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A NOTE FOR COMPACT OPERATORS ON INFINITE TENSOR PRODUCTS

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Abstract. In this study, the compactness of the infinite tensor product of bounded operators defined on the infinite tensor product of Hilbert Spaces has been investigated under some conditions. Also, we prove that some compact operators have only point spectrum.

1. Introduction

The first definitions on infinite tensor product were given and studied by J. von Neumann [**12**]. Several researchers later approached this subject from a somewhat different viewpoint and established a coherent structure for dealing with many similar concepts concerning operators, operator algebras, and functionals [**10, 11, 9, 13, 3, 4, 5**]. H. Sahlmann et al. showed that the infinite tensor product construction enables to give rigorous meaning to the infinite volume (thermodynamic) limit of the theory which has been out of reach so far in [**15**]. Moreover, Thiemann and Winkler applied the theory of the infinite tensor product of Hilbert Spaces to quantum general relativity in their study [**18**], and Tepper constructed the mathematical version of a physical film on which space-time events can evolve by using the infinite tensor product of Hilbert spaces [**17**]. In order to make the theory available for application physics, von Neumann's method to construct infinite tensor product of Banach spaces is extended [**16**].

Also, let $x_1 \in H_1,\ x_2 \in H_2$, and $A_1 \in \mathfrak{B}(H_1), A_2 \in \mathfrak{B}(H_2)$ with H_1 and H_2 being Hilbert spaces, then the definitions of the single tensor product $x_1 \otimes x_2 \in H_1 \otimes H_2$ and the tensor product $A_1 \otimes A_2$ on the tensor product space $H_1 \otimes H_2$ is given and they have been considered variously by a number of authors, (see [**7, 8, 1, 14, 2**] for further references). Also, compactness of $A \otimes B \in \mathfrak{B}(H_1 \otimes H_2)$ was investigated by Zanni and Kubrusly [**19**], and Jinchuan [**6**].

What motivates us to this study is the studies [**19**] and [**6**] in the tensor product space. In these studies, the compactness of the operator, which is defined by the tensor product of two operators, is associated with the compactness of the component operators on the tensor product space. Unfortunately, when we look

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in the literature that deals with in infinite tensor product space, we can not see any results this problem.

In this paper, H_n and $(\cdot\,,\,\cdot)_n$ are separable complex Hilbert spaces and inner products on H_n, for all $n \in \mathbb{N}$, respectively. Also, all linear bounded operators space on H_n are denoted by $\mathfrak{B}(H_n)$.

In this study, it is mentioned firstly that the infinite tensor product space of Hilbert spaces H_n denoted by $\underset{n\in\mathbb{N}}{\otimes^c} H_n$, which was established by giving the definition of $\otimes_x$, including H_n, separable Hilbert space, is given the definition of the infinite tensor product of A_n, denoted by $\underset{n\in\mathbb{N}}{\otimes} A_n$, on $\underset{n\in\mathbb{N}}{\otimes^c} H_n$ from [**12**] and [**10**]. Secondly, the necessary conditions are investigated for this operator to be compact.

2. Preliminaries

First of all, for any complex numbers sequence $\{z_n\}$, the infinite product $\prod_{n=1}^{+\infty} z_n$ is said to converge to the number $z \in \mathbb{C}$ iff for all $\varepsilon > 0$, there exists $n_0(\delta) \in \mathbb{N}$ for all $n \geq n_0$, $n \in \mathbb{N}$, $|\prod_{k=1}^{n} z_k - z| < \varepsilon$, and is quasi-convergent if $\prod_{n=1}^{\infty} |z_k|$ converges. In this case, the value of $\prod_{n=1}^{\infty} |z_k|$ is equal $\prod_{n=1}^{\infty} z_k$ if $\prod_{n=1}^{\infty} z_k$ is even convergent and equal to zero otherwise.

Let $x_n \in H_n$ for all $n \in \mathbb{N}$. Then, is shown by $\otimes_x$ that

$$\otimes_x := \underset{n\in\mathbb{N}}{\otimes} x_n := x_1 \otimes x_2 \otimes \cdots \otimes x_n \otimes \cdots .$$

Similarly, for $\alpha \in \mathbb{C}$, $x_n^{(i)}, x_n, y_n \in H_n$, and $n, i \in \mathbb{N}$, the following notations

$$\begin{aligned}
\alpha\otimes_x &:= \alpha \underset{n\in\mathbb{N}}{\otimes} x_n := x_1 \otimes x_2 \otimes \cdots \otimes x_{i-1} \otimes \alpha x_i \otimes x_{i+1} \cdots \otimes x_n \otimes \cdots ,\\
\left(\otimes_x^1 + \cdots + \otimes_x^i\right)(y_1, y_2, \dots) &= \Big(\underset{n\in\mathbb{N}}{\otimes} x_n^{(1)} + \underset{n\in\mathbb{N}}{\otimes} x_n^{(2)} + \cdots + \underset{n\in\mathbb{N}}{\otimes} x_n^{(i)}\Big)(y_1, y_2, \dots)\\
&= \underset{n\in\mathbb{N}}{\otimes} x_n^{(1)}(y_1, y_2, \dots) + \underset{n\in\mathbb{N}}{\otimes} x_n^{(2)}(y_1, y_2, \dots)\\
&\quad + \cdots + \underset{n\in\mathbb{N}}{\otimes} x_n^{(i)}(y_1, y_2, \dots)
\end{aligned}$$

are used.

Let $x_n \in H_n$ for all $n \in \mathbb{N}$. Then, the infinite singular tensor product $\otimes_x = \otimes_{n\in\mathbb{N}} x_n = x_1 \otimes x_2 \otimes \cdots \otimes x_n \otimes \dots$ means that

$$\otimes_x (y_1, y_2, \dots) = \underset{n\in\mathbb{N}}{\otimes} x_n (y_1, y_2, \dots) := \prod_{n\in\mathbb{N}} (x_n, y_n)_n$$

for every $(y_1, y_2, \dots) \in \times_{n\in\mathbb{N}} H_n$. Similarly, scalar multiplication of $\otimes_x$ and sum of a finite number of $\otimes_x^i, i \in \mathbb{N}$ are defined by

$$\alpha \otimes_x (y_1, y_2, \dots) = \alpha \underset{n\in\mathbb{N}}{\otimes} x_n (y_1, y_2, \dots) = \prod_{n=1}^{i-1} (x_n, y_n)_n (\alpha x_i, y_i)_i \prod_{n=i+1}^{\infty} (x_n, y_n)_n$$

and

$$\left(\otimes_x^1+\otimes_x^2+\ldots+\otimes_x^i\right)(y_1,y_2,\dots) = \Big(\underset{n\in\mathbb{N}}{\otimes} x_n^{(1)}+\underset{n\in\mathbb{N}}{\otimes} x_n^{(2)}+\ldots+\underset{n\in\mathbb{N}}{\otimes} x_n^{(i)}\Big)(y_1,y_2,\dots)$$
$$= \underset{n\in\mathbb{N}}{\otimes} x_n^{(1)}(y_1,y_2,\dots)+\cdots+\underset{n\in\mathbb{N}}{\otimes} x_n^{(i)}(y_1,y_2,\dots)$$
$$= \prod_{n\in\mathbb{N}}\big(x_n^{(1)},y_n\big)_n+\prod_{n\in\mathbb{N}}\big(x_n^{(2)},y_n\big)_n +\cdots+\prod_{n\in\mathbb{N}}\big(x_n^{(i)},y_n\big)_n,$$

respectively.

The set of formed by all finite sum of infinite singular tensor products with convergent condition by ${\otimes^c}_{n\in\mathbb{N}} H_n$, is such that

$$\underset{n\in\mathbb{N}}{\otimes^c} H_n := \Big\{\sum_{i=1}^k \otimes_x^i \;:\; \otimes_x^i = \underset{n\in\mathbb{N}}{\otimes} x_n^{(i)},\ x_n^{(i)}\in H_n,\ n,k\in\mathbb{N},\ 1\le i\le k,$$
$$\prod_{n\in\mathbb{N}}\|x_n^{(i)}\|_n \text{ is convergent}\Big\}.$$

It can be seen that $\underset{n\in\mathbb{N}}{\otimes^c} H_n$ is a linear vector space with scalar multiplication and sum of two elements are defined by

$$\alpha\cdot\otimes_x := \alpha\otimes_x,$$
$$\sum_{i=1}^k \otimes_x^i + \sum_{i=1}^m \otimes_y^i := \sum_{i=1}^{k+m} \otimes_x^i$$

for all $k+1\le i\le k+m$, $\otimes_x^i=\otimes_y^{i-k}$, respectively. Also, it is obviously seen that its zero is $\otimes_0 := x_1^{(0)}\otimes x_2^{(0)}\otimes\ldots$ such that $\prod_{n\in\mathbb{N}}\|x_n^{(0)}\|_n=0$ for every $x_n^{(0)}\in H_n$, for all $n\in\mathbb{N}$.

Lemma 2.1. *For arbitrary $\otimes_x \in {\otimes^c}_{n\in\mathbb{N}} H_n$ such that $\otimes_x\neq 0$, the set defined by*

$$\underset{n\in\mathbb{N}}{\otimes_x^c} H_n := \Big\{\sum_{i=1}^k \otimes_y^i \in \underset{n\in\mathbb{N}}{\otimes^c} H_n : \otimes_y^i = \underset{n\in\mathbb{N}}{\otimes} y_n^{(i)},\ y_n^{(i)}\in H_n,\ n,k\in\mathbb{N},\ 1\le i\le k$$
$$\text{and } \prod_{n\in\mathbb{N}}\Big(x_n,\, y_n^{(i)}\Big)_n \text{ is convergent}\Big\}$$

is a linear subspace of $\underset{n\in\mathbb{N}}{\otimes^c} H_n$.

Proposition 2.1. *Let $\otimes_x,\otimes_y,\otimes_z \in \underset{n\in\mathbb{N}}{\otimes^c} H_n$ be such that all of them are not equal to zero. Then the following statements hold:*

a) $\otimes_x \in \underset{n\in\mathbb{N}}{\otimes_x^c} H_n$.

b) *If* $\otimes_y \in \underset{n\in\mathbb{N}}{\otimes_x^c} H_n$*, then* $\otimes_x \in \underset{n\in\mathbb{N}}{\otimes_y^c} H_n$*.*

c) *If* $\otimes_y \in \underset{n\in\mathbb{N}}{\otimes_x^c} H_n$ *and* $\otimes_z \in \underset{n\in\mathbb{N}}{\otimes_y^c} H_n$*, then* $\otimes_z \in \underset{n\in\mathbb{N}}{\otimes_x^c} H_n$*.*

d) *Let* $\otimes_y$ *and* $\otimes_z \in \underset{n\in\mathbb{N}}{\otimes_x^c} H_n$*.* $\prod_{n\in\mathbb{N}} (y_n, z_n)_n = 0$ *if and only if some* $(y_n,\, z_n)_n = 0$*.*

e) *If* $\otimes_y \in \underset{n\in\mathbb{N}}{\otimes_x^c} H_n$ *and* $\otimes_z \notin \underset{n\in\mathbb{N}}{\otimes_x^c} H_n$*, then* $\prod_{n\in\mathbb{N}} (y_n, z_n)_n = 0$*.*

Proof. It can be seen in [**12**, Definition 3.3.2, Lemma 3.3.3, and Theorem I]. □

Lemma 2.1 and Proposition 2.1 are useed in the proof of the next Lemma 2.2.

Lemma 2.2. $\underset{n\in\mathbb{N}}{\otimes^c} H_n$ *is a pre-Hilbert space with the inner product defined by*

$$(\cdot\,,\,\cdot)_c : \underset{n\in\mathbb{N}}{\otimes^c} H_n \times \underset{n\in\mathbb{N}}{\otimes^c} H_n \longrightarrow \mathbb{C},$$

$$\left(\sum_{i=1}^{k} \otimes_x^i \,, \sum_{j=1}^{m} \otimes_y^j\right)_c = \sum_{i=1}^{k}\sum_{j=1}^{m}\prod_{n\in\mathbb{N}} \left(x_n^{(i)}, y_n^{(j)}\right)_n . \tag{2.1}$$

(for details see in [**12**, Lemma 3.3.2, Lemma 3.4.1, and Lemma 3.4.3]*).*

Proposition 2.2. *If* $\dim H_n \geq 1$*, then* $\dim \otimes^c{}_{n\in\mathbb{N}} H_n = \infty$*, and moreover, it is nonseparable.*

Definition 1. We denote by $\otimes_{n\in\mathbb{N}} H_n$ the Cauchy-completion of the pre-Hilbert space $\otimes^c{}_{n\in\mathbb{N}} H_n$.

Theorem 2.1 ([**10**]). *Let* $A_n \in \mathfrak{B}(H_n)$ *for all* $n \in \mathbb{N}$ *and* $\prod_{n\in\mathbb{N}} \|A_n\|$ *be convergent, then the infinite tensor product of* A_n*, which is denoted by* $\otimes_{n\in\mathbb{N}} A_n$ *on* $\underset{n\in\mathbb{N}}{\otimes} H_n$*, can be defined by*

$$\underset{n\in\mathbb{N}}{\otimes} A_n : \underset{n\in\mathbb{N}}{\otimes} H_n \longrightarrow \underset{n\in\mathbb{N}}{\otimes} H_n,$$

$$\underset{n\in\mathbb{N}}{\otimes} A_n \left(\sum_{i=1}^{k} \otimes_x^i\right) := \sum_{i=1}^{k} \left(\underset{n\in\mathbb{N}}{\otimes} A_n\right) \otimes_x^i := \sum_{i=1}^{k} \underset{n\in\mathbb{N}}{\otimes} A_n x_n^{(i)},$$

and $\otimes_{n\in\mathbb{N}} A_n \in \mathfrak{B}\big(\otimes_{n\in\mathbb{N}} H_n\big)$*. Finally, its norm is*

$$\Big\| \underset{n\in\mathbb{N}}{\otimes} A_n \Big\| = \prod_{n\in\mathbb{N}} \|A_n\|.$$

Proposition 2.3. *Let* $\alpha, \beta \in \mathbb{C}$ *and* A_n*,* B_n*,* $A_n^{(1)}$*,* $A_n^{(2)}$ *be linear operators on* H_n *for all* $n \in \mathbb{N}$*. The following identities are true:*

a) $\alpha\beta \underset{n\in\mathbb{N}}{\otimes} A_n = A_1 \otimes \cdots \otimes \beta A_j \otimes \cdots \otimes \alpha A_i \otimes \dots$*, for arbitrary* $i, j \in \mathbb{N}$*.*

b) $A_1 \otimes \cdots \otimes \left(A_n^{(1)} + A_n^{(2)}\right) \otimes \dots$
$= \left(A_1 \otimes \cdots \otimes A_n^{(1)} \otimes \dots\right) + \left(A_1 \otimes \cdots \otimes A_n^{(2)} \otimes \dots\right).$

Proposition 2.4 ([**10**]). *Let* $A_n \in \mathfrak{B}(H_n)$ *for all* $n \in \mathbb{N}$, *and* $\otimes_{n\in\mathbb{N}} A_n \in \mathfrak{B}\big(\otimes_{n\in\mathbb{N}} H_n\big)$. *Also,* A_n^{-1} *and* A_n^* *are inverse and adjoint of* A_n *for all* $n \in \mathbb{N}$, *respectively. Then, the following identities hold:*

a) $\underset{n\in\mathbb{N}}{\otimes}(A_nB_n) = \Big(\underset{n\in\mathbb{N}}{\otimes} A_n\Big)\Big(\underset{n\in\mathbb{N}}{\otimes} B_n\Big)$.

b) *If* A_n *is invertible and* $\prod_{n\in\mathbb{N}} \|A_n^{-1}\|_n < +\infty$, *then* $\underset{n\in\mathbb{N}}{\otimes} A_n$ *is invertible and* $\Big(\underset{n\in\mathbb{N}}{\otimes} A_n\Big)^{-1} = \underset{n\in\mathbb{N}}{\otimes} A_n^{-1}$.

c) $\Big(\underset{n\in\mathbb{N}}{\otimes} A_n\Big)^* = \underset{n\in\mathbb{N}}{\otimes} A_n^*$.

3. Compact operators

Theorem 3.1. *Let* $\otimes_{i\in\mathbb{N}} A_i^{(n)}$ *be an operator sequence in* $\mathfrak{B}\big(\otimes_{i\in\mathbb{N}} H_i\big)$ *such that* $A_i^{(n)} = A_i$ *for* $i \in \mathbb{N} \setminus \{k_1, k_2, \dots, k_m\}$. *If the sequences* $\{A_i^{(n)}\}$ *converge to* A_i *for all* $i \in \{k_1, k_2, \dots, k_m\}$, *then the operator sequence* $\otimes_{i\in\mathbb{N}} A_i^{(n)}$ *converges to* $\otimes_{i\in\mathbb{N}} A_i$.

Proof. Suppose that $\otimes_{i\in\mathbb{N}} A_i^{(n)}$ is in $\in \mathfrak{B}\big(\otimes^c{}_{i\in\mathbb{N}} H_i\big)$ for all $n \in \mathbb{N}$, defined by $A_i^{(n)} = A_i$ for $i \in \mathbb{N} \setminus \{1, 2\}$, and $A_i^{(n)} \in \mathfrak{B}(H_i)$ for all $i \in \mathbb{N}$. We note that this assumption does not impair the generality of the theorem. If $A_i^{(n)} \to A_i$ for all $i \in \{1, 2\}$ as $n \to \infty$, then

$$
\begin{aligned}
\Big\|\underset{i\in\mathbb{N}}{\otimes} A_i^{(n)} - \underset{i\in\mathbb{N}}{\otimes} A_i\Big\| &= \|A_1^{(n)} \otimes A_2^{(n)} \otimes A_3 \otimes \dots - A_1 \otimes A_2 \otimes A_3 \otimes \dots\| \\
&= \Big\| \Big(\big(A_1^{(n)} - A_1\big) \otimes A_2^{(n)} \otimes A_3 \otimes \dots\Big) \\
&\qquad + \Big(A_1 \otimes \big(A_2^{(n)} - A_2\big) \otimes A_3 \otimes \dots\Big)\Big\| \\
&\le \Big\|\big(A_1^{(n)} - A_1\big) \otimes A_2 \otimes A_3 \otimes \dots \otimes A_i \otimes \dots\Big\| \\
&\quad + \Big\|A_1 \otimes \big(A_2^{(n)} - A_2\big) \otimes A_3 \otimes \dots \otimes A_m \otimes \dots\Big\| \\
&= \Big\|A_1^{(n)} - A_1\Big\| \prod_{i=2}^{\infty} \Big\|A_i^{(n)}\Big\| + \|A_1\| \Big\|A_2^{(n)} - A_2\Big\| \prod_{i=3}^{\infty} \Big\|A_i^{(n)}\Big\| \\
&\longrightarrow 0 \qquad \text{as} \qquad n \longrightarrow \infty.
\end{aligned}
$$

Consequently, it is obtained that the operator sequence $\otimes_{i\in\mathbb{N}} A_i^{(n)}$ converges to $\otimes_{i\in\mathbb{N}} A_i$. □

Lemma 3.1. *Let* $\{x_k^{(n)}\}$ *be a sequence in* H_k *for fixed* $k \in \mathbb{N}$ *and for all* $n \in \mathbb{N}$. *If the sequence* $\otimes_x^n$, *defined by* $\otimes_x^n := x_1 \otimes x_2 \otimes \dots \otimes x_{k-1} \otimes x_k^{(n)} \otimes x_{k+1} \otimes \dots \otimes x_i \otimes \dots \in \otimes_{i\in\mathbb{N}} H_i$ *for all* $x_i \in H_i$ *and* $n, i \in \mathbb{N}$, *converges in* $\otimes_{i\in\mathbb{N}} H_i$ *to* $\otimes_x =$

$x_1 \otimes x_2 \otimes \cdots \otimes x_{k-1} \otimes x_k \otimes x_{k+1} \otimes \cdots \otimes x_i \otimes \cdots \in \otimes_{i\in\mathbb{N}} H_i$ for all $x_i \in H_i$, $i \in \mathbb{N}$, then $\{x_k^{(n)}\}$ converges to x_k in H_k.

Proof. Let $\otimes_x^n := x_1 \otimes x_2 \otimes \cdots \otimes x_{k-1} \otimes x_k^{(n)} \otimes x_{k+1} \otimes \cdots \otimes x_i \otimes \cdots \in \underset{i\in\mathbb{N}}{\otimes} H_i$ for all $i \in \mathbb{N}$, $x_i \in H_i$, where k is a fixed natural number, and for all $n \in \mathbb{N}$, $x_k^{(n)} \in H_k$. If $\otimes_x^n$ converges to $\otimes_x = x_1 \otimes \cdots \otimes x_k \otimes x_{k+1} \otimes \cdots \in \underset{i\in\mathbb{N}}{\otimes} H_i$, then

$$\|\otimes_x^n - \otimes_x\| \longrightarrow 0 \qquad \text{as} \qquad n \longrightarrow \infty.$$

Because of the following equation

$$\begin{aligned}\otimes_x^n - \otimes_x &= \big(x_1 \otimes \cdots \otimes x_k^{(n)} \otimes x_{k+1} \otimes \dots\big) - (x_1 \otimes \cdots \otimes x_k \otimes x_{k+1} \otimes \dots) \\ &= x_1 \otimes \cdots \otimes \big(x_k^{(n)} - x_k\big) \otimes \cdots \otimes x_i \otimes \dots,\end{aligned}$$

it is obtained that

$$\Big\|x_1 \otimes \cdots \otimes \big(x_k^{(n)} - x_k\big) \otimes \cdots \otimes x_i \otimes \dots\Big\| = \Big\|x_k^{(n)} - x_k\Big\|_k \prod_{\substack{i=1\\ i\neq k}}^{+\infty} \|x_i\|_i.$$

From the last relation, it is seen that $\{x_k^{(n)}\}$ converges to x_k in H_k. □

Theorem 3.2. *Let $A_i \in \mathfrak{B}(H_i)$ for all $i \in \mathbb{N}$ and $\otimes_{i\in\mathbb{N}} A_i \in \mathfrak{B}(\otimes_{i\in\mathbb{N}} H_i)$. If $\otimes_{n\in\mathbb{N}} A_i$ is a nonzero compact operator, then A_i is a nonzero compact operator for all $i \in \mathbb{N}$.*

Proof. Suppose that $\otimes_{i\in\mathbb{N}} A_i \in \mathfrak{B}(\otimes_{i\in\mathbb{N}} H_i)$ is a nonzero compact operator. Thus, $\otimes_{i\in\mathbb{N}} A_i x_i \neq \otimes_0$ for some $\otimes_{i\in\mathbb{N}} x_i$, and hence $A_i x_i \neq 0$ for all $i \in \mathbb{N}$. Consider an arbitrary bounded sequence $\{x_k^{(n)}\}$ of vectors in H_k for a fixed number $k \in \mathbb{N}$, and a sequence $\{\otimes_x^n\}$ defined by $\otimes_x^n = x_1 \otimes \cdots \otimes x_{k-1} \otimes x_k^{(n)} \otimes x_{k+1} \otimes \cdots \otimes x_i \otimes \dots$ for all n, $i \in \mathbb{N}$. Since $\{x_k^{(n)}\}$ is bounded and

$$\begin{aligned}\sup_{n\in\mathbb{N}} \|\otimes_x^n\| &= \sup_{n\in\mathbb{N}} \|x_1 \otimes \cdots \otimes x_{k-1} \otimes x_k^{(n)} \otimes x_{k+1} \otimes \cdots \otimes x_i \otimes \dots\| \\ &= \sup_{n\in\mathbb{N}} \Bigg(\|x_k^{(n)}\|_k \prod_{\substack{i=1\\ i\neq k}}^{+\infty} \|x_i\|_i\Bigg) = \prod_{\substack{i=1\\ i\neq k}}^{+\infty} \|x_i\|_i \sup_{n\in\mathbb{N}} \|x_k^{(n)}\|_k,\end{aligned}$$

$\{\otimes_x^n\}$ is a bounded sequence. Moreover, when the operator $\otimes_{i\in\mathbb{N}} A_i$ is compact, then there exists a subsequence $\otimes_x^{m_n}$ of $\otimes_x^n$ such that $(\otimes_{i\in\mathbb{N}} A_i)\otimes_x^{m_n}$ converges in $\otimes_{i\in\mathbb{N}} H_i$. Therefore, there exists an element y_k in H_k and

$$\begin{aligned}\Big(\underset{i\in\mathbb{N}}{\otimes} A_i\Big)\otimes_x^{m_n} &= \Big(\underset{i\in\mathbb{N}}{\otimes} A_i\Big)\big(x_1 \otimes \cdots \otimes x_k^{(m_n)} \otimes \cdots \otimes x_{k+1} \otimes \dots\big) \\ &= A_1x_1 \otimes \cdots \otimes A_k x_k^{(m_n)} \otimes \cdots \otimes A_i x_i \otimes \dots \\ &\longrightarrow A_1x_1 \otimes \cdots \otimes y_k \otimes A_{k+1}x_{k+1} \otimes \dots, \quad \text{as} \quad n \to +\infty.\end{aligned}$$

Using the last result and Lemma 3.1, the subsequence $\{A_k x_k^{(m_n)}\}$ is convergent in H_k. This means that A_k is a compact operator. Similarly, it can be shown that A_i is compact for all $i \in \mathbb{N}$. □

Remark 3.1. The inverse of the last theorem, that is, if each component is compact, it is generally not true that the tensor product of operators is compact. For example, consider $H_n := \mathbb{C}^2$ and $A_n := I$ (the identity operator) for all $n \in \mathbb{N}$. It is known that A_n is compact for all $n \in \mathbb{N}$. However, the tensor product space $\otimes_{n\in\mathbb{N}} H_n$ has infinite dimension, and so the identity operator $\otimes_{n\in\mathbb{N}} I$ on $\otimes_{n\in\mathbb{N}} H_n$ is not compact operator.

Theorem 3.3. *Let A_n be a compact operator for all $n \in \mathbb{N}$ such that $\dim \operatorname{Range}(A_i) \geq 2$ for all $i \in \{k_n : n \in \mathbb{N}\} \subset \mathbb{N}$. If the operator $\otimes_{n\in\mathbb{N}} A_n$ is compact on $\otimes_{n\in\mathbb{N}} H_n$, then $0 \in \sigma_p(\otimes_{n\in\mathbb{N}} A_n)$.*

Proof. If $\otimes_{n\in\mathbb{N}} A_n = 0$, it is apparent. Suppose that $\otimes_{n\in\mathbb{N}} A_n$ is a nonzero compact operator on $\otimes_{n\in\mathbb{N}} H_n$ and $\dim \operatorname{Range}(A_n) \geq 2$ for all $n \in \mathbb{N}$. This assumption does not impair the generality of the theorem. In this case, there is a nonzero element $\otimes_x = x_1 \otimes x_2 \otimes \dots$ in $\otimes_{n\in\mathbb{N}} H_n$ such that $\otimes_{n\in\mathbb{N}} A_n \otimes_x \neq 0$. Moreover, since $\dim \operatorname{Range}(A_n) \geq 2$ for all $n \in \mathbb{N}$, it can be found at least an element $\otimes_y = y_1 \otimes y_2 \otimes \cdots \otimes y_n \otimes \cdots \in \otimes_{n\in\mathbb{N}} H_n$ such that $(A_n x_n, A_n x_y)_n = 0$ for all $n \in \mathbb{N}$, and specially chosen that $\|y_n\|_n = 1$ and $A_n y_n \neq 0$. Now, we can construct a bounded sequence in $\otimes_{n\in\mathbb{N}} H_n$ as follows

$$\otimes_x^n = x_1 \otimes \cdots \otimes x_{n-1} \otimes y_n \otimes x_{n+1} \otimes \dots .$$

Since $\otimes_{n\in\mathbb{N}} A_n$ is a compact operator, there is a $\otimes_x^{(k_n)}$ subsequence such that $\otimes_{i\in\mathbb{N}} A_i \otimes_x^{(k_n)}$ is convergent on $\mathfrak{B}\big(\otimes_{n\in\mathbb{N}} H_n\big)$. In addition, the following equality holds

$$\Big\| \underset{n\in\mathbb{N}}{\otimes} A_n \underset{n\in\mathbb{N}}{\otimes} x^{(k_n)} - \underset{n\in\mathbb{N}}{\otimes} A_n \underset{n\in\mathbb{N}}{\otimes} x^{(k_m)} \Big\|^2 = \Big(\frac{\|A_{k_n} y_{k_n}\|^2}{\|Ak_n x_{k_n}\|^2} + \frac{\|A_{k_m} y_{k_m}\|^2}{\|A_{k_m} x_{k_m}\|^2} \Big) \prod_{n\in\mathbb{N}} \|A_n x_n\|^2$$

for $n \neq m$. Since $\lim_{n\to+\infty} \|A_n x_n\| = 1$ and $\otimes_{n\in\mathbb{N}} A_n \otimes_x^{(k_n)}$ is a Cauchy sequence, it has to be such that $\lim_{n\to+\infty} \|A_{k_n} y_{k_n}\| = 0$. Hence,

$$\Big\| \underset{n\in\mathbb{N}}{\otimes} A_n \otimes_y \Big\| = \prod_{n\in\mathbb{N}} \|A_n y_n\|_n = 0$$

is obtained, and this means that $0 \in \sigma_p(\otimes_{n\in\mathbb{N}} A_n)$. □

Corollary 3.1. *Let A_n be a compact operator for all $n \in \mathbb{N}$ such that $\dim \operatorname{Range}(A_i) \geq 2$ for all $i \in \{k_n : n \in \mathbb{N}\} \subset \mathbb{N}$. If the operator $\otimes_{n\in\mathbb{N}} A_n$ is compact on $\otimes_{n\in\mathbb{N}} H_n$, then $\sigma(\otimes_{n\in\mathbb{N}} A_n) = \sigma_p(\otimes_{n\in\mathbb{N}} A_n)$.*

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