

IMPLICIT FRACTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES VIA PICARD AND WEAKLY PICARD OPERATOR THEORY

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ABSTRACT. In this paper, by employing fixed-point methods, we obtain the existence and uniqueness results for the nonlinear implicit fractional differential equations in Banach spaces. Further, we obtain the uniqueness, dependence of the solution on the initial condition as well as on the functions involved on the right-hand side by means of Picard and weakly Picard operator theory and Pompeiu-Hausdorff functional.

1. INTRODUCTION

The initial and the boundary value problems for fractional differential equations (FDEs) naturally appear in the modeling of various physical phenomena emerging in diverse disciplines of science and engineering [16, 17, 18, 30, 31, 41]. Subsequently, this field has been drawing attention of many mathematicians for researching numerous types of FDEs for its theory and applicability. Basics of fractional calculus, detailed theoretical analysis, and applications of FDEs can be found in interesting monographs by Diethelm [14], Kilbas et al. [22], Lakshmikantham et al. [26], and Podlubny [35]. Fundamental results on the existence and uniqueness of solutions, comparison results, different types of data dependency, the existence of extremal solutions, and global existence of the nonlinear FDEs can be found in [1, 13, 15, 25, 27, 28, 42]. Recently, there have been numerous papers that deals with the existence, uniqueness, controllability, and various qualitative properties of the solution for FDEs with initial and boundary conditions, [12, 19, 21, 36].

The implicit differential equation of the form

$$x^{(n)}(t) = f(t, x(t), x'(t), x''(t), \dots, x^{(n-1)}(t)),$$

with different kind of initial or boundary condition has been analyzed for existence, uniqueness, and various other qualitative properties of the solution, for instance,

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see [29] and the references cited therein. Motivated by the investigations in the theory of integer order implicit differential equation considered above and its different special form, Benchohra et al. [6, 7] and Nieto et al. [32] initiated the study of implicit FDEs of the form

$$D^\alpha x(t) = f(t, x(t), D^\alpha x(t)),$$

with different kinds of initial and boundary conditions. Nieto et al. [32] derived existence and uniqueness results by employing fixed point theory and an approximation method. Benchohra et al. [8, 9, 10] investigated the existence of integrable solutions for implicit fractional functional differential equations with and without delay. Kucche et al. in [23, 24, 40] analyzed implicit FDEs for existence, uniqueness, dependency of the solution on initial data, Ulam–Hyers and Ulam–Hyers–Rassias stabilities.

On the other hand, Rus [37, 39] introduced fundamental results of the weakly Picard operators theory, and employed it to analyze the Cauchy problem, boundary value problem, integral equations, and difference equations. Wang [43] obtained some existence, uniqueness, and data dependence results by means of Picard and weakly Picard operators theory, and the Bielecki norms. Then again, Otrocol and Ilea [33, 34] analyzed differential and integrodifferential equations with abstract Volterra operators using the method of Picard and weakly Picard operators.

Motivated by the works of [39, 43], in the present paper, we obtain the existence, uniqueness, and dependency of solution for the nonlinear implicit FDEs of the form

$$\begin{aligned} (1) \quad & {}_0^c \mathcal{D}_t^\alpha x(t) = f(t, x(t), {}_0^c \mathcal{D}_t^\alpha x(t)), \quad t \in J, 0 < \alpha < 1, \\ (2) \quad & x(0) = x_0 \in X, \end{aligned}$$

in the Banach space $X = (X, \|\cdot\|)$, where $J = [0, T], T > 0$, ${}_0^c \mathcal{D}_t^\alpha$ is the Caputo fractional derivative operator of order α and lower terminal 0, and $f: J \times X \times X \rightarrow X$ is a nonlinear function satisfying certain assumptions.

We obtain the existence and uniqueness of the solution by using fixed-point methods in the space of Lipschitz function and considering the Bielecki norm. The dependency of a solution is obtained through Picard operator theory and Pompeiu–Hausdorff functional. It is observed that the single inequality obtained for the difference of solutions via Picard operator theory gives simultaneously, the uniqueness of solution, the dependence of the solution on the initial condition as well as on the functions involved on the right-hand side of the problem (17)–(18). Whereas, the inequality obtained for the solutions via Pompeiu–Hausdorff functional gives the dependency of the solution on functions involved in the right-hand side of the problem (17)–(18).

This paper is structured in the following way. In Section 2, we present briefly the basic definitions and results of fractional calculus, Picard and weakly Picard operators. In Section 3, we obtain the existence, uniqueness, and dependency of solution for the nonlinear implicit FDEs (17)–(18). In Section 4, we provide example to illustrate our results. Finally, we end the paper with concluding remarks.

2. PRELIMINARIES

In this section, we give some definitions and basic results of Caputo fractional derivative [14, 35], and Picard and weakly Picard operator theory [37, 38, 43].

Definition 2.1. The Riemann-Liouville fractional integral ${}_0\mathcal{I}_t^\beta$ of order $\beta > 0$ of a function $g \in C[0, T]$, $T > 0$ is defined as

$${}_0\mathcal{I}_t^\beta g(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s) ds, \quad t > 0,$$

provided the integral exists.

Definition 2.2. Let $0 < \alpha \leq 1$, then the Caputo fractional derivative ${}_0^c\mathcal{D}_t^\alpha$ of order α with lower terminal 0 of a function $g \in C^1[0, T]$ is defined as

$${}_0^c\mathcal{D}_t^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} g'(s) ds, \quad t > 0.$$

Next, we provide the basics of Picard and weakly Picard operators theory from [37, 38, 43]. Let $X := (X, d)$ be a metric space and $T: X \rightarrow X$ be an operator. Then:

- (i) $P(X) := \{Y \subseteq X : Y \neq \emptyset\}$.
- (ii) $F_T := \{x \in X : T(x) = x\}$ is the fixed point set of T .
- (iii) $I(T) := \{Y \in P(X) : T(Y) \subseteq Y\}$ is a set of T -invariant non-empty subsets X .
- (iv) The Pompeiu-Hausdorff functional $H_d: P(X) \times P(X) \rightarrow \mathbb{R}_+ \cup \{\infty\}$ is defined as

$$H_d(Y, Z) = \max \left\{ \sup_{a \in Y} \inf_{b \in Z} d(a, b), \sup_{b \in Z} \inf_{a \in Y} d(a, b) \right\}.$$

Definition 2.3 ([43]). Let (X, d) be a metric space. An operator $T: X \rightarrow X$ is a Picard operator if there exists $x^* \in X$ such that $F_T = \{x^*\}$ and the sequence $(T^n(x_0))_{n \in \mathbb{N}}$ converges to x^* for all $x_0 \in X$.

Definition 2.4 ([43]). Let (X, d) be a metric space. An operator $T: X \rightarrow X$ is a weakly Picard operator if the sequence $(T^n(x_0))_{n \in \mathbb{N}}$ converges for all $x_0 \in X$, and its limit (which may depend on x_0) is a fixed point of T .

If $T: X \rightarrow X$ is a weakly Picard operator, then the operator $T^\infty: X \rightarrow X$ defined by

$$T^\infty(x) = \lim_{n \rightarrow \infty} T^n(x).$$

Lemma 2.5 ([43]). Let (X, d) be a metric space. Then $T: X \rightarrow X$ is a weakly Picard operator if and only if there exists a partition of X , $X = \bigcup_{\lambda \in \Lambda} X_\lambda$, where Λ is the indices set of partition such that

- (i) $X_\lambda \in I(T)$,
- (ii) $T|_{X_\lambda}: X_\lambda \rightarrow X_\lambda$ is a Picard operator for all $\lambda \in \Lambda$.

Lemma 2.6 ([43]). Let (X, d) be a complete metric space and $T, S: X \rightarrow X$ two operators. We suppose the following:

- (i) T is a contraction with constant α and $F_T = \{x_T^*\}$.
- (ii) S has a fixed points and $x_S^* \in F_S$.
- (iii) There exists $\gamma > 0$ such that $d(T(x), S(x)) \leq \gamma$ for all $x \in X$.

Then,

$$d(x_T^*, x_S^*) \leq \frac{\gamma}{1 - \alpha}.$$

Lemma 2.7 ([43]). *Let (X, d) be a complete metric space and $T, S: X \rightarrow X$ two orbitally continuous operators. We suppose the following:*

- (i) There exists $\alpha \in [0, 1)$ such that

$$d(T^2(x), T(x)) \leq \alpha d(T(x), x),$$

$$d(S^2(x), S(x)) \leq \alpha d(S(x), x)$$

for all $x \in X$.

- (ii) There exists $\gamma > 0$ such that $d(T(x), S(x)) \leq \gamma$ for all $x \in X$.

Then,

$$H_d(F_T, F_S) \leq \frac{\gamma}{1 - \alpha},$$

where H_d denotes Pompeiu-Hausdorff functional.

3. MAIN RESULTS

To derive our main results, we consider the spaces defined in [43].

Let $C := C(J, X)$ be the space of all X -valued continuous functions defined on J . Then $\mathcal{C} := (C, \|\cdot\|_{\mathcal{C}})$ is the Banach space with the Chebyshev norm

$$\|x\|_{\mathcal{C}} := \sup_{t \in J} \{\|x(t)\|\}.$$

Further, $\mathcal{B} = (C, \|\cdot\|_{\mathcal{B}})$ is the Banach space endowed with the Bielecki norm

$$\|x\|_{\mathcal{B}} := \sup_{t \in J} \left\{ \frac{\|x(t)\|}{\mathbb{E}_{\alpha}(\theta t^{\alpha})} \right\}, \quad x \in C \text{ for some } \theta > 0.$$

Let $d_{\mathcal{B}}$ and $d_{\mathcal{C}}$ are the metrics on C generated by the norms $\|\cdot\|_{\mathcal{B}}$ and $\|\cdot\|_{\mathcal{C}}$.

Let $L > 0$ and define the set

$$C_L(J, X) := \{x \in C(J, X) : \|x(t_1) - x(t_2)\| \leq L|t_1 - t_2|, \text{ for all } t_1, t_2 \in J\}.$$

Let $R > 0$, $B_R = S[0, R] = \{x \in X : \|x\| \leq R\}$, and

$$C_L(J, B_R) := \{x \in C(J, B_R) : \|x(t_1) - x(t_2)\| \leq L|t_1 - t_2|, \text{ for all } t_1, t_2 \in J\}.$$

We acquire our main results via equivalent fractional integral equation to the nonlinear implicit FDEs (1)–(2) given in the following lemma.

Lemma 3.1 ([8]). *The solution of the IVP (1)–(2) can be expressed by the integral equation*

$$(3) \quad x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} z(s) ds, \quad t \in J,$$

where z is the solution of the functional integral equation

$$(4) \quad z(t) = f\left(t, x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, z(t)\right), \quad t \in J.$$

3.1. Existence and uniqueness results

Theorem 3.2. *Assume that:*

(A1) $f: J \times X \times X \rightarrow X$ is a continuous function and satisfies Lipschitz type condition

$$(5) \quad \|f(t_1, x_1, y_1) - f(t_2, x_2, y_2)\| \leq M_1|t_1 - t_2| + M_2\|x_1 - x_2\| + M_3\|y_1 - y_2\|$$

for all $t_i \in J$ and $x_i, y_i \in X$, where $M_i > 0, i = 1, 2$, and $0 < M_3 < 1$.

Then, the problem (1)–(2) has unique solution, provided that

$$(6) \quad \frac{M_2}{\Gamma(\alpha + 1)} T^\alpha + M_3 < 1.$$

Further, the solution $z^* \in C_L(J, B_R)$ of functional integral equation (4) can be obtained by successive approximation method starting from any element of $C_L(J, B_R)$. Utilizing this z^* , the solution x^* of (1)–(2) is then acquired from the equation (3).

Proof. In the view of condition (6), we can choose $R > 0$ such that

$$(7) \quad \frac{M_2\|x_0\| + K}{1 - \left(\frac{M_2}{\Gamma(\alpha + 1)} T^\alpha + M_3\right)} \leq R,$$

where $K = \sup_{t \in J} \|f(t, 0, 0)\|$. Further the condition $0 < M_3 < 1$ allows us to choose $L > 0$ such that

$$(8) \quad \frac{M_1 + \frac{2M_2R}{\Gamma(\alpha + 1)}}{1 - M_3} \leq L.$$

With these choices of L and R , consider the space

$$C_L(J, B_R): = \{x \in C(J, B_R) : \|x(t_1) - x(t_2)\| \leq L|t_1 - t_2|, \text{ for all } t_1, t_2 \in J\}.$$

Define the mapping

$$(9) \quad (Tz)(t) = f\left(t, x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, z(t)\right), \quad t \in J, z \in C_L(J, B_R).$$

We show that the operator T is a Picard operator on $C_L(J, B_R)$. We divide the proof into two parts.

Part-(I). First we prove that T is self map on $C_L(J, B_R)$. We give the proof in the following steps.

Step 1. $Tz \in C(J, X)$ for any $z \in C_L(J, B_R)$.

Let any $\delta > 0$. Then for any $z \in C_L(J, B_R)$ and $t \in J$, we have

$$(10) \quad \|z(t + \delta) - z(t)\| \leq L|t + \delta - t| = L\delta \quad \text{and} \quad \|z(t)\| \leq R.$$

By Lipschitz type condition on f given in (A1) and using the inequalities in (10), we obtain

$$\begin{aligned}
& \|(Tz)(t+\delta) - (Tz)(t)\| \\
&= \left\| f\left(t+\delta, x_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t+\delta} (t+\delta-s)^{\alpha-1} z(s) ds, z(t+\delta)\right) \right. \\
&\quad \left. - f\left(t, x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, z(t)\right) \right\| \\
&\leq M_1|(t+\delta) - t| + M_2 \left\| \left(x_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t+\delta} (t+\delta-s)^{\alpha-1} z(s) ds \right) \right. \\
&\quad \left. - \left(x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds \right) \right\| + M_3 \|z(t+\delta) - z(t)\| \\
&\leq M_1\delta + \frac{M_2}{\Gamma(\alpha)} \int_0^t [(t+\delta-s)^{\alpha-1} - (t-s)^{\alpha-1}] \|z(s)\| ds \\
&\quad + \frac{M_2}{\Gamma(\alpha)} \int_t^{t+\delta} (t+\delta-s)^{\alpha-1} \|z(s)\| ds + M_3 \|z(t+\delta) - z(t)\| \\
&\leq (M_1 + M_3L)\delta + \frac{M_2R}{\Gamma(\alpha)} \int_0^t [(t+\delta-s)^{\alpha-1} - (t-s)^{\alpha-1}] ds \\
&\quad + \frac{M_2R}{\Gamma(\alpha)} \int_t^{t+\delta} (t+\delta-s)^{\alpha-1} ds \\
&= (M_1 + M_3L)\delta + \frac{M_2R}{\Gamma(\alpha+1)} \{-\delta^\alpha + (t+\delta)^\alpha - t^\alpha\} + \delta^\alpha \\
&= (M_1 + M_3L)\delta + \frac{M_2R}{\Gamma(\alpha+1)} [(t+\delta)^\alpha - t^\alpha].
\end{aligned}$$

Therefore,

$$\lim_{\delta \rightarrow 0} \|(Tz)(t+\delta) - (Tz)(t)\| = 0.$$

This proves that $Tz \in C(J, X)$.

Step 2. $Tz \in C_L(J, B_R)$ for any $z \in C_L(J, B_R)$.

Let any $z \in C_L(J, B_R)$. Then using inequalities (7) and (10), for any $t \in J$, we obtain

$$\begin{aligned}
\|T(z)(t)\| &= \left\| f\left(t, x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, z(t)\right) \right\| \\
&= \left\| f\left(t, x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, z(t)\right) - f(t, 0, 0) \right\| + \|f(t, 0, 0)\| \\
&\leq M_2 \|x_0\| + \frac{M_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|z(s)\| ds + M_3 \|z(t)\| + K \\
&\leq (M_2 \|x_0\| + K) + \frac{M_2R}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + M_3R
\end{aligned}$$

$$\begin{aligned}
 &= (M_2\|x_0\| + K) + \left[\frac{M_2}{\Gamma(\alpha + 1)} T^\alpha + M_3 \right] R \\
 &\leq R \left[1 - \left(\frac{M_2}{\Gamma(\alpha + 1)} T^\alpha + M_3 \right) \right] + \left[\frac{M_2 R}{\Gamma(\alpha + 1)} T^\alpha + M_3 R \right] = R.
 \end{aligned}$$

This proves that any $z \in C_L(J, B_R)$, $Tz \in C(J, B_R)$.

Step 3. For any $z \in C(J, B_R)$, Tz is Lipschitz with Lipschitz constant L .

Let any $z \in C_L(J, B_R)$ and $t_1, t_2 \in J$ with $t_1 < t_2$. Then by Lipschitz type condition on f given in (A1) and the inequality (10), we obtain

$$\begin{aligned}
 &\|(Tz)(t_1) - (Tz)(t_2)\| \\
 &= \left\| f\left(t_1, x_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} z(s) ds, z(t_1)\right) \right. \\
 &\quad \left. - f\left(t_2, x_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} z(s) ds, z(t_2)\right) \right\| \\
 &\leq M_1 |t_1 - t_2| + \frac{M_2}{\Gamma(\alpha)} \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] \|z(s)\| ds \\
 &\quad + \frac{M_2}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \|z(s)\| ds + M_3 \|z(t_2) - z(t_1)\| \\
 &\leq M_1 |t_1 - t_2| + \frac{M_2 R}{\Gamma(\alpha)} \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] ds \\
 &\quad + \frac{M_2 R}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds + M_3 L |t_2 - t_1| \\
 &= (M_1 + M_3 L) |t_1 - t_2| + \frac{M_2 R}{\Gamma(\alpha + 1)} [(t_2 - t_1)^\alpha + t_1^\alpha - t_2^\alpha] \\
 &\quad + \frac{M_2 R}{\Gamma(\alpha + 1)} (t_2 - t_1)^\alpha \\
 &\leq (M_1 + M_3 L) |t_1 - t_2| + \frac{2M_2 R}{\Gamma(\alpha + 1)} (t_2 - t_1)^\alpha \\
 &= \left(M_1 + \frac{2M_2 R}{\Gamma(\alpha + 1)} + M_3 L \right) |t_2 - t_1| \\
 &\leq (L(1 - M_3) + M_3 L) |t_2 - t_1| = L |t_2 - t_1|.
 \end{aligned}$$

This shows that Tz is Lipschitz function with Lipschitz constant L . From Steps 1 to 3, we get $Tz \in C_L(J, B_R)$. Hence T is self mapping on $C_L(J, B_R)$.

Part-(II). To prove that $T: C_L(J, B_R) \rightarrow C_L(J, B_R)$ is a Picard operator, we prove that T is a contraction. Let any $z_1, z_2 \in C_L(J, B_R)$. Then for any $t \in J$, we have

$$\begin{aligned}
 \|T(z_1)(t) - T(z_2)(t)\| &= \left\| f\left(t, x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} z_1(s) ds, z_1(t)\right) \right. \\
 &\quad \left. - f\left(t, x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} z_2(s) ds, z_2(t)\right) \right\|
 \end{aligned}$$

$$\begin{aligned} &\leq \frac{M_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|z_1(s) - z_2(s)\| ds + M_3 \|z_1(t) - z_2(t)\| \\ &\leq \frac{M_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathbb{E}_\alpha(\theta s^\alpha) \sup_{s \in J} \left(\frac{\|(z_1 - z_2)(s)\|}{\mathbb{E}_\alpha(\theta s^\alpha)} \right) ds \\ &\quad + M_3 \mathbb{E}_\alpha(\theta t^\alpha) \sup_{t \in J} \left(\frac{\|(z_1 - z_2)(t)\|}{\mathbb{E}_\alpha(\theta t^\alpha)} \right) \\ &= \frac{M_2 \|z_1 - z_2\|_B}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathbb{E}_\alpha(\theta s^\alpha) ds + M_3 E_\alpha(\theta t^\alpha) \|z_1 - z_2\|_B. \end{aligned}$$

But

$${}_0I_t^\alpha E_\alpha(\theta t^\alpha) = \frac{1}{\theta} [E_\alpha(\theta t^\alpha) - 1] \leq \frac{1}{\theta} E_\alpha(\theta t^\alpha), \quad t \in J.$$

This gives

$$\|(Tz_1)(t) - (Tz_2)(t)\| \leq \frac{M_2 \|z_1 - z_2\|_B}{\theta} \mathbb{E}_\alpha(\theta t^\alpha) + M_3 E_\alpha(\theta t^\alpha) \|z_1 - z_2\|_B, \quad t \in J.$$

Therefore,

$$\begin{aligned} \|Tz_1 - Tz_2\|_B &= \sup_{t \in J} \left(\frac{\|(Tz_1 - Tz_2)(t)\|}{\mathbb{E}_\alpha(\theta t^\alpha)} \right) \\ &\leq \left(\frac{M_2}{\theta} + M_3 \right) \|z_1 - z_2\|_B, \quad t \in J. \end{aligned}$$

Since $0 < M_3 < 1$, we can choose sufficiently large θ so that $\frac{M_2}{\theta} + M_3 < 1$. This proves that $T: C_L(J, B_R) \rightarrow C_L(J, B_R)$ is a contraction and hence Picard operator. By contraction principle, there is z^* in $C_L(J, B_R)$ such that $z^* = Tz^*$. This z^* is the unique solution of functional integral equation (4). Further for any $z \in C_L(J, B_R)$, $\|T^n z - z^*\|_B \rightarrow 0$ as $n \rightarrow \infty$. Substituting this z^* in (3), we obtain the unique solution x^* of the nonlinear implicit FDEs (1)–(2). \square

3.2. Dependency of the solution through Picard operator theory

To investigate the data dependency of the solution of the nonlinear implicit FDEs (1)–(2), we consider another nonlinear implicit FDEs of the form

$$(11) \quad {}_0\mathcal{D}_t^\alpha x(t) = g(t, x(t), {}_0\mathcal{D}_t^\alpha x(t)), \quad t \in [0, T],$$

$$(12) \quad x(0) = y_0 \in X,$$

where $g: J \times X \times X \rightarrow X$ is any nonlinear function need not equal to f .

Theorem 3.3. *Assume that the functions f and g satisfy the hypothesis (A1). Let there exist $\eta \in L^1(J, \mathbb{R}_+) \cap C(J, \mathbb{R}_+)$ such that*

$$\|f(t, x, y) - g(t, x, y)\| \leq \eta(t), \quad t \in J, x, y \in X.$$

If the condition (6) holds and $\theta > 0$ is such that $\frac{M_2}{\theta} + M_3 < 1$, then the solution x^* of (1)–(2) and the solution y^* of (11)–(12) satisfy the inequality

$$(13) \quad \|x^* - y^*\|_B \leq \|x_0 - y_0\| + \frac{K_\eta}{\theta \left[1 - \left(\frac{M_2}{\theta} + M_3\right)\right]},$$

where $K_\eta = \max \{\eta(t) : t \in J\}$.

Proof. By Lemma 3.1, the equivalent integral equation to the nonlinear implicit FDEs (11)–(12) is given by

$$(14) \quad x(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, \quad t \in J,$$

where z is the solution of the functional integral equation

$$(15) \quad z(t) = g\left(t, y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, z(t)\right), \quad t \in J.$$

Since $g: J \times X \times X \rightarrow X$ satisfies the conditions of Theorem 3.2, proceeding as in the proof of Theorem 3.2, the condition (6) allows us to choose $\tilde{L} > 0$ and $\tilde{R} > 0$ such that the mapping

$$S: C_{\tilde{L}}(J, B_{\tilde{R}}) \rightarrow C_{\tilde{L}}(J, B_{\tilde{R}})$$

defined by

$$S(z)(t) = g\left(t, y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, z(t)\right), \quad t \in J,$$

has a unique fixed point $F_S \in C_{\tilde{L}}(J, B_{\tilde{R}})$. Define $L^* = \max\{L, \tilde{L}\}$. Then one can verify that $F_T, F_S \in C_{L^*}(J, B_{R^*})$.

Now, for any $z \in C_{L^*}(J, B_{R^*})$ and $t \in J$, we have

$$\begin{aligned} \|(Tz)(t) - (Sz)(t)\| &= \left\| f\left(t, x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, z(t)\right) \right. \\ &\quad \left. - g\left(t, y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, z(t)\right) \right\| \leq \eta(t) \leq K_\eta. \end{aligned}$$

Since $\mathbb{E}_\alpha(\theta t^\alpha) \geq 1$ for all $t \in J$, we have

$$\|Tz - Sz\|_B = \sup_{t \in J} \left(\frac{\|(Tz - Sz)(t)\|}{\mathbb{E}_\alpha(\theta t^\alpha)} \right) \leq K_\eta.$$

Since the operators T, S satisfy the conditions of Lemma 2.6, by applying it, we obtain

$$(16) \quad \|F_T - F_S\|_B \leq \frac{K_\eta}{1 - \left(\frac{M_2}{\theta} + M_3\right)}.$$

Note that

$$x^*(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F_T(s) ds, \quad t \in J,$$

and

$$y^*(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F_S(s) ds, \quad t \in J,$$

are the unique solutions of nonlinear implicit FDEs (1)–(2), and (11)–(12), respectively. Then using inequality (16), for any $t \in J$, we have

$$\begin{aligned} & \|x^*(t) - y^*(t)\| \\ & \leq \|x_0 - y_0\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|F_T(s) - F_S(s)\| ds \\ & \leq \|x_0 - y_0\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathbb{E}_\alpha(\theta s^\alpha) \sup_{s \in J} \left(\frac{\|F_T(s) - F_S(s)\|}{\mathbb{E}_\alpha(\theta s^\alpha)} \right) ds \\ & = \|x_0 - y_0\| + \frac{\|F_T - F_S\|_B}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathbb{E}_\alpha(\theta s^\alpha) ds \\ & = \|x_0 - y_0\| + \|F_T - F_S\|_B \left(\frac{\mathbb{E}_\alpha(\theta t^\alpha) - 1}{\theta} \right) \\ & \leq \|x_0 - y_0\| + \frac{\mathbb{E}_\alpha(\theta t^\alpha) K_\eta}{\theta \left[1 - \left(\frac{M_2}{\theta} + M_3 \right) \right]}. \end{aligned}$$

This gives

$$\|x^* - y^*\|_B = \sup_{t \in J} \left(\frac{\|(x^* - y^*)(t)\|}{\mathbb{E}_\alpha(\theta t^\alpha)} \right) \leq \|x_0 - y_0\| + \frac{K_\eta}{\theta \left[1 - \left(\frac{M_2}{\theta} + M_3 \right) \right]},$$

which is the desired inequality (13). \square

Remarks

- (1) Theorem 3.3 gives the dependence of the solution of the problem (1)–(2) on the initial condition as well as on the functions involved on the right-hand side.
- (2) If $K_\eta = 0$ in (13), that is when $f = g$, Theorem 3.3 gives the dependency of solution of (1)–(2) on the initial condition.
- (3) If $x_0 = y_0$ and $K_\eta \neq 0$ in (13), then Theorem 3.3 gives the dependency of solution of (1)–(2) on functions involved in the right hand side of equation.
- (4) If $x_0 = y_0$ and $K_\eta = 0$ in (13), Theorem 3.3 gives the uniqueness of solution of the problem (1)–(2).

3.3. Dependency of the solution through Pompeiu–Hausdorff functional

Next, we consider an equation of the form

$$(17) \quad x_z(t) = z(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, \quad t \in J,$$

where z is a solution of fractional functional equation

$$(18) \quad z(t) = f\left(t, z(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, z(t)\right), \quad t \in J,$$

where f is as in the problem (1)–(2). From equations (17)–(18), it follows that $x_z(0) = z(0) = f(0, z(0), z(0))$.

Theorem 3.4. *Let $f: J \times X \times X \rightarrow X$ satisfies the assumptions (A1) and $f(0, x, x) = x$ for all $x \in X$. Then the equation (17) has a solution in $C_L(J, B_R)$ for some $L, R > 0$. If $S \subset C_L(J, B_R)$ is its solution set, then $\text{card } S = \text{card } B_R$.*

Proof. Consider the operator $T_*: (C_L(J, B_R), \|\cdot\|_B) \rightarrow (C_L(J, B_R), \|\cdot\|_B)$ defined by

$$(19) \quad T_*(z)(t) = f\left(t, z(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, z(t)\right), \quad t \in J.$$

Proceeding as in proof of Theorem 3.2, there exist constants $L, R > 0$ such that T_* is a continuous operator on $C_L(J, B_R)$. One can verify that the operator T_* does not satisfy Lipschitz condition, and hence T_* is not a Picard operator.

Next, we apply Lemma 2.5 to prove that the operator T_* is a weakly Picard operator. For each $\alpha \in B_R$, we define $X_\alpha = \{z \in C_L(J, B_R) : z(0) = \alpha\}$. Clearly, $C_L(J, B_R) = \bigcup_{\alpha \in B_R} X_\alpha$. By assumption for any $z \in X_\alpha$, we have

$$T_*(z)(0) = f(0, z(0), z(0)) = f(0, \alpha, \alpha) = \alpha.$$

Therefore, for each $\alpha \in B_R$, X_α is invariant under T_* . Proceeding as in the proof of Theorem (3.2), one can verify that for each $\alpha \in B_R$, the operator $T_*|_{X_\alpha}: X_\alpha \rightarrow X_\alpha$ is a Picard operator. Using Lemma 2.5, T_* is weakly Picard operator. By Definition 2.3, for any $x_0 \in C_L(J, B_R)$, the sequence $(T_*^n(x_0))$ converges to the fixed point of T_* , which is the solution of fractional functional equation (18), from which we can obtain the solution of (17). Following the similar steps of [43, Theorem 4.1], one can complete the proof of

$$\text{card } S = \text{card } B_R. \quad \square$$

In order to discuss the data dependency of solution, we consider the equations (17)–(18) and the equations

$$(20) \quad x_z(t) = z(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, \quad t \in J,$$

where z is a solution of functional equation

$$(21) \quad z(t) = g\left(t, \tilde{z}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, z(t)\right), \quad t \in J,$$

and $g \in C(J \times X \times X, X)$ is any other nonlinear function need not equal to f .

Theorem 3.5. *Assume that the functions f and g satisfy the hypothesis (A1) and $f(0, x, x) = x = g(0, x, x)$ for all $x \in X$. Let there exist $K > 0$ such that*

$$\|f(t, x, y) - g(t, x, y)\| \leq K \mathbb{E}_\alpha(\theta t^\alpha), \quad t \in J, x, y \in X,$$

for sufficiently large $\theta > 0$ satisfying $\frac{M_2}{\theta} + M_3 < 1$. If \mathcal{S} and $\hat{\mathcal{S}}$ are the solution sets of the equation (18) and (20), respectively, then there exist $L^*, R^* > 0$ such that

$$(22) \quad H_{\|\cdot\|_B}(\mathcal{S}, \hat{\mathcal{S}}) \leq \frac{KT^\alpha}{\Gamma(\alpha + 1) \left[1 - \left(\frac{M_2}{\theta} + M_3 \right) \right]},$$

where $H_{\|\cdot\|_B}$ is the Pompeiu-Hausdorff functional with respect to the norm $\|\cdot\|_B$ on $C_{L^*}(J, B_{R^*})$.

Proof. Since $g: J \times X \times X \rightarrow X$ satisfies the conditions of Theorem 3.4 by proceeding as in the proof of Theorem 3.4, we can choose $\tilde{L} > 0$ and $\tilde{R} > 0$ such that the mapping

$$S_* : (C_{\tilde{L}}(J, B_{\tilde{R}}), \|\cdot\|_B) \rightarrow (C_{\tilde{L}}(J, B_{\tilde{R}}), \|\cdot\|_B),$$

defined by

$$(23) \quad S_*(z)(t) = g\left(t, z(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, z(t)\right), \quad t \in J,$$

has a fixed point $F_{S_*} \in C_{\tilde{L}}(J, B_{\tilde{R}})$. Define $L_* = \max\{L, \tilde{L}\}$ and $R_* = \max\{R, \tilde{R}\}$, then one can verify that $F_{T_*}, F_{S_*} \in C_{L_*}(J, B_{R_*})$. Now, for any $z \in C_{L_*}(J, B_{R_*})$ and $t \in J$, we have

$$\begin{aligned} & \|T_*^2(z)(t) - T_*(z)(t)\| = \|T_*(T_*(z)(t)) - T_*z(t)\| \\ & = \left\| f\left(t, T_*(z)(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T_*(z(s)) ds, T_*(z(t))\right) \right. \\ & \quad \left. - f\left(t, z(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, z(t)\right) \right\| \\ & \leq M_2 \left\| \left(z(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T_*(z(s)) ds \right) \right. \\ & \quad \left. - \left(z(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds \right) \right\| + M_3 \|T_*(z)(t) - z(t)\| \\ & \leq \frac{M_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|T_*(z(s)) - z(s)\| ds + M_3 \|T_*(z)(t) - z(t)\| \\ & \leq \frac{M_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathbb{E}_\alpha(\theta s^\alpha) \frac{\|T_*(z(s)) - z(s)\|}{\mathbb{E}_\alpha(\theta s^\alpha)} ds \\ & \quad + M_3 \mathbb{E}_\alpha(\theta t^\alpha) \frac{\|T_*(z(t)) - z(t)\|}{\mathbb{E}_\alpha(\theta t^\alpha)} \\ & \leq \frac{M_2}{\Gamma(\alpha)} \|T_*(z) - z\|_B \int_0^t (t-s)^{\alpha-1} \mathbb{E}_\alpha(\theta s^\alpha) ds + M_3 \mathbb{E}_\alpha(\theta t^\alpha) \|T_*(z) - z\|_B \\ & \leq M_2 \|T_*(z) - z\|_B \frac{\mathbb{E}_\alpha(\theta t^\alpha)}{\theta} + M_3 \mathbb{E}_\alpha(\theta t^\alpha) \|T_*(z) - z\|_B. \end{aligned}$$

This gives

$$\frac{\|T_*^2 z(t) - T_* z(t)\|}{\mathbb{E}_\alpha(\theta t^\alpha)} \leq \frac{M_2}{\theta} \|T_* z - z\|_B + M_3 \|T_* z - z\|_B.$$

Therefore,

$$\|T_*^2 z - T_* z\|_B \leq \left(\frac{M_2}{\theta} + M_3\right) \|T_* z - z\|_B, \quad z \in C_{L^*}(J, B_{R^*}).$$

On the similar line, we have

$$\|S_*^2 z - S_* z\|_B \leq \left(\frac{M_2}{\theta} + M_3\right) \|S_* z - z\|_B, \quad z \in C_{L^*}(J, B_{R^*}).$$

Further, for any $z \in C_{L^*}(J, B_{R^*})$ and $t \in J$, we have

$$\begin{aligned} \|T_*(z)(t) - S_*(z)(t)\| &= \left\| f\left(t, z(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, z(t)\right) \right. \\ &\quad \left. - g\left(t, z(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds, z(t)\right) \right\| \\ &\leq K \mathbb{E}_\alpha(\theta t^\alpha). \end{aligned}$$

This gives

$$\|T_* z - S_* z\|_B = \sup_{t \in J} \frac{\|T_* z(t) - S_* z(t)\|}{\mathbb{E}_\alpha(\theta t^\alpha)} \leq K, \quad z \in C_{L^*}(J, B_{R^*}).$$

Since, the operators T_* , S_* fulfill the requirements of Lemma (2.7), by applying them, we obtain

$$H_{\|\cdot\|_B}(F_{T_*}, F_{S_*}) \leq \frac{K}{1 - \left(\frac{M_2}{\theta} + M_3\right)}.$$

Note that

$$x_z^*(t) = z(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F_{T_*}(s) ds, \quad t \in J,$$

and

$$y_z^*(t) = z(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F_{S_*}(s) ds, \quad t \in J,$$

are the solutions of (17) and (20), respectively. Therefore, for any $t \in J$, we have

$$\|x_z^*(t) - y_z^*(t)\| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|F_{T_*}(s) - F_{S_*}(s)\| ds.$$

This gives

$$\begin{aligned} H_{\|\cdot\|_B}(x_z^*, y_z^*) &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} H_{\|\cdot\|_B}(F_{T_*}, F_{S_*}) ds \\ &\leq \frac{K}{\Gamma(\alpha) \left[1 - \left(\frac{M_2}{\theta} + M_3\right)\right]} \int_0^t (t-s)^{\alpha-1} ds \end{aligned}$$

$$= \frac{Kt^\alpha}{\Gamma(\alpha + 1)\left[1 - \left(\frac{M_2}{\theta} + M_3\right)\right]} \leq \frac{KT^\alpha}{\Gamma(\alpha + 1)\left[1 - \left(\frac{M_2}{\theta} + M_3\right)\right]}.$$

Therefore,

$$H_{\|\cdot\|_B}(\mathcal{S}, \hat{\mathcal{S}}) \leq \frac{KT^\alpha}{\Gamma(\alpha + 1)\left[1 - \left(\frac{M_2}{\theta} + M_3\right)\right]},$$

which is the desired inequality (22). □

Remark. Theorem 3.5, gives the dependency of solution on functions involved in the right hand side of the problem (17)–(18). In particular, if $K = 0$, then $f = g$, and we obtain uniqueness of the solution.

4. EXAMPLE

Consider the nonlinear implicit FDEs of the form

$$(24) \quad {}_0^c \mathcal{D}_t^{\frac{1}{2}} x(t) = \frac{\sqrt{\pi}}{4} - \frac{1}{2}t^{\frac{1}{2}} + \frac{1}{2} \left(x(t) + |{}_0^c \mathcal{D}_t^{\frac{1}{2}} x(t)| \right), \quad t \in J := [0, 0.5],$$

$$(25) \quad x(0) = 1 \in X,$$

in the Banach space $X = (\mathbb{R}, |\cdot|)$. Define $f: J \times X \times X \rightarrow X$ by

$$f(t, x, y) = \frac{\sqrt{\pi}}{4} - \frac{1}{2}t^{\frac{1}{2}} + \frac{1}{2}(x + |y|).$$

For any $t_1, t_2 \in J$ and $x_i, y_i \in X, i = 1, 2$, we have

$$\begin{aligned} & |f(t_1, x_1, y_1) - f(t_2, x_2, y_2)| \\ &= \left| \left(\frac{\sqrt{\pi}}{4} - \frac{1}{2}t_1^{\frac{1}{2}} + \frac{1}{2}(x_1 + |y_1|) \right) - \left(\frac{\sqrt{\pi}}{4} - \frac{1}{2}t_2^{\frac{1}{2}} + \frac{1}{2}(x_2 + |y_2|) \right) \right| \\ &\leq \frac{1}{2}|t_1^{\frac{1}{2}} - t_2^{\frac{1}{2}}| + \frac{1}{2}(|x_1 - x_2| + ||y_1| - |y_2||) \\ &\leq \frac{1}{2}|t_1 - t_2| + \frac{1}{2}(|x_1 - x_2| + |y_1 - y_2|). \end{aligned}$$

Therefore, the function f satisfies hypothesis (A1) with constants $M_1 = M_2 = M_3 = \frac{1}{2}$. Note that

$$\frac{M_2 T^\alpha}{\Gamma(\alpha + 1)} + M_3 = \frac{\frac{1}{2}(0.5)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} + M_3 = \sqrt{\frac{0.5}{\pi}} + \frac{1}{2} = 0.8989 < 1.$$

Since the function f satisfies all conditions of Theorem 3.2, the problem (24)–(25) has a unique solution. By direct calculation, one can verify that the solution of the problem (24)–(25) is

$$x^*(t) = t^{\frac{1}{2}} + \mathbb{E}_{\frac{1}{2}}(t^{\frac{1}{2}}), \quad t \in J.$$

Next, we consider another implicit FDEs

$$(26) \quad {}_0^c \mathcal{D}_t^{\frac{1}{2}} x(t) = \frac{t^{\frac{1}{2}}}{2} + \frac{1}{2} \left(x(t) + |{}_0^c \mathcal{D}_t^{\frac{1}{2}} x(t)| \right), \quad t \in [0, 0.5],$$

$$(27) \quad x(0) = 1 - \frac{\sqrt{\pi}}{2} \in X.$$

Define $g: J \times X \times X \rightarrow X$ by

$$g(t, x, y) = \frac{t^{\frac{1}{2}}}{2} + \frac{1}{2} \left(x(t) + |{}_0^c \mathcal{D}_t^{\frac{1}{2}} x(t)| \right), \quad t \in J.$$

One can verify that the function $g: J \times X \times X \rightarrow X$ defined by

$$g(t, x, y) = \frac{t^{\frac{1}{2}}}{2} + \frac{1}{2} (x + |y|),$$

satisfies all conditions of Theorem 3.2 with constants $M_1 = M_2 = M_3 = \frac{1}{2}$. One can verify that

$$y^*(t) = t^{\frac{1}{2}} + \mathbb{E}_{\frac{1}{2}}(t^{\frac{1}{2}}) - \frac{\sqrt{\pi}}{2}, \quad t \in J,$$

is an exact solution of (26)–(27). Note that for any $t \in J$ and $x, y \in X$

$$\begin{aligned} |f(t, x, y) - g(t, x, y)| &= \left| \left(\frac{\sqrt{\pi}}{4} - \frac{1}{2} t^{\frac{1}{2}} + \frac{1}{2} (x + |y|) \right) - \left(\frac{t^{\frac{1}{2}}}{2} + \frac{1}{2} (x + |y|) \right) \right| \\ &= \left| \frac{\sqrt{\pi}}{4} - t^{\frac{1}{2}} \right| \leq \frac{\sqrt{\pi}}{4} + t^{\frac{1}{2}} := \eta(t). \end{aligned}$$

Since f, g satisfy all conditions of Theorem 3.3, by applying it, we obtain

$$(28) \quad \|x^* - y^*\|_B \leq \|x_0 - y_0\| + \frac{K_\eta}{\theta \left[1 - \left(\frac{M_2}{\theta} + M_3 \right) \right]},$$

where $K_\eta = \max_{t \in J} \eta(t) = \max_{t \in J} \left(\frac{\sqrt{\pi}}{4} + t^{\frac{1}{2}} \right) = \frac{\sqrt{\pi}}{4} + (0.5)^{\frac{1}{2}} = 1.1502$. Taking $\theta = 2$, we have $\frac{M_2}{\theta} + M_3 < 1$. With this choice of θ , from (28), we get

$$(29) \quad \|x^* - y^*\|_B \leq \left| 1 - \left(1 - \frac{\sqrt{\pi}}{2} \right) \right| + \frac{1.1502}{2 [1 - 0.75]} = \frac{\sqrt{\pi}}{2} + 2.3004.$$

Using the exact solution and by actual calculation, we have

$$(30) \quad \|x^* - y^*\|_B = \sup_{t \in J} \frac{|x^* - y^*|}{\mathbb{E}_{\frac{1}{2}}(\theta t^{\frac{1}{2}})} \leq |x^* - y^*| = \frac{\sqrt{\pi}}{2} < \frac{\sqrt{\pi}}{2} + 2.3004.$$

Remark. From the inequalities (29) and (30), it is observed that the difference of solution calculated by applying Theorem 3.3 is similar to the difference of solution calculated actually.

5. CONCLUDING REMARKS

Observing the applicability of Caputo–Fabrizio (CF) fractional derivative operator [3, 4, 11, 20] and Atangana–Baleanu–Caputo (ABC) fractional derivative operator [2, 5] that possesses a non-singular kernel, one can analyze and extend the results of the present paper for the implicit FDEs with CF and ABC fractional derivative operators.

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