dfrac{(1-\tau)(1-\delta)}{1-\delta(1-\tau)} > \sigma(\overline{b}+h^*)$.

(iii) $\tau \underset{(+)}{y_b^*} + \left\{\overline{\pi} + \dfrac{\varepsilon_0 + n_s}{1-\varepsilon}\right\}\{1+\psi(r^*)\underset{(+)}{y_b^*}\} > r^*(1-\tau)$.

Assumption 1(i) means that $y_r^* = (\partial y/\partial r)^* < 0$, and in this case, we have $F_{42} = (\partial \dot{b}/\partial r)^* > 0$ (see equations (10) and (A2)). *Assumption 1*(ii) means that $F_{41} = (\partial \dot{b}/\partial g)^* > 0$ (see Eq. (A1)). On the other hand, under *Assumption 1*(iii) we have $F_{44} = (\partial \dot{b}/\partial b)^* < 0$ *if the fiscal policy parameter* θ *is close to* 1.

Assumption 1(i) implies the inequality $F_{42} > 0$, but it does not determine the sign of F_{52}. In this paper, however, we *assume* that the inequality $F_{52} = (\partial \dot{e}/\partial r)^* < 0$ is in fact satisfied. This means that we posit the following additional assumption (see Eq. (A7) in Appendix A).

Assumption 2.

$$\underset{(+)}{y_b^*}\ \underset{(+)}{F_{42}} < (1-\varepsilon)\left|i^*_{r-\pi^e}\right| y^*.$$

It is easy to see that F_{55} becomes a linear decreasing function of the parameters β_1 and β_2 if the value of y_b is sufficiently small.

Assumption 3. The value of y_b is so small that we have $\partial F_{55}/\partial \beta_1 < 0$ and $\partial F_{55}/\partial \beta_2 < 0$.

Proposition 2 (Stability Proposition)**.** *In addition to Assumptions 1, 2 and 3, let us assume that the following set of conditions is satisfied:*
(1) *Fiscal policy parameter θ is less than* 1*, but it is close to* 1.
(2) *Fiscal policy parameter α is fixed at any positive value.*
(3) *Either of the monetary policy parameters β_1 or β_2 is sufficiently large.*
(4) *Credibility parameter of the central bank's inflation targeting ξ is close to* 1 *(including the case of $\xi = 1$).*
Then, the equilibrium point of the system (9) *is locally asymptotically stable.*

Proof. See Appendix C. □

Proposition 1 characterizes the *inappropriate* policy mix of fiscal and monetary policies, and Proposition 2 characterizes the *appropriate* policy mix. These propositions say that the equilibrium point of the system tends to be locally unstable if all four parameters θ, β_1, β_2, and ξ are sufficiently small, and it tends to be locally stable if all of them are sufficiently large under some additional conditions. Needless to say, the stabilization by the simultaneous increase of these four parameters does not necessarily mean that an increase of a single parameter has a similar effect.[5] Nevertheless, in some cases, it may be possible that the increase of one of such parameters changes the unstable system into the stable system. In this case, there exists at least one 'bifurcation point' of such a parameter value, at which the switching from the unstable region to the stable region occurs. At

[5]This fact was pointed out by the editor of this journal.

such a bifurcation point, the characteristic equation (15) must have at least one root with zero real part.

Incidentally, we can prove that the coefficient a_5 defined by Eq. (18) is always positive as long as $0 \leqq \theta < 1$, $\beta_1 > 0$, $\beta_2 > 0$, and $0 \leqq \xi \leqq 1$ (for the proof, see *Appendix D*). Therefore, we have

$$\Gamma(0) = a_5 > 0, \tag{20}$$

which means that the characteristic equation (15) does *not* have the real root such as $\lambda = 0$. It follows from this fact that the characteristic equation (15) has a pair of pure imaginary roots at the 'bifurcation point'. In this case, the *cyclical fluctuations* around the equilibrium point occur at some range of parameter values that are near the bifurcation point, because of the existence of a pair of complex roots. Obviously, in case of the subcritical bifurcation the bifurcating cycle is unstable, and in this case, the permanent cyclical fluctuation will not appear so that the cyclical fluctuations will be 'invisible' in reality.[6] In the next section, however, we provide a numerical example in which the cyclical fluctuation is 'visible'.

5. Numerical simulations

Next, we provide some numerical simulations that support the analytical results of the previous sections. Let us assume the following parameter values, functional forms and initial values.[7]

(1) Fixed parameters
$\delta = 0.6$, $c_0 = 0.08$, $\tau = 0.2$, $t_0 = 0.5$, $\alpha = 0.95$, $\kappa = 0.8$, $\sigma = 0.6$, $\gamma = 0.9$, $\varepsilon = 0.3$, $n_s = 1\%$ (annual growth rate), $\overline{e} = 0.95$, $\overline{b} = 0.1$, $\overline{\pi} = 0.2\%$ per period.

(2) Functional forms
Consumption function: $c = \delta(y + rb - t) + c_0 = 0.6(y + rb - t) + 0.08$,
Income tax function: $t = \tau(y + rb) - t_0 = 0.2(y + rb) - 0.5$,
Investment function: $i = -a_1(r - \pi^e) + i_0 = -0.8(r - \pi^e) + 1.5$,
LM equation: $h = \psi(r)y = \dfrac{a_2}{a_3 r + a_4} y = \dfrac{0.05}{0.8r + 1} y$,
Technical progress function: $\dot{a}/a = \varepsilon(\dot{K}/K) + \varepsilon_0 = 0.3(\dot{K}/K) + 0.5$.

(3) Initial values
$g(0) \approx 1.51$, $r(0) \approx 0.541\%$ per period, $\pi^e(0) \approx 0.15\%$ per period, $b(0) = 0.15$, $e(0) = 0.9$.

[6]This fact was also pointed out by the editor of this journal.

[7]In this simulation, we introduce the exogenous constraint $0.4 \leqq e \leqq 1$, and we assume that the unit time period is 0.1 year. In other words, $t = 100$ means 10 years. The values π, π^e, and r denote per cent per unit time period. For example, $\overline{\pi} = 0.2$ and $r^* = 0.454$ mean that the target rate of inflation is 2% per year and the equilibrium nominal interest rate is 4.54% per year. We conducted numerical simulations by means of the software *Mathematica* by using the fully nonlinear system without linear approximation.

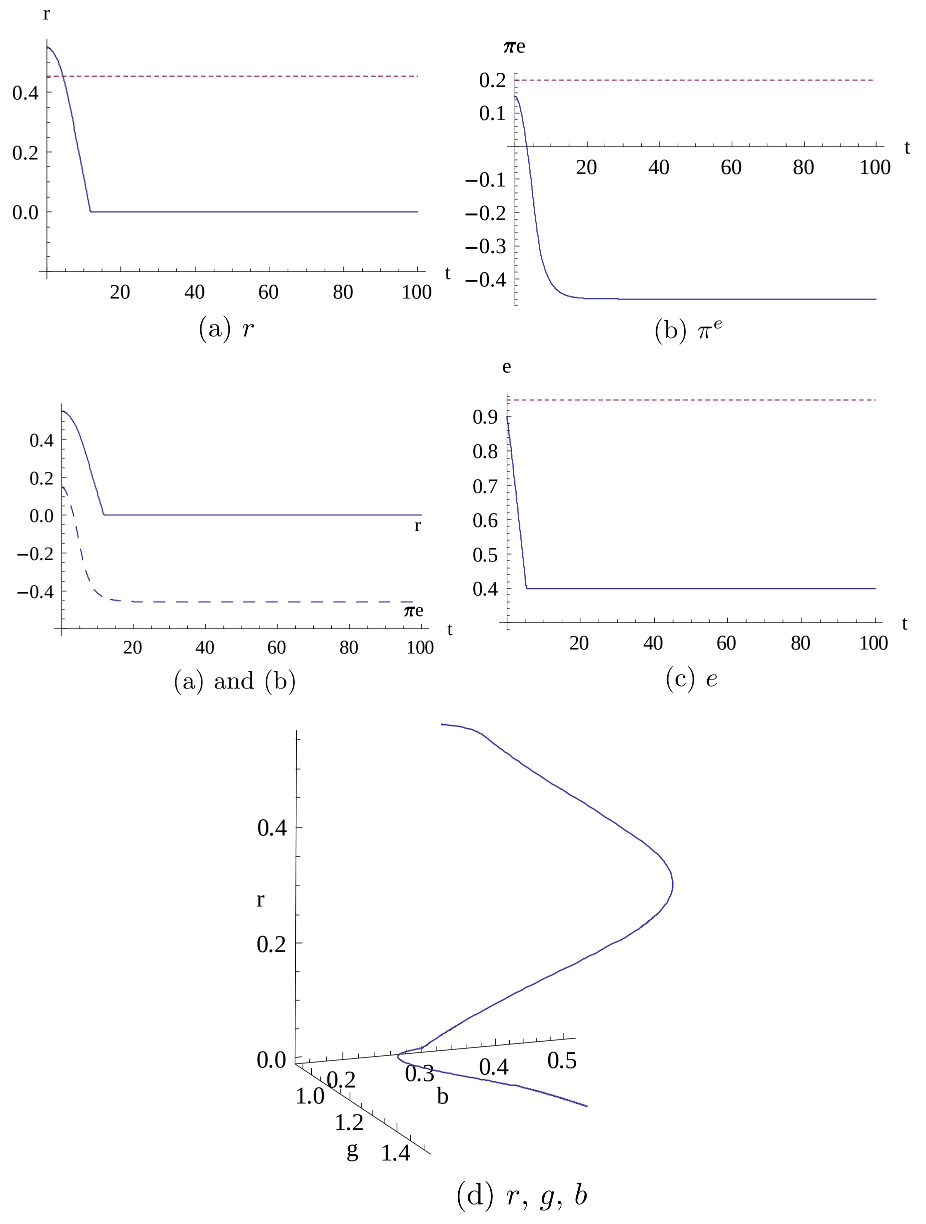


Figure 1. Case A: Unstable Case (ξ =0.4, θ =0.3, β_1 =0.04, β_2 =0.04).

We have the equilibrium values $g^* \approx 0.41$ and $r^* \approx 0.454\%$ per period. Furthermore, we select four parameters ξ, θ, β_1 and β_2 as bifurcation parameters.

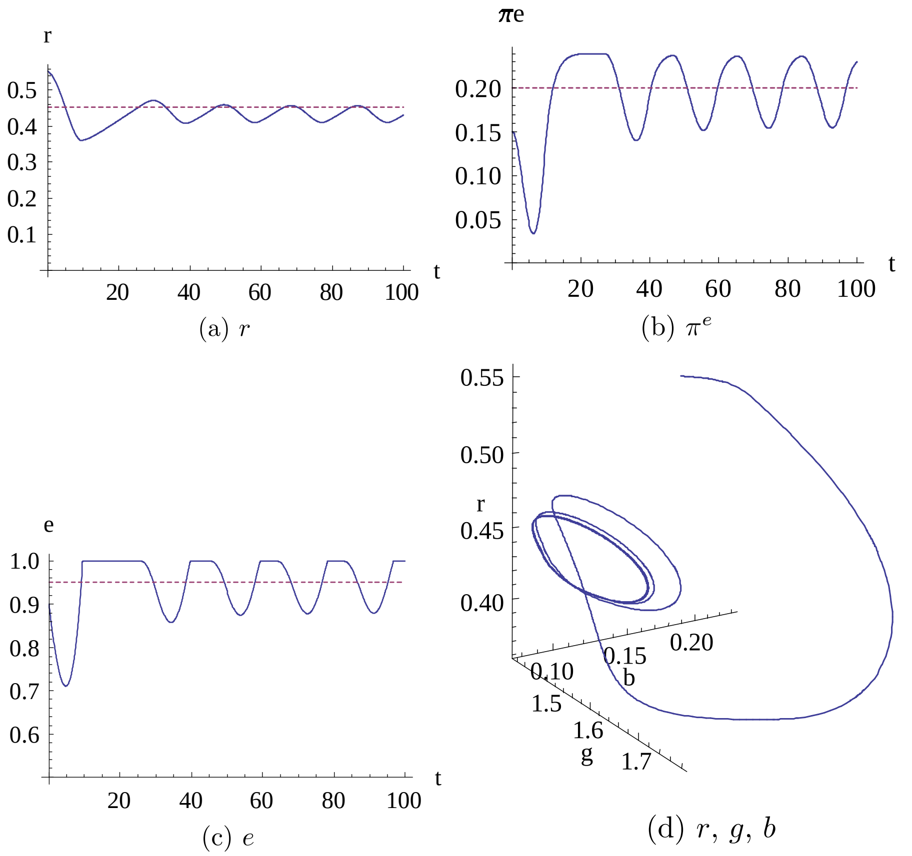
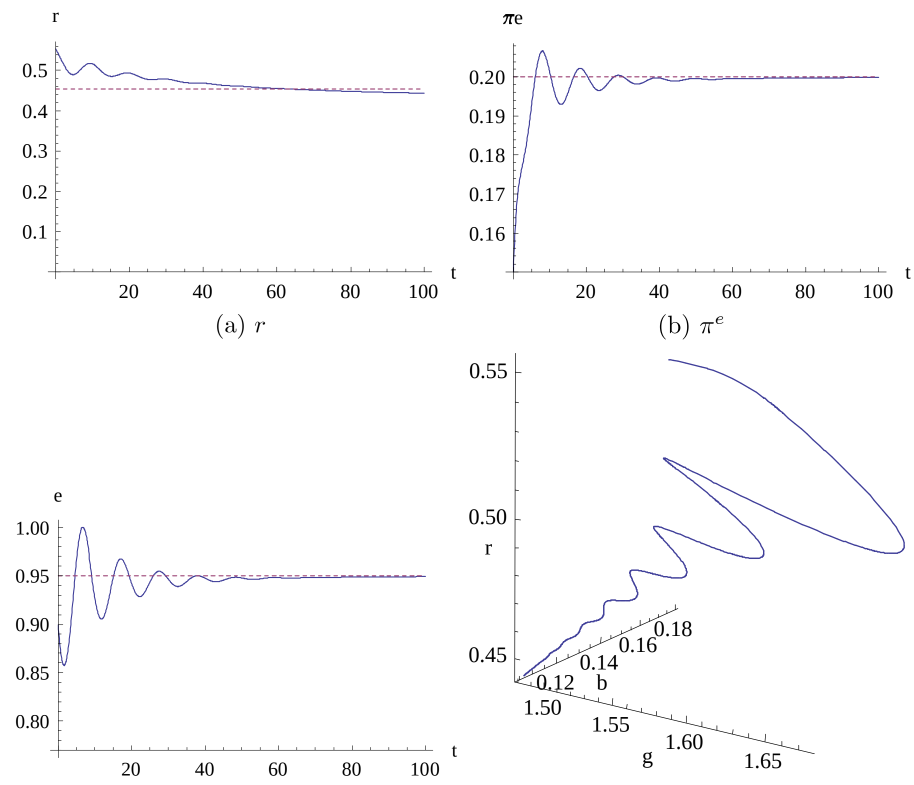


Figure 2. Case B: Case of Limit Cycles (ξ =0.5, θ =0.5, β_1 =0.05, β_2 =0.05).

Figure 1 (Case A) describes the unstable case in which the parameter values ξ, θ, β_1 and β_2 are sufficiently small ($\xi = 0.4$, $\theta = 0.3$, $\beta_1 = \beta_2 = 0.04$). In this case, the equilibrium point becomes strongly unstable and the 'deflationary depression with the liquidity trap' emerges. It is worth noting that the expected real interest rate ($r - \pi^e$) becomes considerably high because of the deflation even if the nominal interest rate (r) is fallen to its lower bound, and the public debt-capital ratio (b) continues to rise in the process of the deflationary depression. In this case, the nominal interest rate is forced to fall to its lower bound not because of the active monetary policy, but because of the inactive monetary policy of the central bank.

Figure 2 (Case B) describes the case of limit cycles in which the above parameter values are intermediate values ($\xi = \theta = 0.5$, $\beta_1 = \beta_2 = 0.05$). Figure 3 (Case C) is the stable case in which the above parameter values are sufficiently large ($\xi = \theta = 0.8$, $\beta_1 = \beta_2 = 0.1$). These numerical examples support our analytical results.



(a) r (b) π^e (c) e (d) r, g, b

Figure 3. Case C: Stable Case (ξ =0.8, θ =0.8, β_1 =0.1, β_2 =0.1).

6. Concluding remarks

Now, we shall summarize the main conclusion of this paper. Proposition 1 characterizes a typical inappropriate policy mix. This proposition says that the macroeconomic system tends to be dynamically unstable if (1) the government expenditure responds sensitively to the changes of public debt rather than the changes of employment, (2) the central bank's monetary policy is relatively inactive, and (3) the central bank's inflation targeting is relatively incredible so that the public form the inflation expectation rather adaptively (in a backward-looking way). Surprisingly enough, however, this inappropriate policy mix is often adopted by some policy makers especially in the period of 'deflationary depression' in Japan during such a long period of 20 years (the 1990s and the 2000s), which is called 'lost twenty years' (see Krugman 1998 and General Introduction of Asada (ed.) 2014).

On the other hand, Proposition 2 characterizes a typical appropriate policy mix. This proposition says that the macroeconomic system tends to be dynamically stable if (1) the government expenditure responds sensitively to the changes of employment rather than the changes of public debt, (2) the central bank's monetary policy is relatively active, and (3) the central bank's inflation targeting is so credible that the public can form the inflation expectation in a forward-looking way on the basis of the announced target rate of inflation. We can consider that this is the rationale of new macroeconomic policy in Japan that was initiated by Abe administration in 2013, which is called 'Abenomics' (see General Introduction of Asada(ed.) 2014).

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Appendix A. Partial Derivatives:

$$\text{(A1)}\quad F_{41} = (\partial F_4/\partial g)^* = \left\{\frac{(1-\tau)(1-\delta)}{1-\delta(1-\tau)} - \sigma(\bar{b}+h^*)\right\}/\{1+\psi(r*)\underset{(+)}{y_b^*}\}.$$

$$\text{(A2)}\quad F_{42} = (\partial F_4/\partial r)^* = [-\tau y_r^* + \{(1-\tau) - \underset{(-)}{i^*_{r-\pi^e}}\}\bar{b} - \underset{(-)}{i^*_{r-\pi^e}} h^*]/\{1+\psi(r^*)\underset{(+)}{y_b^*}\}.$$

$$\begin{aligned}\text{(A3)}\quad F_{43} = (\partial F_4/\partial \pi^e)^* = \Big[&-\tau \underset{(+)}{y^*_{\pi^e}} - \bar{b} + \underset{(-)}{i^*_{r-\pi^e}}(\bar{b}+h^*) - \Big\{\bar{\pi} + \frac{\varepsilon_0+n_s}{1-\varepsilon}\Big\}\psi(r^*)\underset{(+)}{y^*_{\pi^e}} \\ &- \{\underset{(-)}{\psi'(r^*)}y^* + \psi(r^*)y_r^*\}\beta_1 + \psi(r^*)\underset{(+)}{y^*_{\pi^e}}\gamma\xi\Big]/\{1+\psi(r^*)\underset{(+)}{y_b^*}\}.\end{aligned}$$

$$\begin{aligned}\text{(A4)}\quad F_{44} = (\partial F_4/\partial b)^* = \Big[&-\tau\underset{(+)}{y_b^*} + \Big\{r^*(1-\tau) - \bar{\pi} - \frac{\varepsilon_0+n_s}{1-\varepsilon}\Big\} \\ &- \Big\{\bar{\pi} + \frac{\varepsilon_0+n_s}{1-\varepsilon}\Big\}\psi(r^*)\underset{(+)}{y_b^*} + \psi(r^*)\underset{(+)}{y_g^*}\alpha(1-\theta)\Big]/\{1+\psi(r^*)\underset{(+)}{y_b^*}\}.\end{aligned}$$

$$\begin{aligned}\text{(A5)}\quad F_{45} = (\partial F_4/\partial e)^* = [&-\kappa - \{\underset{(-)}{\psi'(r^*)} + \psi(r^*)y_r^*\}(\kappa\beta_1+\beta_2) \\ &+ \psi(r^*)\{-\underset{(+)}{y^*_{\pi^e}}\gamma(1-\xi)\kappa + \underset{(+)}{y_g^*}\alpha\theta\}]/\{1+\psi(r^*)\underset{(+)}{y_b^*}\}.\end{aligned}$$

$$\text{(A6)}\quad F_{51} = (\partial F_5/\partial g)^* = \bar{e}[\underset{(+)}{y_b^*}F_{41}/y^* + (1-\varepsilon)\sigma].$$

$$\text{(A7)}\quad F_{52} = (\partial F_5/\partial r)^* = \bar{e}[\underset{(+)}{y_b^*}F_{42}/y^* + (1-\varepsilon)\underset{(-)}{i^*_{r-\pi^e}}].$$

$$\text{(A8)}\quad F_{53} = (\partial F_5/\partial \pi^e)^* = (\bar{e}/y^*)[y_r^*\beta_1 - \underset{(+)}{y^*_{\pi^e}}\gamma\xi + \underset{(+)}{y_b^*}F_{43} - (1-\varepsilon)\underset{(-)}{i^*_{r-\pi^e}}].$$

(A9) $F_{54} = (\partial F_5/\partial b)^* = (\bar{e}/y^*)[\underset{(+)}{y_b^*} F_{44} - \underset{(+)}{y_g^*} \alpha(1-\theta)]$.

(A10)
$$F_{55} = (\partial F_5/\partial e)^* = (\bar{e}/y^*)[y_r^*(\kappa\beta_1+\beta_2) + \underset{(+)}{y_{\pi^e}^*}\gamma(1-\xi)\kappa + \underset{(+)}{y_b^*} F_{45} - \underset{(+)}{y_g^*} \alpha\theta].$$

The asterisk($*$) means that the values are evaluated *at the equilibrium point*.

Appendix B.

Proof of Proposion 1. Suppose that $\theta = \beta_1 = \beta_2 = \xi = 0$. In this case, we have the following expression (see equations (16), (A4), (A5) and (A10)).

(B1)
$$\begin{aligned} a_1 = &\left[\tau \underset{(+)}{y_b^*} - r^*(1-\tau) + \left\{\bar{\pi} + \frac{\varepsilon_0 + n_s}{1-\varepsilon}\right\}\{1 + \psi(r^*)\underset{(+)}{y_b^*}\} - \psi(r*)\underset{(+)}{y_g^*}\alpha\right] \\ &/\{1 + \psi(r^*)\underset{(+)}{y_b^*}\} + (\bar{e}/y^*)[-\underset{(+)}{y_{\pi^e}^*}\gamma\kappa + \underset{(+)}{y_b^*}\{\kappa + \psi(r^*)\underset{(+)}{y_{\pi^e}^*}\gamma\kappa\}/\{1 + \psi(r^*)\underset{(+)}{y_b^*}\}]. \end{aligned}$$

It follows from Eq. (B1) that we have $a_1 < 0$ for all sufficiently large values of $\alpha > 0$, which means that one of the necessary conditions for the local stability is violated for all sufficiently large values of $\alpha > 0$ if $\theta = \beta_1 = \beta_1 = \xi = 0$. By continuity, this conclusion is qualitatively unaffected even if $0 < \theta < 1$, $\beta_1 > 0$, $\beta_2 > 0$ and $0 < \xi < 1$, as long as all of them are sufficiently close to 0. □

Appendix C.

Proof of Proposion 2. Assume Assumptions 1 and 2, and suppose that $\xi = 1$. In this case, the Jacobian matrix (14) becomes as follows.

(C1)
$$J = \begin{bmatrix} 0 & 0 & 0 & -\alpha(1-\theta) & -\alpha\theta \\ 0 & 0 & \beta_1 & 0 & \kappa\beta_1 + \beta_2 \\ 0 & 0 & -\gamma & 0 & 0 \\ F_{41} & F_{42} & F_{43} & F_{44} & F_{45} \\ F_{51} & F_{52} & F_{53} & F_{54} & F_{55} \end{bmatrix}$$

Then, the characteristic equation (15) becomes

(C2)
$$\Gamma(\lambda) = |\lambda I - J| = |\lambda I - J_4|\,(\lambda + \gamma) = 0,$$

where

(C3)
$$J_4 = \begin{bmatrix} 0 & 0 & -\alpha(1-\theta) & -\alpha\theta \\ 0 & 0 & 0 & \kappa\beta_1 + \beta_2 \\ F_{41} & F_{42} & F_{44}(\theta) & F_{45}(\beta_1, \beta_2, \theta) \\ F_{51} & F_{52} & F_{54}(\theta) & F_{55}(\beta_1, \beta_2, \theta) \end{bmatrix}.$$

Under Assumptions 1 and 2, we have

(C4)
$$F_{41} > 0,\ F_{42} > 0,\ F_{44}(1) < 0,\ F_{51} > 0,\ F_{52} < 0,\ F_{54}(1) < 0.$$

The characteristic equation (C2) has a negative real root $\lambda_5 = -\gamma$, and other four roots are determined by the following equation

$$\Gamma_4(\lambda) = |\lambda I - J_4| = \lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4 = 0, \tag{C5}$$

where

$$b_1 = -\operatorname{trace} J_4 = -F_{44}(\theta) - F_{55}(\beta_1, \beta_2, \theta), \tag{C6}$$

$$\begin{aligned} b_2 = {} & \text{sum of all pricipal second-order minors of } J_4 \\ = {} & \alpha(1-\theta)\underset{(+)}{F_{41}} + \alpha\theta \underset{(+)}{F_{51}} - (\kappa\beta_1 + \beta_2)\underset{(-)}{F_{52}} + F_{44}(\theta)F_{55}(\beta_1, \beta_2, \theta) \\ & - F_{45}(\beta_1, \beta_2, \theta)F_{54}(\theta), \end{aligned} \tag{C7}$$

$$\begin{aligned} b_3 = {} & -(\text{sum of all pricipal third-order minors of } J_4) \\ = {} & (\kappa\beta_1 + \beta_2)\{-F_{42}F_{54}(\theta) + F_{44}(\theta)F_{52}\} \\ & + \alpha(1-\theta)A(\beta_1, \beta_2, \theta) + \alpha\theta B(\theta) \\ = {} & (\kappa\beta_1 + \beta_2)[(1-\varepsilon)\underset{(-)}{i^*_{r-\pi^e}}F_{44}(\theta) - \underset{(+)}{y^*_g}\alpha(1-\theta)\underset{(+)}{F_{42}}] \\ & + \alpha(1-\theta)A(\beta_1, \beta_2, \theta) + \alpha\theta B(\theta), \end{aligned} \tag{C8}$$

$$\begin{aligned} b_4 = {} & \det J_4 = \alpha(1-\theta)(\kappa\beta_1 + \beta_2)(-F_{41}F_{52} + F_{42}F_{51}) \\ & = \alpha(1-\theta)(\kappa\beta_1 + \beta_2)(1-\varepsilon)\bar{e}[\ -\underset{(-)}{i^*_{r-\pi^e}}\underset{(+)}{F_{41}} + \sigma\underset{(+)}{F_{42}}\ ] > 0 \\ & \text{if} 0 < \theta < 0. \end{aligned} \tag{C9}$$

It is well known that the Routh-Hurwitz conditions for stable roots of the characteristic equation (C5) are given by the following set of inequalities (see mathematical appendices of Asada, Chiarella, Flaschel and Franke 2003, 2010).

$$b_j > 0 \quad \text{for all } j \in \{1, 2, 3, 4\}, \quad \Phi = b_1b_2b_3 - b_1^2b_4 - b_3^2 > 0. \tag{C10}$$

Suppose that the parameter θ is close to 1 and either of the parameters β_1 or β_2 is sufficiently large. Then, it is easy to see that the inequalities $b_1 > 0$, $b_2 > 0$, and $b_3 > 0$ are satisfied under Assumptions 1 and 2. On the other hand, we always have $b_4 > 0$ as long as $0 < \theta < 1$.

Furthermore, we have

$$\lim_{\theta \to 1} \Phi = \lim_{\theta \to 1}(b_1b_2 - b_3)b_3 \tag{C11}$$

because of the fact that $\lim\limits_{\theta \to 1} b_4 = 0$. Next, suppose that the parameter β_2 is fixed at any positive value. In this case, we can easily see that $b_1b_2 - b_3$ becomes a quadratic function of β_1 and the coefficient of β_1^2 becomes positive. Furthermore, in this case, b_3 becomes a linear increasing function of β_1. It follows from the above considerations that we obtain

$$\lim_{\theta \to 1} \Phi > 0 \tag{C12}$$

for all sufficiently large values $\beta_1 > 0$. It is worth noting that we can easily interchange the roles of β_1 and β_2 in the above reasoning.

Therefore, all the Routh-Hurwitz conditions for local stability are satisfied under the conditions that (1) the parameter θ is less than 1, but it is close to 1, (2) parameter α is fixed at any positive value, (3) either the parameter β_1 or β_2 is sufficiently large, and (4) $\xi = 1$. By continuity, however, the local stability result also applies in case of $0 < \xi < 1$, as long as ξ is sufficiently close to 1. □

Appendix D. Calculation of the Coefficient a_5 in the General Case

From equations (14) and (18), we obtain the following relationship.

$$\text{(D1)} \quad \begin{aligned} a_5 &= -\det J = \alpha\gamma(1-\theta)\{\xi(\kappa\beta_1+\beta_2)+(1-\xi)\kappa\beta_1\}(-F_{41}F_{52}+F_{42}F_{51}) \\ &= \alpha(1-\theta)(1-\varepsilon)\{\xi(\kappa\beta_1+\beta_2)+(1-\xi)\kappa\beta_1\}[-\underset{(-)}{i^*_{r-\pi^e}}F_{41}+\sigma F_{42}]. \end{aligned}$$

Under Assumptions 1 and 2, we have $F_{41} > 0$ and $F_{42} > 0$ (cf. equations (A1) and (A2)). In this case, we always have $a_5 > 0$ as long as $0 \leqq \theta < 1$, $\beta_1 > 0$, $\beta_2 > 0$, and $0 \leqq \xi \leqq 1$.

References

1. Asada, T. (ed.), *The Development of Economics in Japan: From the Inter-war Period to the 2000s*, Routledge, London 2014.

2. Asada, T., Chiarella, C., Flaschel, P. and Franke, R., *Open Economy Macrodynamics: An Integrated Disequilibrium Approach*, Springer, Berlin 2003.

3. Asada, T., Chiarella, C., Flaschel, P. and Franke, R., *Monetary Macrodynamics*, Routledge, London 2010.

4. Asada, T. and Ouchi, M., *Fiscal and Monetary Policies in a Keynesian Model of Endogenous Growth Cycle with Public Debt Accumulation*, Discussion Paper (Econ) No. 2, Faculty of International Politics and Economics, Nishogakusha University, Tokyo (2013). http://www.nishogakusha-u.ac.jp/ks.htm

5. Gandolfo, G., *Economic Dynamics, Fourth Edition*, Springer, Berlin 2009.

6. Kaldor, N., *A Model of Economic Growth*, Economic Journal **67** (1957), 591–624.

7. Krugman, P., *It's Baaack: Japan's Slump and the Return of the Liquidity Trap*, Brookings Papers on Economic Activity **2** (1998), 137–205.

8. Taylor, J. B., *Discretion versus Policy Rules in Practice*, Carnegie-Rochester Conference Series on Public Policy **39** (1993), 195–214.

9. Woodford, M., *Interest and Prices: Foundations of a Theory of Monetary Policy*, Princeton University Press, Princeton 2003.

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