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EMPIRICAL ESTIMATES IN STOCHASTIC PROGRAMS WITH PROBABILITY AND SECOND ORDER STOCHASTIC DOMINANCE CONSTRAINTS

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Abstract. Stochastic optimization problems with an operator of the mathematical expectation in the objective function, probability and stochastic dominance constraints belong to "deterministic" problems depending on a probability measure (for a definition of the probability and stochastic dominance constraints, see, e.g., [**1**], [**12**] or [**18**]). Complete knowledge of the probability measure is a necessary condition for solving these problems. However, since this assumption is very rarely fulfilled (in applications), problems are mostly solved on the basis of data. Mathematically it means that the "underlying" probability measure is replaced by an empirical one (determined by the corresponding data). Stochastic estimates of an optimal value and an optimal solution can only then be obtained. Properties of these estimates have been investigated many times, mostly in the case of constraint sets not depending on the probability measure. Our results generalize such estimates to two separate cases (already mentioned above) when the constraint sets do depend on the probability measure.

We focus on the case of heavy-tailed distributions. First we try to emphasize the results achieved (for the above-mentioned problems) in the cases of the both light- and heavy-tailed distributions. However, the aim of this paper is mainly to analyze the case of second order stochastic dominance constraints for heavy tailed distributions. Namely, it seems that troubles that are not usual in the case of the light-tailed distributions can arise. The heavy-tailed distributions (and especially stable distributions; for the definition, see, e.g., [**8**]) correspond to many economic and financial applications (see, e.g., [**10**], [**13**]). Consequently, to include their case in the investigation is evidently very desirable. Theoretical analysis is completed by a simulation investigation.

1. Introduction

Let $(\Omega, \mathcal{S}, P)$ be a probability space, $\xi := \xi(\omega) = (\xi_1(\omega), \ldots, \xi_s(\omega))$ a random vector of dimension s defined on $(\Omega, \mathcal{S}, P)$, $F := F_\xi$ the distribution function of ξ, P_F, and Z_F the probability measure and the support corresponding to F, respectively. Let, moreover, $g_0 : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^1$ be a real-valued function, $X_F \subset X \subset \mathbb{R}^n$

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a nonempty set generally depending on F, $X \subset \mathbb{R}^n$ a nonempty "deterministic" set. If E_F denotes the operator of mathematical expectation corresponding to F and if for $x \in X$, there exists $\mathsf{E}_F g_0(x, \xi)$, then a rather general one-stage (static) stochastic optimization problem can be introduced in the form, to find

$$\varphi(F, X_F) = \inf\{\mathsf{E}_F g_0(x, \xi) | x \in X_F\}. \tag{1}$$

In this paper, we focus on two types of the constraint sets:

1. *individual probability constraints*

$$X_F := X_F(\alpha) := \bigcap_{i=1}^{s} \{x \in X : P[\omega | g_i(x) \le \xi_i] \ge \alpha_i\}, \tag{2}$$

   where $\alpha_i \in (0, 1)$, $i = 1, \dots, s$, $\alpha = (\alpha_1, \dots, \alpha_s)$, and $g_i \colon \mathbb{R}^n \to \mathbb{R}^1$, $i = 1, \dots, s$;

2. *second-order stochastic dominance constraints.* Let $g \colon \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^1$, $Y \colon \mathbb{R}^s \to \mathbb{R}^1$ be such that $g(x, \xi)$ is a random value for every $x \in X$. If

$$F^{(2)}_{g(x,\xi)}(u) := \int_{-\infty}^{u} F_{g(x,\xi)}(y) \mathrm{d}y, \qquad F^{(2)}_{Y}(u) := \int_{-\infty}^{u} F_Y(y) \mathrm{d}y, \quad u \in \mathbb{R}^1,$$

   then a second order stochastic dominance constraint can be defined by

$$X_F := \{x \in X : F^{(2)}_{g(x,\xi)}(u) \le F^{(2)}_{Y}(u) \text{ for every } u \in \mathbb{R}^1\}. \tag{3}$$

   (For more information about stochastic dominance, see, e.g., [**18**].)

In applications we often must replace the measure P_F with an empirical measure P_{F^N} determined from a random sample (not necessary independent) corresponding to the measure P_F. Consequently, instead of Problem (1), the following problem is often solved

$$\text{to find } \varphi(F^N, X_{F^N}) = \inf\{\mathsf{E}_{F^N} g_0(x, \xi) : x \in X_{F^N}\}. \tag{4}$$

Solving (4), we obtain (empirical) estimates of the optimal value and optimal solutions of Problem (1). The aim of this paper is firsty, to investigate the empirical estimates in the case of light- and heavy-tailed distributions. We shall measure the quality of this approximation by the convergence rate of the optimal value approximation. Of course, by employing the growth condition, we can obtain results concerning the optimal solution. To this end, the approach of [**15**] can be employed. However, the aim of this paper is to focus on analysis of the problems with second order stochastic dominance constraints. This is where very serious troubles can arise.

According to the above-mentioned plan, the paper is organized as follows. First, we recall some auxiliary assertions (Section 2), and a brief survey of empirical estimates (corresponding to the constraint sets (2) and (3)) can be found in Section 3. Section 4 is devoted to an analysis of the second order stochastic dominance problems in the case of heavy-tailed distributions. There a new serious problem will be introduced. The paper is completed with a conclusion (Section 5).

2. Some Definitions and Auxiliary Assertions

Problem (1) depends on the distribution function F. Replacing F by another s-dimensional distribution function G, we obtain a modified problem. Employing triangular inequality, we have

$$(5)\qquad |\varphi(F, X_F) - \varphi(G, X_G)| \le |\varphi(F, X_F) - \varphi(G, X_F)| + |\varphi(G, X_F) - \varphi(G, X_G)|.$$

To recall the first auxiliary assertion, let $\mathcal{P}(\mathbb{R}^s)$ denote the set of all (Borel) probability measures on $\mathbb{R}^s$ and let the system $\mathcal{M}_1^1(\mathbb{R}^s)$ be defined by the relation

$$(6)\qquad \mathcal{M}_1^1(\mathbb{R}^s) := \left\{ \nu \in \mathcal{P}(\mathbb{R}^s) | \int_{\mathbb{R}^s} \|z\|_1^s \mathrm{d}\nu(z) < \infty \right\},$$

where $\|\cdot\|_1^s$ denotes $\mathcal{L}_1$ norm in $\mathbb{R}^s$. If the assumption A.0 is defined as

A.0 $g_0(x, z)$ is, for $x \in X$, a Lipschitz function of $z \in \mathbb{R}^s$ with the Lipschitz constant L (corresponding to the $\mathcal{L}_1$ norm) not depending on x,

and if F_i, $i = 1, \dots, s$ denotes one-dimensional marginal distribution functions corresponding to F, then the following assertion has been proven.

Proposition 1 ([**5**]). *Let $P_F, P_G \in \mathcal{M}_1^1(\mathbb{R}^s)$. If Assumption A.0 is fulfilled, then*

$$(7)\qquad |\mathsf{E}_F g_0(x, \xi) - \mathsf{E}_G g_0(x, \xi)| \le L \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \mathrm{d}z_i, \quad x \in X.$$

To recall equivalent forms of the constraint sets X_F, let $k_F(\alpha)$ be defined by the following relations:

$$(8)\qquad \begin{aligned} k_F(\alpha) &= (k_{F_1}(\alpha_1), \dots, k_{F_s}(\alpha_s)),\ \alpha = (\alpha_1, \dots, \alpha_s), \\ k_{F_i}(\alpha_i) &= \sup\bigl\{ z_i | \mathbb{P}_{F_i}\{\omega | z_i \le \xi_i(\omega)\} \ge \alpha_i \bigr\}, \quad \alpha_i \in (0, 1),\ i = 1, \dots, s. \end{aligned}$$

Lemma 1 ([**4**]). *Let $g_i(x)$, $i = 1, \dots, s$ be continuous functions defined on $\mathbb{R}^n$, P_{F_i}, $i = 1, \dots, s$ absolutely continuous w. r. t. the Lebesgue measure on $\mathbb{R}^1$. Let, moreover, X_F be defined by (2), then*

$$X_F = \bar{X}(k_F(\alpha)),$$

where

$$(9)\qquad \bar{X}(v) = \bigcap_{i=1}^{s} \{x \in X : g_i(x) \le v_i\}, \quad v = (v_1, \dots, v_s),\ v \in \mathbb{R}^s.$$

Lemma 2 ([**7**]). *Let for every $x \in X$, $g(x, z)$, $Y(z)$ be Lipschitz functions of $z \in \mathbb{R}^s$ with the Lipschitz constant L_g not depending on $x \in X$. Let, moreover, $P_F, P_G \in \mathcal{M}_1^1(\mathbb{R}^s)$. If X_F is defined by the relation (3), then*

1. *$X_F = \{x \in X : E_F(u - g(x, \xi))^+ \le \mathsf{E}_F(u - Y(\xi))^+,\ u \in \mathbb{R}^1\}$;*
2. *$(u - g(x, z))^+$, $(u - Y(z))^+$ are Lipschitz functions of $z \in \mathbb{R}^s$ with the Lipschitz constant L_g not depending on $u \in \mathbb{R}^1$, $x \in \mathbb{R}^n$;*

3. *for* $u \in \mathbb{R}^1$, $x \in X$, *it holds that*

$$|\mathsf{E}_F(u - g(x,\xi))^+ - \mathsf{E}_G(u - g(x,\xi))^+| \le L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \mathrm{d}z_i,$$

$$|\mathsf{E}_F(u - Y(\xi))^+ - \mathsf{E}_G(u - Y(\xi))^+| \le L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \mathrm{d}z_i.$$

To complete this auxiliary section, let us recall the notions of light- and heavy-tailed distributions. The distribution function F_η of a random variable η (defined on $(\Omega, \mathcal{S}, P)$) has light tails if its finite moment generating function exists in a neighbourhood of zero; η has the distribution function with heavy tails if its finite moment generating function does not exist (for the definition of the moment generating function see, e. g., [**2**]). Consequently, a light-tailed distribution has all moments and they are finite. On the other hand, finiteness of all moments does not guarantee a finite moment generating function. Other (not quite equivalent) definitions of light- and heavy-tailed distributions can be found in the literature. However, stable distributions (with the exception of normal ones) always belong to the class of the heavy-tailed distributions.

Stable distributions are characterized by four parameters: index of stability $\nu \in (0, 2]$ that says how heavy the tails of the distributions are; the scale parameter $\sigma \ge 0$; the skewness parameter $\beta \in [-1, 1]$, and the shift parameter $\mu \in \mathbb{R}^1$. The stable distribution with parameters ν, σ, β, μ can be denoted by the symbol $S_\nu(\sigma,\ \beta,\ \mu)$. A stable distribution is Gaussian when $\nu = 2$, and in this case σ, is proportional to the standard deviation, β can be taken as zero and μ is the mean. It is known that probability densities of stable random variables exist and are continuous but with a few exceptions they are not known in closed forms. Moreover, it is known that a finite first moment exists if $\nu > 1$.

The following simple assertion is important for the problem of portfolio selection [**18**]. We shall see (in Section 4) that this assertion will be very important for the demonstration of our new problem introduced there.

Lemma 3. *Let* ξ_i, $i = 1, \dots s$ *be independent and moreover,* $S_\nu(\sigma_i,\ \beta_i,\ \mu_i)$, $x_i \ge 0$, $i = 1, \dots, s$, $x = (x_1, \dots, x_n)$. *If* $s = n$, $g(x, \xi) = \sum_{i=1}^{s} x_i \xi_i$, $Y(\xi) = \sum_{i=1}^{s} \frac{1}{s} \xi_i$, *then*

$g(x, \xi)$ *is* $S_\nu(\sigma, \beta, \mu)$, $\quad x \in X \quad$ *with*

$$\sigma = \Big[\sum_{i=1}^{s} (x_i \sigma_i)^\nu\Big]^{1/\nu}, \qquad \beta = \frac{\sum_{i=1}^{s} \operatorname{sgn}(x_i) \beta_i (x_i \sigma_i)^\nu}{\sum_{i=1}^{s} (x_i \sigma_i)^\nu}, \qquad \mu = \sum_{i=1}^{s} x_i \mu_i. \tag{10}$$

Consequently,

$$Y(\xi) \text{ is } \quad S_\nu(\sigma, \beta, \mu) \quad \text{with}$$

$$\sigma = \frac{1}{s}\Big[\sum_{i=1}^{s} \sigma_i^\nu\Big]^{1/\nu}, \ \beta = \frac{\sum\limits_{i=1}^{s} \beta_i(\sigma_i)^\nu}{\sum\limits_{i=1}^{s} (\sigma_i)^\nu}, \ \mu = \sum_{i=1}^{s} \frac{1}{s}\mu_i. \tag{11}$$

Proof. The assertion of Lemma 3 follows immediately from the standard results presented in [**17**, pp. 10–11]. □

(The symbol α is usually employed in the literature to represent index stability. We employ the symbol ν because the symbol α is already employed in the definition of probability constraints. For more details about stable distributions, see, e.g., [**8**]).

If we replace G by F^N in the previous section, then we can formulate auxiliary assertions concerning empirical estimates (in the case of $X_F = X$) that will be useful for the investigation of Problem (1) with more general constraint sets X_F. To this end, we first introduce the following system of assumptions:

A.1 $g_0(x, z)$ is either a uniformly continuous function on $X \times \mathbb{R}^s$, or X is a convex bounded set and there exists $\varepsilon > 0$ such that $g_0(x, z)$ is a convex bounded function on $X(\varepsilon)$, where $X(\varepsilon)$ denotes the ε-neighborhood of the set X;

A.2
- $\{\xi^i\}_{i=1}^{\infty}$ is an independent random sequence corresponding to F,
- F^N is an empirical distribution function determined by $\{\xi^i\}_{i=1}^{N}$, $N = 1, 2, \dots$,

A.3 P_{F_i}, $i = 1, \dots, s$ are absolutely continuous w. r. t. the Lebesgue measure on $\mathbb{R}^1$,

A.4 For every $i \in \{1, \dots, s\}$, there exist $\delta > 0$ and $\vartheta > 0$ such that $f_i(z_i) > \vartheta$ for $z_i \in Z_{F_i}$, $|z_i - k_{F_i}(\alpha_i)| < 2\delta$. ($f_i$ denotes the probability density corresponding to F_i, $i = 1, \dots, s$.)

Theorem 1 ([**3**]). *Let $P_F \in \mathcal{M}_1^1(\mathbb{R}^s)$, X be a compact set. If the assumptions A.0, A.1, A.2 and A.3 are fulfilled, then*

$$P\{\omega | |\varphi(F^N, X) - \varphi(F, X)| \xrightarrow[N\to\infty]{} 0\} = 1.$$

According to Theorem 1, we can see that $\varphi(F^N, X)$ is a a consistent estimate of $\varphi(F, X)$ in the case of "underlying" distribution F with finite first moments (under some additional assumptions). It means that these estimates are also consistent in the case of the heavy-tailed distributions (including stable distributions) as long as first absolute moments exist.

Theorem 2 ([**3**]). *Let it hold, for a certain $r > 2$, that $\mathsf{E}_{F_i}|\xi_i|^r < +\infty$, $i = 1, \dots, s$. Let, moreover, the constant γ fulfil the inequalities $0 < \gamma < 1/2 - 1/r$. If Assumptions A.2 and A.3 are fulfilled, and $P_F \in \mathcal{M}_1^1(\mathbb{R}^s)$, $t > 0$, then*

$$P\Big\{\omega : N^\gamma \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - F_i^N(z_i)| \mathrm{d}z_i > t\Big\} \xrightarrow[N\to\infty]{} 0. \tag{12}$$

If, moreover, X is a compact set, and Assumptions A.0 and A.1 are fulfilled, then also

$$P\big\{\omega : N^{\gamma}|\varphi(F, X) - \varphi(F^N, X)| > t\big\} \xrightarrow[N\to\infty]{} 0. \tag{13}$$

It follows from Theorem 2 that the rate of convergence γ (in the case of "deterministic" constraint sets) depends on the existence of finite moments. The following weaker assertion (covering stable distributions with $\nu \in (1, 2)$) is valid.

Theorem 3 ([**6**]). *Let Assumptions A.0, A.1, A.2 and A.3 be fulfilled, $P_F \in \mathcal{M}_1^1(\mathbb{R}^s)$, $M > 0$, and X be a compact set. Let the one-dimensional components ξ_i, $i = 1, \dots, s$ of the random vector ξ have stable distribution functions F_i with the indices of stability tails parameter $\nu_i \in (1, 2)$ fulfilling the relations*

$$\sup_{t>0} t^{\nu_i}\mathbb{P}\{\omega : |\xi_i| > t\} < \infty, \quad i = 1, \dots, s,$$

then

$$\lim_{M\to\infty}\sup_N \mathbb{P}\Big\{\omega : \frac{N}{N^{1/\nu}}|\varphi(F^N, X) - \varphi(F, X)| > M\Big\} = 0 \tag{14}$$

with $\nu = \min(\nu_1, \dots, \nu_s)$.

3. Empirical estimates

In this section, we deal with the empirical estimates in the case of the problems with constraint sets defined by relations (2) and (3).

3.1. Probability Constraints

Theorem 4 ([**7**]). *Let Assumptions A.2, A.3 and A.4 be fulfilled, $\alpha = (\alpha_1, \dots, \alpha_s)$, $\alpha_i \in (0, 1)$, $i = 1, \dots, s$, $t > 0$, and $X_F = X_F(\alpha)$ be defined by the relation (2). If*

1. *$\bar{X}(v)$ (defined by relation (9)) is a nonempty set for every $v \in Z_F$, and moreover, there exists a constant $\hat{C} > 0$ such that*

$$\Delta_n[\bar{X}(v(1)), \bar{X}(v(2))] \le \hat{C}\|v(1) - v(2)\|_2^s \qquad \text{for } v(1), v(2) \in Z_F,$$

2. *there exists $\gamma \in (0, 1/2)$ such that*

$$P\big\{\omega|\, N^{\gamma}|\varphi(F, X_F) - \varphi(F^N, X_F)| > t\big\} \xrightarrow[N\to\infty]{} 0, \tag{15}$$

3. *$g_0(x, z)$ is a Lipschitz function on X with the Lipschitz constant L' not depending on $z \in Z_F$,*

then

$$P\big\{\omega|\, N^{\gamma}|\inf_{X_F(\alpha)} \mathsf{E}_F g_0(x, \xi) - \inf_{X_{F^N}(\alpha)} \mathsf{E}_{F^N} g_0(x, \xi)| > t\big\} \xrightarrow[N\to\infty]{} 0.$$

Here $\Delta_n[\cdot, \cdot]$ denotes the Hausdorff distance of subsets $\mathbb{R}^n$; for the definition of the Hausdorff distance see, e.g., [**14**]; and $\|\cdot\|_2^s$ denotes the Euclidean norm in $\mathbb{R}^s$.

Remarks.

1. Assumption 3 of Theorem 4 can be replaced by the assumption: X is a nonempty, convex and bounded set, $g_0(x, z)$ is a convex function on X and there exist $M > 0, \varepsilon > 0$ such that $|g_0(x, z)| \le M$ for $x \in X(\varepsilon)$, $z \in Z_F$.
2. If the assumptions of Theorem 4 are fulfilled, and moreover, there exists a function $\hat{g}_0$ $(:= \hat{g}_0(x))$ defined on X such that $g_0(x, z) := \hat{g}_0(x)$, $x \in X$, $z \in Z_F$, then the assertions of Theorem 4 are valid with $\gamma \in (0, 1/2)$.

Assertion 2 in Remarks above follows from [**7**, Proposition 3.5 and Lemma 13], see also [**4**]. According to this assertion (when the objective function in Problem (1) does not depend on the probability measure), the convergence rate (under the corresponding assumptions) γ does not depend on the tails distribution, and $\gamma \in (0, 1/2)$ holds.

3.2. Second Order Stochastic Dominance Constraints

Let the finite $\mathsf{E}_F g(x, \xi)$, $\mathsf{E}_F Y(\xi)$ exists for $x \in X$. First we define the sets X^{ε}, $\varepsilon \in \mathbb{R}^1$ by

$$X_F^{\varepsilon} = \{x \in X : \mathsf{E}_F(u - g(x, \xi))^+ - \mathsf{E}_F(u - Y(\xi))^+ \le \varepsilon,\ u \in \mathbb{R}^1\}. \tag{16}$$

Theorem 5. *Let $P_F \in \mathcal{M}_1^1(\mathbb{R}^s)$, $t > 0$, X be a compact set, and Assumptions A.0, A.1, A.2 and A.3 be fulfilled. If*

1. - *$g(x, z)$ is a Lipschitz function of $z \in Z_F$ with the Lipschitz constant not depending on $x \in X$,*
   - *$g_0(x, z)$ is a Lipschitz function on X with the Lipschitz constant L' not depending on $z \in Z_F$,*
2. *there exists $\varepsilon_0 > 0$ such that X_F^{ε} are nonempty compact sets for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, and moreover, there exists a constant $\hat{C} > 0$ such that*

$$\Delta_n[X_F^{\varepsilon}, X_F^{\varepsilon'}] \le \hat{C}|\varepsilon - \varepsilon'| \qquad \text{for } \varepsilon, \varepsilon' \in [-\varepsilon_0, \varepsilon_0],$$

3. *for $r > 2$ it holds, that $\mathsf{E}_{F_i}|\xi_i|^r < +\infty$, $i = 1, \ldots, s$ and a constant γ fulfills the inequality*

$$0 < \gamma < 1/2 - 1/r,$$

then

$$P\{\omega : N^{\gamma}|\varphi(F, X_F^0) - \varphi(F^N, X_{F^N}^0)| > t\} \xrightarrow[N \to +\infty]{} 0. \tag{17}$$

Proof. Setting $G = F^N$ in relation (5), we obtain

$$\begin{aligned} &|\varphi(F, X_F^0) - \varphi(F^N, X_{F^N}^0)| \\ &\le |\varphi(F, X_F^0) - \varphi(F^N, X_F^0)| + |\varphi(F^N, X_F^0) - \varphi(F^N, X_{F^N}^0)|. \end{aligned} \tag{18}$$

According to Assumption 2, X_F^0 is a nonempty compact set. Consequently, from Theorem 2, it follows that

$$P\{\omega : N^{\gamma}|\varphi(F, X_F^0) - \varphi(F^N, X_F^0)| > t\} \xrightarrow[N \to +\infty]{} 0. \tag{19}$$

From Lemma 2, it follows

$$X_{F^N}^{\delta-\varepsilon(N)} \subset X_F^{\delta} \subset X_{F^N}^{\delta+\varepsilon(N)}$$

$$\text{with} \quad \varepsilon(N) = 2L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - F_i^N(z_i)| \mathrm{d}z_i, \ \delta \geq 0. \tag{20}$$

Since the Hausdorff distance is a metric in the space of compact subsets in $\mathbb{R}^n$ (see, e.g., [**16**]) we have

$$\Delta[X_F^0, X_{F^N}^0] \leq \Delta[X_F^0, \ X_F^{-\varepsilon(N)}] + \Delta[X_F^{-\varepsilon(N)}, X_{F^N}^0]. \tag{21}$$

According to (20), we can obtain that

$$X_F^{-\varepsilon(N)} \ \subset \ X_{F^N}^0 \ \subset X_F^{\varepsilon(N)} \tag{22}$$

and consequently, according to the definition of the Hausdorff distance, Assumption 2 and the relation (21), we can successively obtain

$$\begin{aligned} &\Delta[X_F^{-\varepsilon(N)}, X_{F^N}^0] \leq \Delta[X_F^{-\varepsilon(N)}, X_F^{\varepsilon(N)}] \leq C''\varepsilon(N), \\ &\Delta[X_F^0, X_{F^N}^0] \leq \bar{D}\varepsilon(N) \qquad \text{for } \varepsilon(N) < \varepsilon_0 \text{ and certain } \bar{D}, \ C'' > 0. \end{aligned} \tag{23}$$

Furthermore, since it follows from Assumption 1 that $\mathsf{E}_{F^N} g_0(x, \xi)$ is a Lipschitz function of $x \in X$ with the Lipschitz constant not depending on $\xi^1, \ldots, \xi^N$, and therefore also on $\omega \in \Omega$. Consequently, employing a slightly modified assertion of [**4**, Proposition 1] and Assumption 2, we obtain

$$|\varphi(F^N, X_F^0) - \varphi(F^N, X_{F^N}^0)| \leq D\varepsilon(N) \tag{24}$$

for $\varepsilon(N) < \varepsilon_0$ and a certain $D > 0$. According to Theorem 2 also

$$p\{\omega : \ N^{\gamma}|\varphi(F^N, X_F^0) - \varphi(F^N, X_{F^N}^0)| > t\} \xrightarrow[N \to \infty]{} 0.$$

Now the assertion of Theorem 5 follows from the last relation, and relations (18) and (19). □

From the assumptions of Theorem 5, it follows that for relation (17) to be valid, it is necessary that $r > 2$ holds. It means that a moment of random vector ξ has to be finite for $r > 2$. Evidently, stable distributions with the exception of normal ones do not fulfil this condition. A rather weaker assertion is valid if only finite first moments exist and are finite.

Theorem 6. *Let $P_F \in \mathcal{M}_1^1(\mathbb{R}^s)$, and X be a compact set. Let, moreover, Assumptions A.0, A.1, A.2 and A.3 be fulfilled. If*

1. • *$g(x, z)$ is a Lipschitz function of $z \in Z_F$ with the Lipschitz constant not depending on $x \in X$,*
   • *$g_0(x, z)$ is a Lipschitz function on X with the Lipschitz constant L' not depending on $z \in Z_F$,*
2. *there exists $\varepsilon_0 > 0$ such that X_F^{ε} are nonempty compact sets for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, and, moreover, there exists a constant $\hat{C} > 0$ such that*

$$\Delta_n[X_F^{\varepsilon}, \ X_F^{\varepsilon'}] \leq \hat{C}|\varepsilon - \varepsilon'| \quad \textit{for } \varepsilon, \varepsilon' \in [-\varepsilon_0, \varepsilon_0],$$

then

$$P\{\omega : |\varphi(F, X_F^0) - \varphi(F^N, X_{F^N}^0)| \xrightarrow[N\to\infty]{} 0.\} = 1. \tag{25}$$

Proof. Following the proof of Theorem 5, we can see that relation (24) can be written in the form

$$|\varphi(F^N, X_{F^N}^0) - \varphi(F^N, X_F^0)| \le D\varepsilon(N) \qquad \text{for some } D > 0,$$
$$\text{and} \quad \varepsilon(N) = 2L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - F_i^N(z_i)| \mathrm{d}z_i < \varepsilon_0.$$

Simultaneously, it follows from Proposition 1 that successively

$$|\mathsf{E}_F g_0(x, \xi) - \mathsf{E}_{F^N} g_0(x, \xi)| \le L \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - F_i^N(z_i)| \mathrm{d}z_i \qquad \text{for } x \in X,$$
$$|\varphi(F, X_F^0) - (F^N, X_F^0)| \le L \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - F_i^N(z_i)| \mathrm{d}z_i.$$

Consequently, we obtain

$$|\varphi(F, X_F^0) - \varphi(F^N, X_{F^N}^0| \le D^* \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - F_i^N(z_i)| \mathrm{d}z_i$$
$$\text{for some} \quad D^* > 0, \quad \text{and} \quad 2L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - F_i^N(z_i)| \mathrm{d}z_i < \varepsilon_0.$$

Since it has been proven in [**19**] that

$$P\{\omega | \int_{-\infty}^{\infty} |F_i(z_i) - F_i^N(z_i)| dz_i \xrightarrow[N\to\infty]{} 0\} = 1 \qquad i = 1, \ldots, s,$$

we can see that the assertion of Theorem 6 is valid. □

In Section 3, we introduced rather "pleasant" properties of empirical estimates in the case of the stochastic programming problems with individual probability constraints and second order stochastic dominance constraints. While it seems that in the case of the probability constraints no special problems arise dependent on the type of the tails, the situation can be much worse in the case of stochastic dominance constraints. Evidently, a new very serious problem can arise in the case if the "underlying" distribution has heavy tails. An especially horrible situation can appear in the case of "underlying" stable distributions.

4. Analysis of Second Order Stochastic Dominance Constraints

Let the assumptions of Lemma 2 be fulfilled. Employing the first result of Lemma 2, we can see that Problem (1) with constraint set (3) can be rewritten in the form, to find

$$\begin{aligned} &\varphi(F, X_F) = \inf\{\mathsf{E}_F g_0(x, \xi) : x \in X_F\}, \\ &\text{where } X_F = \{x \in X : \mathsf{E}_F(u - g(x, \xi))^+ - \mathsf{E}_F(u - Y(\xi))^+ \le 0, \ u \in \mathbb{R}^1\}. \end{aligned} \tag{26}$$

4.1. Approximation

Problem (26) is a problem of infinitesimal programming. It follows from the analysis presented, e.g., in [**1**] or in [**9**] that the Slater's condition is not fulfilled in this case. Consequently, the authors suggest replacing Problem (26) with a rather relaxed one, to find

$$\begin{aligned} &\varphi(F, X_F) = \inf\{\mathsf{E}_F g_0(x, \xi) : x \in X_F\}, \\ &\text{where } X_F = \{x \in X : \mathsf{E}_F(u - g(x, \xi))^+ - \mathsf{E}_F(u - Y(\xi))^+ \le 0, \ u \in [a, b]\} \end{aligned} \tag{27}$$

without a recommendation for how to choose constants $a, b \in \mathbb{R}^1$.

Let us now analyze relations (26) and (27) with respect to a choice of the parameters a, b. Evidently in relation (27), the following events are neglected: $Y(\xi) > b$ and $g(x, \xi) > b$ (for certain $x \in X$). It means that events with probability

$$P\{\omega | \, Y(\xi) > b \bigcup g(x, \xi) > b \quad \text{for certain} \quad x \in X\}$$

are neglected.

Evidently, this probability is 'connected' (in many cases) with the quantiles of $Y(\xi)$ and $g(x, \xi)$ following the relations (10), (11). The next table presents the quantiles corresponding to distributions with the densities of the following distributions: normal, Weibul, Pareto and lognormal, given by the corresponding following densities:

1. n – Normal distribution
2. W – Weibull distribution with probability density
$$f(z) = \begin{cases} \frac{c}{\nu}(\frac{z - z_0}{\nu})^{c-1} \exp\{-(z - z_{0)}/\nu)^c & \text{for } z > z_0 \\ 0 & \text{for } z \le z_0 \end{cases}$$
with $\nu = 1$, $z_0 = 0$, $c = 0$.
3. P – Pareto distribution is given with probability density
$$f(z) = \begin{cases} \alpha C^\alpha z^{-\alpha-1} & \text{for } z \ge C \\ 0 & \text{for } z < C \end{cases}$$
with $C = 1$, $\alpha = 2$.
4. L – lognormal distribution with $\mu = 0$, $\sigma = 0.5$, $k_1 = 0$.

The values in the next table are calculated for mean value 2 and variance 4 and corresponding distributions.

%	0.999999	0.99999	0.9999	0.999	0.99
p	0.0 – 179.07	0.0 – 68.99	0.0 – 26.58	0.0 – 10.24	0.0 – 3.95
w	0.0 – 27.63	0.0 – 23.03	0.0 – 1842	0.0 – 13.82	0.0 – 9.21
l	0.0 – 1455.05	0.0 – 739.18	0.0 – 346.82	0.0 – 145.06	0.0 – 50.31
n	−17.01 – 21.01	−15.06 – 19.06	−12.88 – 16.88	−10.36 – 14.36	−7.31 – 11.31

Table 1. Quantiles for normal, Weibull, Pareto and log-normal distributions..

It means that three of these distributions belong to the class of distributions with heavy tails. However, they do not belong to the class of stable distributions. Of course, the normal distribution belongs to the class of stable distributions, however, those with light tails.

Remark. Numerical results above were obtained by K. Odintsov [**11**].

Table 2 presents a similar analysis for stable distributions. In this case, the situation is much more relevant. The dependence of the values of the quantiles on the parameters values is crucial. Consequently, the wrong choice of the parameters a,b can cause a very bad approximation of the optimal value and the optimal solution.

In Table 2, we take four quantiles of the stable distribution $S_2(1,0,0) \sim N(0,\sqrt{2})$:

1. 95.00% quantile ($q_{95.00}$),
2. 99.00% quantile ($q_{99.00}$),
3. 99.50% quantile ($q_{99.50}$),
4. 99.99% quantile ($q_{99.99}$)

and calculate the value of distribution functions $F_{S_\alpha(1,0,0)}(\cdot)$ in these points for $\alpha = 1.05, 1.1, 1.15, \dots, 1.95, 2$. From Table 2, it follows that we have to take into account the role of heavy tails.

4.2. Crossing

To investigate Problem (1) within the constraints set X_F, it is necessary to require the set X_F to be nonempty. Let us investigate this requirement in the case of X_F given by the relation (3) with $g(x,\xi) = \sum_{i=1}^{s} x_i\xi_i$; $Y(\xi) = \sum_{i=1}^{s} \frac{1}{s}\xi_i$ and ξ_i, $i = 1, \dots s$, independent $S_\nu(\sigma_i,\, \beta_i,\, \mu_i)$, $X = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1,\ x_i \geq 0,$ $i = 1, \dots, n\}$. This constraint set evidently has its origin in the "classical" problem portfolio selection [**?**]. To find

$$x_i \geq 0,\, i = 1,\, \dots,\, s \quad \text{such that} \quad \sum_{i=1}^{s} x_i \leq 1 \tag{28}$$

$$\text{and simultaneously,} \quad \sum_{i=1}^{s} \mathsf{E}_F \xi_i x_i \quad \text{is maximized,}$$

where x_i is a proportion of the unit wealth invested in the asset i and ξ_i the return of the asset.

The distribution of $Y(\xi)$, $g(x, \xi)$, $x \in X$ can be determined by Lemma 3. To investigate the constraint sets X_F, we employ a simulation technique with $n = s = 2$, $\xi_1 \sim S_{1.5}(0.5, 0, 6)$ and $\xi_2 \sim S_{1.5}(1, 0, 0)$ and their linear combination $\zeta = a\xi_1 + (1-a)\xi_2$ for $a = 0.2, 0.5, 0.9$. In this context, by crossing of two variables U and Y, we mean that functions $F_U^{(2)}(x) = \int_{-\infty}^{x} F_U(y)\mathrm{d}y$ and $F_Y^{(2)}(x) = \int_{-\infty}^{x} F_Y(y)\mathrm{d}y$ cross. In Figure 1, the blue line corresponds to ξ_1 and the red line corresponds to ξ_2.
Let us consider cases where β is non zero:
$\xi_1 \sim S_{1.5}(1, 0.7, 0)$ and $\xi_2 \sim S_{1.5}(1, -0.8, 2)$. In Figure 2, we can see that ζ crosses ξ_1 and ξ_2 for $a = 0.2$ and $a = 0.5$ and besides these ζ-s (two green lines, Figure 2) cross each other.

In Figure 2, the blue line corresponds to ξ_1 and the red line corresponds to ξ_2.

Hence, it follows that from these results, the stochastic dominance of the first and second order can be violated in a portfolio selection.

α	$F_{S_\alpha(1,0,0)}(q_{95})$	$F_{S_\alpha(1,0,0)}(q_{99})$	$F_{S_\alpha(1,0,0)}(q_{99.5})$	$F_{S_\alpha(1,0,0)}(q_{99.99})$
1.05	0.876452	0.912226	0.920843	0.953008
1.10	0.881909	0.918085	0.926646	0.957888
1.15	0.887148	0.923668	0.932149	0.962372
1.20	0.892181	0.928994	0.937373	0.966491
1.25	0.897015	0.934080	0.942336	0.970273
1.30	0.901660	0.938941	0.947056	0.973743
1.35	0.906122	0.943594	0.951548	0.976926
1.40	0.910409	0.948050	0.955829	0.979842
1.45	0.914525	0.952325	0.959913	0.982513
1.50	0.918477	0.956428	0.963812	0.984956
1.55	0.922267	0.960371	0.967540	0.987188
1.60	0.925903	0.964165	0.971108	0.989225
1.65	0.929387	0.967817	0.974526	0.991081
1.70	0.932727	0.971336	0.977805	0.992769
1.75	0.935925	0.974728	0.980953	0.994301
1.80	0.938988	0.978001	0.983979	0.995689
1.85	0.941922	0.981160	0.986891	0.996943
1.90	0.944732	0.984210	0.989694	0.998072
1.95	0.947422	0.987155	0.992395	0.999085
2.00	0.950000	0.990000	0.995000	0.999990

Table 2. The values of the stable distribution functions $F_{S_\alpha(1,0,0)}(\cdot)$ in normal quantiles.

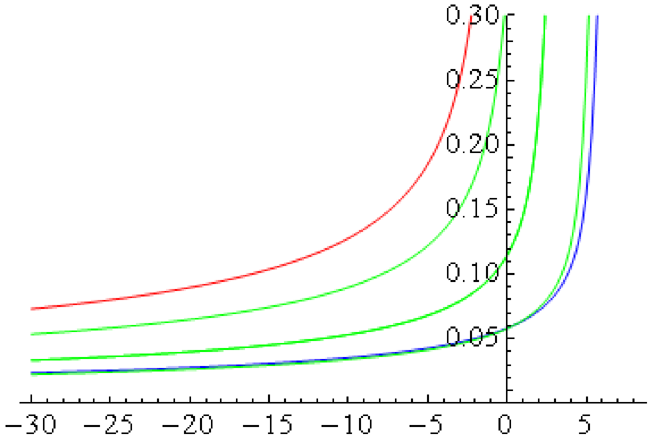

Figure 1. Example of crossing of portfolios with each other and both assets.

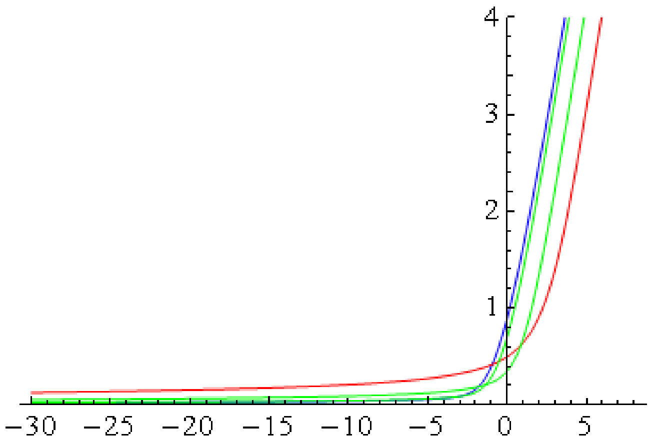

Figure 2. Example of crossing of portfolios..

5. Conclusion

The goal of this paper is to investigate stochastic programming problems with probability and stochastic dominance constraints, mostly in the cases when the "underlying" distributions belong to the class of heavy-tailed distributions. While some results for the case of the individual probability constraints are only recalled, a great attention is paid to problems with stochastic dominance constraints. First, a relationship between "theoretical" characteristics and those obtained on the basis of the data are investigated. Consistency is proven (under the assumption of the finite first moment) and furthermore, the convergence rate in dependence on finite moments is introduced. However, the most attention is paid to the approach of relaxation technique introduced in [**1**]. It is shown that the mentioned relaxation has to be done (in the case of heavy-tailed distributions) very carefully. Moreover, the paper reveals the difficulties in stochastic programming problems with first and second order stochastic dominance constraints. A dangerous of crossing arises here and it is rather imminent.

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