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FINITE DIFFERENCE SCHEMES FOR A NONLINEAR BLACK-SCHOLES MODEL WITH TRANSACTION COST AND VOLATILITY RISK

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Abstract. Several nonlinear Black-Scholes models have been proposed to take transaction cost, large investor performance and illiquid markets into account. One of the most comprehensive models introduced by Barles and Soner in [**4**] considers transaction cost in the hedging strategy and risk from an illiquid market. In this paper, we compare several finite difference methods for the solution of this model with respect to precision and order of convergence within a computationally feasible domain allowing at most 200 space steps and 10000 time steps. We conclude that standard explicit Euler comes out as the preferred explicit method and standard Crank Nicolson with Rannacher time stepping as the preferred implicit method.

1. Introduction

The classical linear Black-Scholes model for option pricing assumes a complete market without transaction cost, illiquidity or feedback issues like large investor performance. Several nonlinear Black-Scholes models have been proposed in recent years to deal with these inadequacies. Nonlinearity in the nonlinear Black-Scholes models always arises from a nonlinear volatility function depending not only on time t and underlying asset price S but also on the Greek Gamma, i.e., the second derivative of the option price $V(S,t)$ with respect to S. Hence the nonlinear Black-Scholes model equation is

$$(1)\quad \frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2\Big(t,S,\frac{\partial^2 V}{\partial S^2}\Big)S^2\frac{\partial^2 V}{\partial S^2}+(r-\gamma)S\frac{\partial V}{\partial S}-rV=0,\quad (S,t)\in(0,S_{\max})\times(0,T)$$

with the following terminal and boundary Dirichlet conditions

$$(2)\qquad V(S,T)=\kappa(S,T),\qquad V(0,t)=\kappa(0,t),\qquad V(S_{\max},t)\simeq\kappa(S_{\max},t),$$

where we use the utility function

$$(3)\quad \kappa(S,t)=\begin{cases}\max\{S\,\mathrm{e}^{-\gamma(T-t)}-K\,\mathrm{e}^{-r(T-t)},0\} & \text{for the } \textit{call option},\\ \max\{K\,\mathrm{e}^{-r(T-t)}-S\,\mathrm{e}^{-\gamma(T-t)},0\} & \text{for the } \textit{put option},\\ B\,\mathrm{e}^{-r(T-t)}\,\mathcal{H}(S-K) & \text{for the } \textit{bet option}.\end{cases}$$

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γ, σ and r are the *dividend yield*, *volatility* (on the underlying risky asset) and *market interest rate* (on the riskfree asset), respectively, K is the *Strike Price*, $\mathcal{H}$ the Heaviside function and $S_{\max} \gg K$ is the upper bound on the computational domain in the S variable.

Two known "numerical issues" from the linear case are expected to carry over to the nonlinear case:

First of all, many methods oscillate either around the strike price $S = K$ or around the upper bound for the computational domain $S = S_{\max}$. One way to eliminate such oscillations is to use a monotone method that cannot oscillate. Alternatively oscillations near $S = S_{\max}$ are easily observed and removed simply by increasing $S_{\max}$, of course at the cost of an increase in computational time. Initiating methods oscillating around $S = K$ by smaller timesteps (4 initial quartersteps has been suggested as optimal) with a nonoscillating method tends to remove the oscillations without decreasing the convergence rate of the oscillating method. Most notable example is probably Crank-Nicolson with 4 initial quartersteps by implicit Euler explored in [**11**].

The second "numerical issue" is the degradation in observed convergence order caused by the singular terminal condition for options unless extremely small stepsizes are used. Such stepsizes often lie outside what is computationally feasible.

In this article, we will investigate these two problems for a number of finite difference schemes and one particular nonlinear model proposed by Barles and Soner in [**4**].

The nonlinearity of the nonlinear case may provide "numerical issues" of its own. Uniqueness of solution is typically an issue for nonlinear problems just as the "nice" smoothening feature of linear heat conduction may be lost. Such nonlinear features will not be dealt with here, but let us just note, that no practical problems in this direction have been observed.

In section 2, we review the Barles and Soner model for nonlinear volatility. In section 3, we present a number of different finite difference methods for the Barles and Soner model. In section 4, we present some numerical results with the different finite difference schemes for the Barles and Soner model. Finally in section 5, we discuss our results and present the conclusions.

2. The Barles and Soner nonlinear volatility model

Barles and Soner [**4**] considers both transaction cost and risk from volatile portfolios. They take an approach based on utility maximization which results in the following adjustment of the volatility

$$\sigma_{BS}^2\Big(t, S, \frac{\partial^2 V}{\partial S^2}\Big) = \sigma_0^2\left[1 + \Psi\left(\mathrm{e}^{r(T-t)}\, aS^2 \frac{\partial^2 V}{\partial S^2}\right)\right]. \tag{4}$$

Here $a = \kappa^2 R$, where κ is the "Leland transaction cost" (denoted μ in [**4**]) and R is a *risk aversion factor* (denoted γ in [**4**]). Finally, $\Psi(x)$ is the solution of the nonlinear ODE

$$\Psi'(x) = \frac{\Psi(x) + 1}{2\sqrt{x\Psi(x)} - x}, \qquad x \neq 0 \tag{5}$$

with the initial condition $\Psi(0) = 0$. In appendix A of [**4**], the existence of a unique continuous viscosity solution to this problem was shown. It was also shown that $\Psi \geq 0$, i.e., that the adjustment factor to σ_0^2 is nonnegative for any argument of Ψ. For the numerical experiments an unspecified explicit time stepping finite difference scheme is used with small time steps near maturity ($t = T$) and larger time steps away from maturity. Lesmana and Wang [**10**] present an implicit first order time stepping and upwind asset price stepping finite difference method that we here denote **ImpUp**. Zhou et al [**13**] present a positivity-preserving scheme that we denote **PosPre**. Both schemes are used to solve the Barles and Soner model.

Note that σ_0 — the volatility of the underlying asset — is assumed constant and if we take $\sigma(t, S, V_{SS}) = \sigma_0$, we have the classical linear Black-Scholes model.

3. Finite Difference Schemes

Arenas et al [**2**] present a nonstandard explicit finite difference scheme for the numerical pricing of options in an illiquid market modeled by Frey et al in [**8**], [**7**] and [**6**]. González et al [**9**] then apply the same scheme for the parameterized model by Bakstein and Howison [**3**]. The method is shown to be nonnegative, nondecreasing, stable and consistent for both model problems. Here we consider this method for solving the option pricing problem with the Barles and Soner model (4–5) which has not previously been attempted. For short the method is denoted **NFDM** and we compare to NFDM to ImpUp and PosPre, both previously used to solve the Barles and Soner model. Finally, we compare two standard methods, namely explicit first order in time and central second order in S denoted **FtCS** and Crank-Nicolson denoted **CN**. CN is stabilized with "Rannacher time stepping" (see [**11**]) — starting up with 4 quarter steps using the fully implicit first order in time and second order in S "Implicit Euler" scheme — and hence the notation **CNR** is used. Also we apply $\boldsymbol{K_\alpha}$**-optimization** to all 5 methods minimizing the error by adjusting stepsizes so that the strike price K is situated in an optimal position in the element that it resides in (see [**11**]).

The volatility function in (1) is then taken to be σ_{BS} from (4–5) and in order to follow the original presentation dividend is not considered ($\gamma = 0$) and time is reversed by replacing the time variable t by $\tau = T - t$, and consequently $V(S, t)$ by $U(S, \tau)$. This transforms equation (1) into

$$\frac{\partial U}{\partial \tau} - \frac{1}{2}\sigma_{BS}^2\Big(T - \tau, S, \frac{\partial^2 U}{\partial S^2}\Big)S^2\frac{\partial^2 U}{\partial S^2} - rS\frac{\partial U}{\partial S} + rU = 0 \tag{6}$$

with σ_{BS} given by (4–5) whereas the terminal and boundary conditions (2) for V are transformed into the obvious initial and boundary conditions for U

(7) $U(S,0) = \kappa(S,T),\quad U(0,\tau) = \kappa(0,T-\tau),\quad U(S_{\max},\tau) \simeq \kappa(S_{\max},T-\tau)$

with κ still given by (3).

We describe our 5 FDM's using the following finite difference operators:

$$
\begin{aligned}
&\delta_\tau^+ u_j^n = \frac{u_j^{n+1}-u_j^n}{k}, &&\tilde{\delta}_\tau^+ u_j^n = \frac{u_j^{n+1}-u_j^n}{\theta(k)},\\
&\delta_S^+ u_j^n = \frac{u_{j+1}^n-u_j^n}{h}, &&\delta_S^0 u_j^n = \frac{u_{j+1}^n-u_{j-1}^n}{2h},\\
&\delta_{SS}^0 u_j^n = \frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{h^2}, &&\tilde{\delta}_{SS}^0 u_j^n = \frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{\phi(h)},\\
&\bar{\delta}_{SS}^0 u_j^n = \frac{u_{j+1}^n-2u_j^{n+1}+u_{j-1}^n}{h^2},
\end{aligned}
\tag{8}
$$

where $\phi(h) = (\mathrm{e}^{\sqrt{r}h} - 2 + \mathrm{e}^{-\sqrt{r}h})/r = h^2 + \mathcal{O}(h^4)$ and $\theta(k) = (1-\mathrm{e}^{-rk})/r = k + \mathcal{O}(k^2)$. Noting that the initial and boundary conditions are the same for all methods, our 5 FDM's are described by their main update equations:

$$
\begin{aligned}
&\text{NFDM} && \tilde{\delta}_\tau^+ u_j^n - \frac{1}{2}\sigma_{BS}^2(T-\tau_n, S_j, \tilde{\delta}_{SS}^0 u_j^n) S_j^2 \tilde{\delta}_{SS}^0 u_j^n - rS_j\delta_S^+ u_j^n + ru_j^n = 0\\
&\text{ImpUp} && \delta_\tau^+ u_j^n - \frac{1}{2}\sigma_{BS}^2(T-\tau_{n+1}, S_j, \delta_{SS}^0 u_j^{n+1}) S_j^2 \delta_{SS}^0 u_j^{n+1} - rS_j\delta_S^+ u_j^{n+1}\\
& && + ru_j^{n+1} n = 0\\
&\text{PosPre} && \delta_\tau^+ \widehat{u}_j^n - \frac{1}{2}\sigma_{BS}^2(T-\tau_n, x_j^n, \bar{\delta}_{xx}^0 \widehat{u}_j^n)(x_j^n)^2 \bar{\delta}_{xx}^0 \widehat{u}_j^n = 0\\
&\text{FtCS} && \delta_\tau^+ u_j^n - \frac{1}{2}\sigma_{BS}^2(T-\tau_n, S_j, \delta_{SS}^0 u_j^n) S_j^2 \delta_{SS}^0 u_j^n - rS_j\delta_S^0 u_j^n + ru_j^n = 0\\
&\text{CN} && \delta_\tau^+ u_j^n - \frac{1}{2}\sigma_{BS}^2\Big(T-\tau_{n+\frac{1}{2}}, S_j, \delta_{SS}^0\Big(\frac{u_j^{n+1}+u_j^n}{2}\Big)\Big) S_j^2\\
& && \cdot\,\delta_{SS}^0\Big(\frac{u_j^{n+1}+u_j^n}{2}\Big) - rS_j\delta_S^0\Big(\frac{u_j^{n+1}+u_j^n}{2}\Big) + r\Big(\frac{u_j^{n+1}+u_j^n}{2}\Big) = 0
\end{aligned}
\tag{9}
$$

PosPre is solving $\widehat{u}_\tau - \frac{1}{2}\sigma_{BS}^2 x^2 \widehat{u}_{xx} = 0$ arising from equation (6) through the transformation $x(S,\tau) = \mathrm{e}^{r\tau} S$ and $\widehat{u}(x,\tau) = \mathrm{e}^{r\tau} U(S,\tau)$ used in [**13**]. PosPre for (6) is highly non-standard.

NFDM is explicit, i.e., has short computational time, but is only conditionally stable. It is non negative and non oscillating and has order of convergence $\mathcal{O}(\mathrm{d}t + \mathrm{d}S)$.

ImpUp is implicit, i.e., has long computational time, but is unconditionally stable. It is non oscillating and has order of convergence $\mathcal{O}(\mathrm{d}t + \mathrm{d}S)$.

PosPre is explicit, i.e., has short computational time, and is unconditionally stable. It is non-negative and non-oscillating and has order of convergence $\mathcal{O}(\mathrm{d}t + \mathrm{d}S^2 + \frac{\mathrm{d}t}{\mathrm{d}S^2})$, and hence is only conditionally consistent with the condition $\frac{\mathrm{d}t}{\mathrm{d}S^2} \longrightarrow 0$

as $\mathrm{d}t$ and $\mathrm{d}S$ go to zero, but the tough condition $\mathrm{d}t = \mathcal{O}(\mathrm{d}S^4)$ in order not to lose orders of convergence compared to the standard FtCS method.

FtCS is explicit, i.e., has short computational time, but is only conditionally stable. It may oscillate (but in practice only around $S = S_{\max}$) and has order of convergence $\mathcal{O}(\mathrm{d}t + \mathrm{d}S^2)$.

CN is implicit, i.e., has long computational time, but is unconditionally stable. It in not L-Stable and may oscillate (in practice both at $S = K$ and at $S = S_{\max}$) and has order of convergence $\mathcal{O}(\mathrm{d}t^2 + \mathrm{d}S^2)$.

The proof of the properties of NFDM, FtCS and CN for the Barles and Soner model will be presented elsewhere. The properties of ImpUp are shown in [**10**] and the properties of PosPre are shown in [**13**].

4. Numerical Results

We reuse the parameter values from [**10**] $\gamma = 0$, $r = 0.1$, $\sigma_0 = 0.2$, $K = 40$, $T = 1$, $S_{\max} = 80$ and $B = 1$. We compare results obtained with NFDM, ImpUp, PosPre, FtCS and CNR (see (8–9)) with these parameter values. For the transaction cost parameter a, we consider the values 0 (linear Black-Scholes), 0.02 and 0.05 (considered in [**10**]) and also 0.1 and 0.4 (as extreme values). As it turns out, $S_{\max} = 2K = 80$ is sufficient to avoid oscillations in FtCS and CN at $S = S_{\max}$. Since in any case a computational domain including $S \in [0, 2K]$ would seem reasonable, the non-oscillatory methods provide no advantage when it comes to reducing the size of the computational domain.

The first step in solving the Barles and Soner model is the solution of the nonlinear ODE (5). This equation is solved in [**1**] with the "ode45" solver in MATLAB based on the well-known Ruge-Kutta-Fehlberg 45 scheme. Instead, we follow the approach from [**10**] using an implicit exact solution derived in [**5**]. We then use Maple's `fsolve` command to find specific values of Ψ. The implicit exact solution takes the form

$$\sqrt{|x|} = \begin{cases} \dfrac{-\sinh^{-1}(\sqrt{\Psi(x)})}{\sqrt{\Psi(x)+1}} + \sqrt{\psi(x)} & \text{for } x > 0, \\[2ex] \dfrac{-\sin^{-1}(\sqrt{-\Psi(x)})}{\sqrt{\Psi(x)+1}} - \sqrt{-\psi(x)} & \text{for } x < 0. \end{cases} \tag{10}$$

We begin our numerical experiments by illustrating the effect of the transaction cost parameter a on the initial option price at $t = 0$ that we compute with NFDM for $h = \mathrm{d}S = 2$, $k = \mathrm{d}t = 0.00078125$ and $a = 0, 0.02, 0.05, 0.1$ and 0.4. The results for the bet (digital call) option are very similar to those for the put and call which are almost identical because of the put-call-parity. Hence we only show results for the put option. In the top left of Figure 1, we show the option price $V(S, 0) = U(S, T)$ for the put option and in the top right we show the difference between the nonlinear put option solved by NFDM and the exact solution to the linear put option ($a = 0$). Similarly, the bottom row shows the difference between two Greeks for the nonlinear put option and the exact Greeks for the linear put option. The

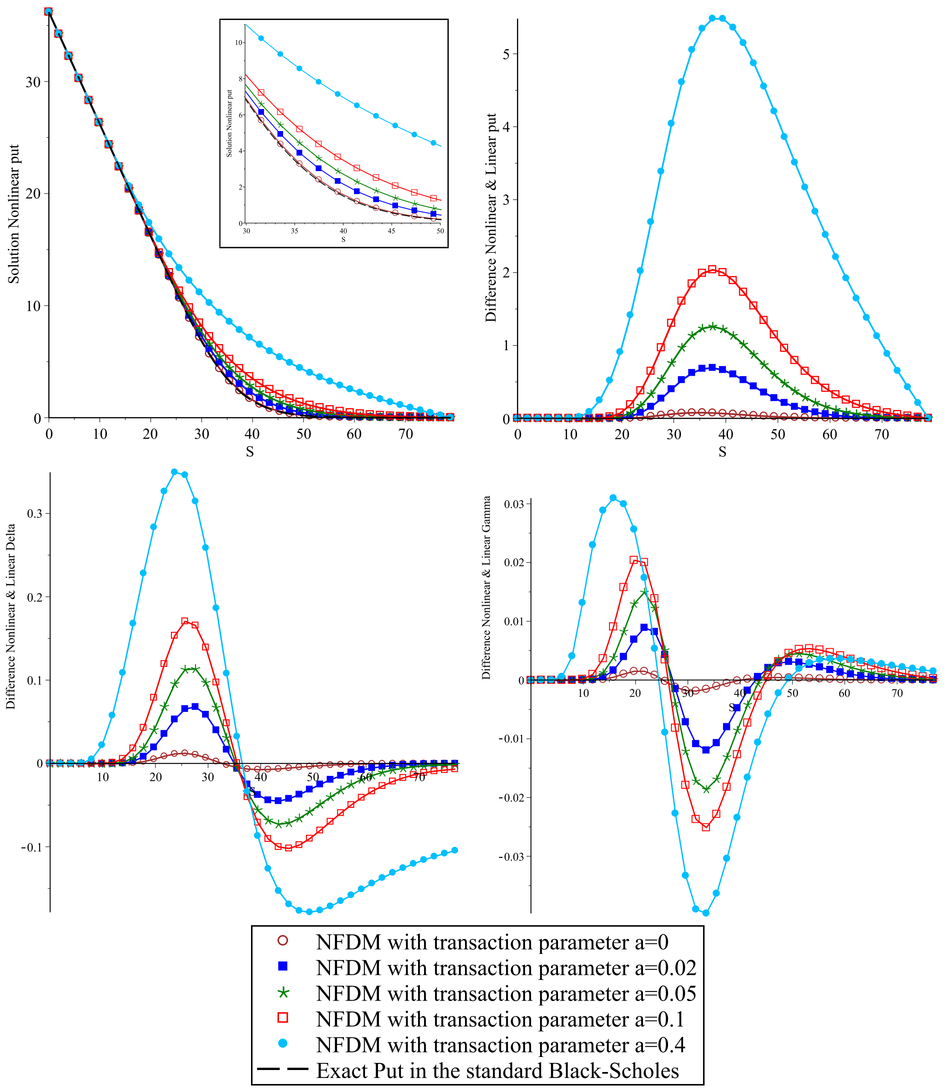

Figure 1. Nonlinear European Put option solved with NFDM for different transaction cost parameters at time 0 and differentiated with second order finite differences. Top left: Numerical option price. (Insert shows $(S, V) \in [30, 50] \times [0, 11]$). Top right: Numerical option price minus exact option price in the linear case $a = 0$. Bottom left: Numerical option Delta minus exact option delta in the linear case $a = 0$. Bottom right: Numerical option Gamma minus exact option gamma in the linear case $a = 0$.

Greeks (Delta ($\frac{\partial V}{\partial S}(S, 0) = \frac{\partial U}{\partial S}(S, T)$) and Gamma ($\frac{\partial^2 V}{\partial S^2}(S, 0) = \frac{\partial^2 U}{\partial S^2}(S, T)$)) are computed from the NFDM solution using second order finite differences. Note that only for the case $a = 0$, (the curves marked by circles in Figure 1) the differences actually constitute a numerical error. For other values of a, the difference is only used for scaling. Figure 1 shows that bigger transaction cost parameters cause more extreme solution values without otherwise changing the overall picture. It should be noted, however, that increasing the transaction cost parameter a heavily influences the stability condition for NFDM. For $\mathrm{d}S = 2$ and $a \leq 0.1$ $\mathrm{d}t = 0.0125$ is sufficient, whereas for $a = 0.4$, it has been necessary to take $\mathrm{d}t = 0.00078125$ in order to get a stable numerical solution. In Figure 2 we show the stability regions for NFDM for $a = 0.02$ and $a = 0.4$, respectively, with the boundary consisting of unstable points $(\mathrm{d}S, \mathrm{d}t = 0.1 \cdot 2^{-j})$ such that $(\mathrm{d}S, \mathrm{d}t = 0.1 \cdot 2^{-(j+1)})$ is stable. For comparison we have included also the stability boundary for the classical FtCS. The results indicate that explicit FDM's become increasingly problematic with increasing transaction cost parameter a. In order to maintain a reasonably sized stability region within the computational domain, we compare the various methods for a low transaction cost parameter $a = 0.02$ below.

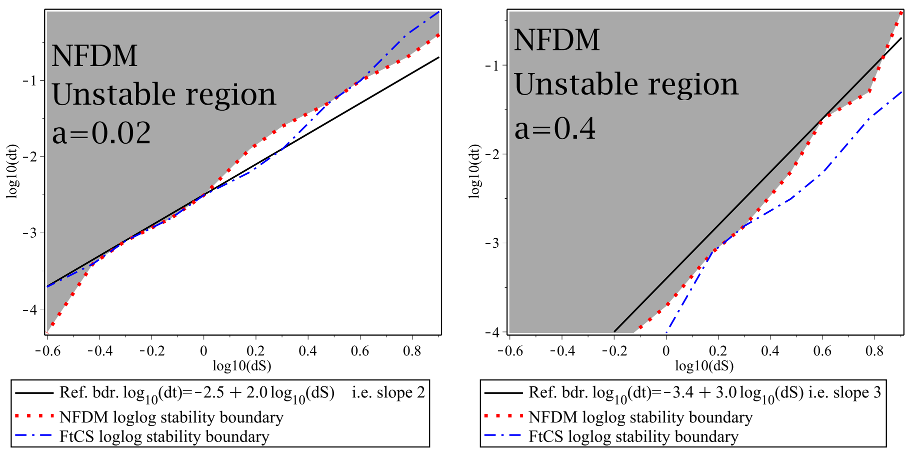

Figure 2. Log-Log-Stability region for NFDM and FtCS with transaction cost parameter $a = 0.02$ (Left figure) and $a = 0.4$ (Right figure).

For a systematic comparison of the 5 finite difference methods we shall compute *base solutions* (**BasSol**) with a series of meshes with $\mathrm{d}S \in [0.5, 8.0]$ and $\mathrm{d}t \in [0.0002, 0.1]$. We need an estimate of the exact solution which is not known for the nonlinear models. The estimate will be based on a *Fine mesh reference solution* (**RefSol**) computed on a mesh with $\mathrm{d}S = 0.375$ and $\mathrm{d}t \simeq 0.0001$ which is finer than the base meshes. The fine mesh is defined as our limit for computational feasibility and takes several days to compute in our Maple setup. RefSol will either be computed with the method itself or with the CNR method which is found to be the better performing method for the linear problem ($a = 0$). (These results will be presented elsewhere).

We consider the *local error estimate* $E(S) = \mathrm{RefSol} - \mathrm{BasSol}$ at any nodal point $S \in [0, S_{\max}]$ and at time $t = 0$ which is a time discretization point for all methods.

Instead the S grid of RefSol is finer than that of BasSol and generally does not have overlapping nodes. When nodes are not overlapping a linear interpolation between the 2 closest neighbors is then performed resulting in a RefSol in the same nodal points as BasSol. We also consider the *global error estimate* $E_\infty = \max_S |E(S)|$, the maximum taken over all S-nodal points for BasSol.

The local error estimate $E(S)$ with transaction cost parameter $a = 0.02$ at time $t = 0$ and with CNR as reference method is shown in Figure 3 for step sizes $h = \mathrm{d}S = 2$ and $k = \mathrm{d}t = 0.0125$. For this particular snapshot FtCS is providing

Call Bet

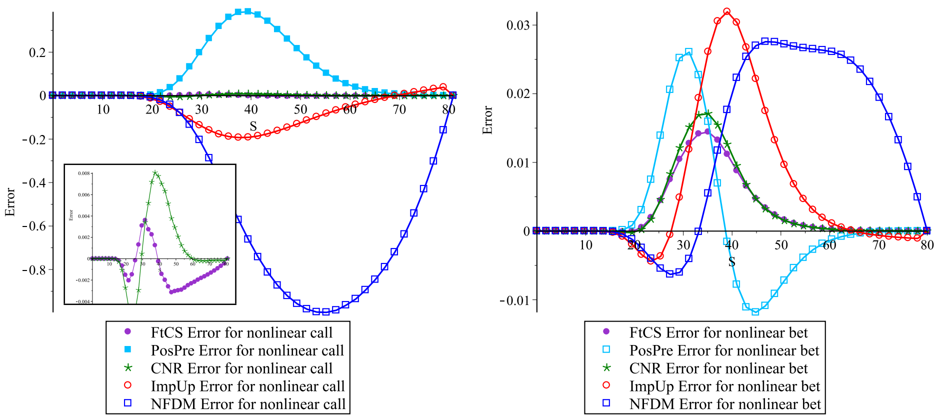

Figure 3. Local estimated errors $E(S)$ for nonlinear European option solved with NFDM, ImpUp, PosPre, FtCS and CNR with $\mathrm{d}S = 2$ and $dt = 0.0125$ for transaction cost parameter $a = 0.02$. The reference solution is computed with CNR for a fine mesh with $\mathrm{d}S = 0.375$ and $\mathrm{d}t = 0.00009765625$. Left: Call option. (Insert shows FtCS and CNR with $(S, E(S)) \in [0, 80] \times [-0.0055, 0.008]$). Right: Bet option.

the smallest global error estimate, closely followed by CNR while NFDM, ImpUp and PosPre are falling significantly behind. Note that the 10 times smaller $E(S)$ for the bet option than for the call option is countered by the 40 times smaller maximal solution value for the bet ($V_{\max}^{\text{bet}} = 1$) than for the call ($V_{\max}^{\text{call}} = 40$).

For a more thorough investigation, the global error estimate is computed for call and bet options for all 5 FDM's, and for all BasSol using the FDM itself for RefSol. A selection of convergence plots with logarithmic axes showing the global error estimate E_∞ for the call option solved with NFDM, PosPre, FtCS and CNR, and for all BasSol are shown in Figure 4. The bet options are omitted since they show very similar results. ImpUp is omitted since in the linear case ($a = 0$) it shows a behavior very similar to CNR only with significantly bigger errors (smaller order of convergence).

Since the exact solution is not known for the nonlinear options, we estimate convergence orders based on the numerical solutions in two different ways: Tables 1–2 illustrate the convergence of the error $e_R \simeq CR^q$ (not knowing whether

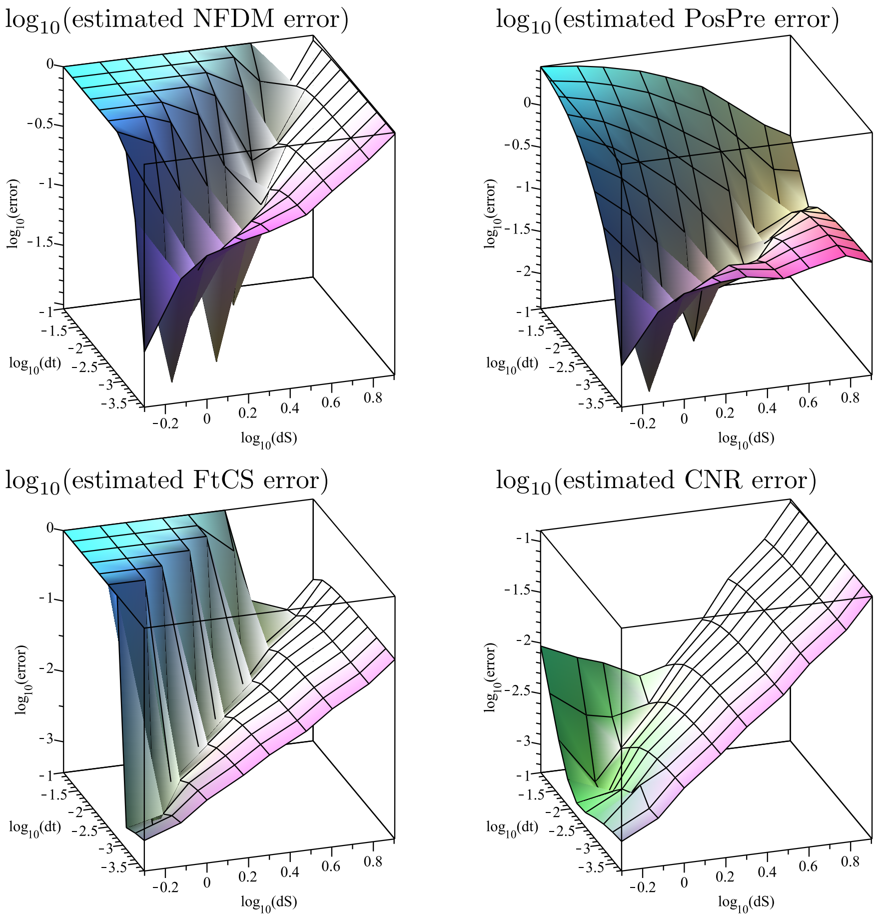

Figure 4. Global estimated errors E_∞ for nonlinear European Call options when solved with NFDM (top left), PosPre (top right), FtCS (bottom left) and CNR (bottom right) for transaction cost parameter $a = 0.02$ at time $t = 0$. The reference solution is computed with NFDM, PosPre, FtCS and CNR, respectively, for a fine mesh with $\mathrm{d}S = 0.375$ and $\mathrm{d}t = 0.00009765625$.

R is $\mathrm{d}S$ or $\mathrm{d}t$) for NFDM, PosPre, FtCS and CNR when halving both step sizes in each iteration so that Ratio $= 2^q = \frac{|e_h - e_{h/2}|}{|e_{h/2} - e_{h/4}|}$. We obtain quadratic convergence for FtCS and CNR, linear but decreasing order of convergence for NFDM and sublinear but increasing order of convergence for PosPre. Clearly, neither NFDM nor PosPre stabilize with respect to order of convergence within the computationally feasible domain. We also propose a more thorough approach focusing on convergence order in both variables S and t simultaneously and utilizing data from all BasSol in a weighted least squares approximation over all stable data points

Table 1. Convergence results for the nonlinear call option solved with NFDM and FtCS.

Nodes		NFDM			FtCS		
S	t	*Error*	*Difference*	*Ratio*	*Error*	*Difference*	*Ratio*
10	320	0.969558			0.126505		
20	640	0.362126	0.607432		0.032240	0.094265	
40	1280	0.161670	0.200456	3.03	0.009176	0.023064	4.09
80	2560	0.090007	0.071663	2.80	0.002706	0.006470	3.56
160	5120	0.026759	0.063248	1.13	0.000986	0.001720	3.76

Table 2. Convergence results for the nonlinear call option solved with CNR and PosPre.

Nodes		*CNR*			*PosPre*		
S	t	*Error*	*Difference*	*Ratio*	*Error*	*Difference*	*Ratio*
10	320	0.127837			0.077035		
20	640	0.032952	0.094885		0.075434	0.001601	
40	1280	0.009476	0.023476	4.04	0.065135	0.010299	0.16
80	2560	0.002836	0.006640	3.53	0.040396	0.024739	0.42
160	5120	0.001026	0.001810	3.66	0.011714	0.028682	0.86

$(\mathrm{d}S_i, \mathrm{d}t_j, (E_\infty)_{i,j})$ of the form

$$\min_{a,b,\alpha,\beta} \sum_{i,j} w_{i,j} \cdot ((E_\infty)_{i,j} - (a \cdot \mathrm{d}S_i^\alpha + b \cdot \mathrm{d}t_j^\beta))^2. \tag{11}$$

The stepsizes are recorded so that they decrease with increasing index, i.e., $\mathrm{d}S_{i+1} \leq \mathrm{d}S_i$ and $\mathrm{d}t_{j+1} \leq \mathrm{d}t_j$, and the simple weight function $w_{i,j} = i \cdot j$ putting higher weight on smaller step sizes is applied. Obviously, selecting a different weight function may change the results somewhat. Points where the error is increasing with decreasing $\mathrm{d}S$ are omitted. In the least squares minimizations the side conditions $0 \leq a$, $0 \leq b$, $0 \leq \alpha \leq 2.5$ and $0 \leq \beta \leq 2.5$ are imposed. The following convergence orders are computed. For the implicit PosPre and CNR the results in the square braces $[\cdot]$ are for the "bubbles" for large $\mathrm{d}t$ and small $\mathrm{d}S$ (see figure 4):

$$\begin{array}{lll} \text{NFDM} & E_\infty^{\text{Call}} = 0.067\mathrm{d}S^{1.3} & E_\infty^{\text{Bet}} = 0.012\mathrm{d}S^{1.1} \\ \text{FtCS} & E_\infty^{\text{Call}} = 0.002\mathrm{d}S^{2.0} & E_\infty^{\text{Bet}} = 0.008\mathrm{d}S^{1.1} \\ \text{CNR} & E_\infty^{\text{Call}} = 0.002\mathrm{d}S^{2.0}\ [0.089\mathrm{d}t^{1.1}] & E_\infty^{\text{Bet}} = 0.007\mathrm{d}S^{1.2} \\ \text{PosPre} & E_\infty^{\text{Call}} = 0.058\ [7.170\mathrm{d}t^{0.5}] & E_\infty^{\text{Bet}} = \text{"Oscillating error"} \end{array} \tag{12}$$

The oscillating error for the PosPre bet is consistent with the conditional consistency condition $\frac{\mathrm{d}t}{\mathrm{d}S^2} \longrightarrow 0$, indicating that smaller $\mathrm{d}t$'s are required.

We venture the following conclusions based on the graphs and the least squares calculations: The implicit methods PosPre and CNR show convergence with decreasing $\mathrm{d}t$ of order 0.5 and 1.1, respectively, in the bubble. For the explicit methods NFDM and FtCS, the bubbles are covered by the instability region. In order to get errors below 0.001, smaller values of $\mathrm{d}S$ are required which is computationally infeasible. A better alternative to investigate is obviously mesh grading the S-mesh, or some transformation method as suggested in [**12**] (ch. 11, for a nonlinear jumping volatility model) but this is beyond the scope of the current article. For the call option FtCS and CNR show the expected quadratic convergence with $\mathrm{d}S$ outside the bubble whereas NFDM only shows an order slightly above linear and PosPre shows no convergence at all. For the bet option both FtCS, CNR and NFDM show orders of convergence in $\mathrm{d}S$ slightly above linear. The stability area for FtCS is comparable to the one for NFDM (see Figure 2). Comparing the methods clearly FtCS and CNR stand out above the rest, the choice being whether to go for the fast computational times for the explicit FtCS, dealing with the instability area increasing in size with the transaction cost parameter a and reducing the usable area for the explicit methods to approximately $0 \leq a \leq 0.1$, or whether to accept the longer computational times of the implicit CNR, avoiding the concerns about feasible $\mathrm{d}t$–$\mathrm{d}S$ combinations for the implicit, absolutely stable CNR.

5. Conclusions

The "classical Explicit Euler" is the better explicit method and the "classical with a twist Crank-Nicolson with Rannacher time stepping" is the better implicit method among the FDM's tested on the Barles and Soner nonlinear Black-Scholes model. The non-oscillating methods do not offer any enhancement of the performance in the cases considered. The explicit methods suffer from an instability region growing with the transaction cost parameter a rendering them somewhat useless for $a \geq 0.1$. None of the methods considered shows significant convergence with $\mathrm{d}t$ — the error from the S-direction dominating except for the coarsest time step sizes. Because of computer time limitations, the brute force solution (smaller $\mathrm{d}S$) seems out of reach pointing instead towards graded S-meshes as the obvious solution to get in the ideal zone where the errors from t and S are balanced.

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