Acta Math. Univ. Comenianae
Vol. XC, 4 (2021), pp. 421–436

EXISTENCE RESULTS FOR FRACTIONAL IMPULSIVE INTEGRO-DIFFERENTIAL EQUATIONS WITH INTEGRAL CONDITIONS OF KATUGAMPOLA TYPE

P. KARTHIKEYAN, K. VENKATACHALAM AND S. ABBAS

Abstract. We study the existence and uniqueness of solutions of impulsive fractional integro-differential equations of order $\alpha_1 \in (2, 3]$ with the Katugampola integral boundary conditions. Krasnoselkii's fixed point theorem and Banach contraction principle are used to prove the existence and uniqueness results. An example is also presented at the end.

1. Introduction

Fractional derivative was discovered by Leibnitz in 1695 and it has received in more attention of many fields of natural science, mathematical science, biological science, and physical science etc. The most of the generally used in fractional calculus definitions in basic mathematics research are the Riemann-Liouville calculus definition, Caputo differential definition, Grunwald-Letnikov differential definition, and so on. Fractional differential equations are commonly used as mathematical modeling in the dynamical system. It is tool for finding the property materials, gene, and etc., for more details, see [**3, 4, 6, 26, 30**]. Many researchers have dedicated themselves to the study of fractional-order differential equations can more accurately explain objective laws and the nature of things than integral differential equations.

In particular, impulsive differential equations have been useful in explaining the dynamics of populations, subject to sudden changes, as well as other phenomena such as harvesting, diseases, and so on. For the fundamentals of impulsive differential equations theory, the reader, can refer to the books [**8, 12, 20**] and the papers [**2, 7, 11, 14, 18**].

The Katugampola introduced a new fractional integral in 2011, which is the combination of the Riemann-Liouville and Hadamard integrals into a single form. The integrals are special cases when a parameter is defined at various values; when $\rho \to 0$, the Riemann-Liouville operators are obtained; when $\rho \to 1$, the Hadamard

Received October 11, 2020; revised May 12, 2021.
2020 *Mathematics Subject Classification.* Primary 34A08, 26A33, 34A12, 47H10.
Key words and phrases. Fractional differential equations; Katugampola derivative; fractional derivative and integrals; existence and uniqueness; fixed point theorems.

operators are obtained (see [**15**]). In [**16**], the authors presented two Katugampola derivative representations of the generalised derivative. In [**17**], the authors proved the existence and uniqueness of solutions to the initial value problem for a class of generalised fractional differential equations. In [**4**], the authors studied fractional boundary value problems with generalized Riemann-Liouville nonlocal integral boundary conditions. A study of fractional differential equations involving generalized Caputo-type derivatives with integral and multipoint boundary conditions was discussed in [**30**]. In [**14**], the authors discussed impulsive fractional differential equations via Katugampola fractional derivatives.

Further, in [**3**], the authors studied the fractional differential equations with Stieltjes and fractional integral boundary conditions using the generalized derivatives of the form

$$
\begin{aligned}
&{}^{\rho}\mathcal{D}^{\alpha_1}y(t) = f(t,y(t)), \qquad t = [0,T],\\
&y(0)=0, \qquad \int_0^T y(s)\mathrm{d}H(s) = \iota\frac{\rho^{1-\gamma}}{\Gamma(\gamma)}\int_0^{\xi}\alpha_i\frac{s^{\rho_i-1}x(s)}{(\xi^{\rho}-s^{\rho})^{1-\gamma}}\mathrm{d}s,
\end{aligned}
$$

where ${}^{\rho}\mathcal{D}^{\alpha_1}$-generalized fractional derivative and ${}^{\rho}I^{\gamma}$-generalized fractional integral and H-continuous function.

In [**6**], the authors discussed Caputo-type fractional differential equations with Katugampola type generalized fractional integral boundary conditions of the form

$$
\begin{aligned}
&{}^{\rho}_{c}\mathcal{D}^{\alpha_1}_{0+}y(t) = f(t,y(t)), && t\in J:=[0,T],\ \ 1<\alpha_1\le 2,\\
&y(T) = \sum_{i=1}^{m}\sigma_i\ {}^{\rho}I^{\beta}_{0+}y(\eta_i)+k, && \delta y(0)=0,\ \ \eta_i\in(0,T),
\end{aligned}
$$

where ${}^{\rho}_{\mathtt{c}}\mathcal{D}^{\alpha_1}_{0+}$ denotes the Caputo fractional derivative and $\mathtt{f}$ is a continuous function.

In [**23**], the authors discussed Katugampola-Caputo fractional differential equations with nonlocal initial value problems involving time scales

$$
\begin{aligned}
&{}^{\rho}\Delta^{\mathfrak{p}}_{t_0}x(t) = f(t,x(t)), \qquad t\in I,\\
&x(0)+g(x) = x_0,
\end{aligned}
$$

where $0<\mathfrak{p}<1$.

In [**26**], the authors studied Riemann-Liouville fractional differential equations with non local Erdelyi-Kober integral conditions

$$
\begin{aligned}
&\mathcal{D}^{q}x(t) = f(t,x(t)), && t\in(0,T),\ 1<q\le 2,\\
&x(0)=0, && x(T)=\sum_{i=1}^{m}\beta_i I^{\gamma_i,\delta_i}_{\eta_i}x(\xi_i)
\end{aligned}
$$

where $\mathcal{D}^{q}$ is the Riemann-Liouville fractional derivative and the function is continuous.

Motivated by the works, consider the fractional integro-differential equations with Katugampola integral boundary conditions of the form:

$$
{}^{c}\mathcal{D}^{\alpha_1}x(t) = f(t,x(t),Bx(t)), \qquad t\in[0,T],\ 2<\alpha_1\le 3, \tag{1}
$$

(2) $$x(t_k^+) = x(t_k^-) + y_k, \qquad y_k \in \mathbb{R}, \quad k = 1, \dots, m,$$

(3) $$x(T) = \vartheta {}^{\varrho}I^q x(\tau), \qquad 0 < \tau < T,$$

(4) $$x'(T) = \chi {}^{\varrho}I^q x'(\nu), \qquad 0 < \nu < T,$$

(5) $$x''(T) = \iota {}^{\varrho}I^q x''(\zeta), \qquad 0 < \zeta < T,$$

where $\mathcal{D}^{\alpha_1}$ is the Caputo fractional derivative, ${}^{\varrho}I^q$-Katugampola integral of $q > 0$, $\varrho > 0$, $\vartheta, \chi, \iota \in \mathbb{R}$, $f\colon [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $Bx(t) = \int_0^t k(t,s,x(s))\mathrm{d}s$, $k\colon \Delta \times [0,T] \to \mathbb{R}$, $\Delta = \{(t,s) : 0 \le s \le t \le T\}$, $0 = t_0 < t_1 < t_2 < \cdots < t_m = 1$, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, and $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0^-} x(t_k + h)$ represent the right and left hand limits of $x(t)$ at $t = t_k$.

Thus, the main motivation for this work is to present a new class of impulsive fractional integro-differential equations with Katugampola boundary conditions, by means of the Caputo fractional derivative, and to investigate the existence and uniqueness of the solutions of equations (1)–(5), using Krasnoselkii's and Banach's fixed point theorem. We extend the results in ([**21**]) by involving integral terms in nonlinear functions and impulsive conditions.

The paper is structured as follows: In Section 2, we present some spaces with norm and important definitions, lemma that we need to develop the paper. The existence and uniqueness results of impulsive fractional integro-differential equations are discussed in Section 3. An example to illustrate the findings is given in Section 4.

2. Auxiliary results

Consider the set of functions $\mathfrak{PC}(J,\mathbb{R}) = \{x\colon J \to \mathbb{R} : x \in C(t_k, t_{k+1}], \mathbb{R}\}$, $k = 0, \dots, m$ and there exist $x(t_k^-)$, $x(t_k^+)$ with $x(t_k^-) = x(t_k^+)$ and endowed with the norm

$$\|x\|_{\mathfrak{PC}} = \sup\{|x(t)| : 0 \le t \le 1\}.$$

Definition 2.1 (Caputo fractional derivative [**21**])**.** The Caputo derivative of order q for the function $f\colon [0,\infty) \to \mathbb{R}$ is defined as

$${}^c\mathcal{D}^q f(t) = \mathcal{D}^q_{0^+}\Big(f(t) - \sum_{k=0}^{\mathfrak{n}-1} \frac{t^k}{k!} f^{(k)}(0)\Big), \qquad t > 0,\ \mathfrak{n} - 1 < q < \mathfrak{n}.$$

Remark. If $f(t) \in \mathfrak{C}^n[0,\infty)$, then

$${}^c\mathcal{D}^q f(t) = \frac{1}{\Gamma(\mathfrak{n} - q)} \int_0^t \frac{f^{(\mathfrak{n})}(s)}{(t-s)^{q+1-\mathfrak{n}}}\mathrm{d}s = I^{\mathfrak{n}-q} f^{(\mathfrak{n})}(t), \qquad t > 0, \mathfrak{n} - 1 < q < \mathfrak{n}.$$

Definition 2.2 (Katugampola fractional derivative [**15**])**.** The Katugampola fractional derivative corresponding to the Katugampola fractional integral is defined by

$$({}^{\mu}\mathcal{D}^{\alpha_1}_{a^+} f(x) = \Big(x^{1-\mu}\frac{\mathrm{d}}{\mathrm{d}x}\Big)^{\mathfrak{n}} \big({}^{\mu}I^{\mathfrak{n}-\alpha_1}_{a^+} f\big)(x)$$

$$= \frac{\mu^{\alpha_1-\mathfrak{n}+1}}{\Gamma(\mathfrak{n}-\alpha_1)}\Big(x^{1-\mu}\frac{\mathrm{d}}{\mathrm{d}x}\Big)^{\mathfrak{n}}\int_a^x \frac{s^{\mu-1}}{(t^{\mu}-s^{\mu})^{\alpha_1-\mathfrak{n}+1}}\mathrm{d}s.$$

Definition 2.3 (Katugampola fractional integral [**15**])**.** Katugampola fractional integral of order $q>0$ and $\varrho>0$ of a function f is defined by

$${}^{\varrho}I^{q}f(t)=\frac{\varrho^{1-q}}{\Gamma(q)}\int_0^t \frac{s^{\varrho-1}f(s)}{(t^{\varrho}-s^{\varrho})^{1-q}}\mathrm{d}s.$$

Definition 2.4 (Riemann-Liouville fractional integral [**19**])**.** The Riemann-Liouville fractional integral of order $\varrho>0$ of a continuous function f is given by

$$\mathcal{J}^{q}f(t)=\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s)\mathrm{d}s,$$

where Γ is defined by $\Gamma(q)=\int_0^{\infty}\mathrm{e}^{-s}s^{q-1}\mathrm{d}s$.

Definition 2.5 (Riemann-Liouville fractional derivative [**19**])**.** The Riemann-Liouville fractional derivative of order $q>0$ of a continuous function f given by

$$\mathcal{D}^{q}_{0+}f(t)=\frac{1}{\Gamma(\mathfrak{n}-q)}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big)^{\mathfrak{n}}\int_0^t (t-s)^{\mathfrak{n}-q-1}f(s)\mathrm{d}s,\quad \mathfrak{n}-1<q<n.$$

Lemma 2.6 ([**4**])**.** *The given constant* $p,q>0$ *and* $p>0$ *then*

$${}^{\varrho}I^{q}t^{p}=\frac{\Gamma(\frac{p+\varrho}{\varrho})}{\Gamma(\frac{p+\varrho q+\varrho}{\varrho})}\frac{t^{p+\varrho q}}{\varrho^{q}}.$$

Lemma 2.7 ([**19**])**.** *For* $q>0$ *and* $x\in\mathfrak{C}(0,T)\cap\mathfrak{L}(0,T)$*. Then,* ${}^{c}D^{q}x(t)=0$ *has a unique solution*

$$x(t)=c_0+c_1t+\cdots+c_{n-1}t^{n-1}$$

and the following formula holds

$$I^{q}D^{q}x(t)=x(t)+c_0+c_1t+\cdots+c_{n-1}t^{n-1},$$

where $c_i\in\mathbb{R}, i=0,1,\ldots,n-1,$ *and* $n-1\le q\le n$.

Theorem 2.8 (Krasnoselkii's, [**13**])**.** *Let* $\mathfrak{K}$ *be a closed, bounded, convex, and nonempty subset of a Banach space* $\mathfrak{X}$*. Let* Ξ_1 *and* Ξ_2 *be two operators such that:*

(i) $\Xi_1x+\Xi_1y\in\mathfrak{K}$ *for any* $x,y\in\mathfrak{K}$,
(ii) Ξ_1 *is compact and continuous,*
(iii) Ξ_2 *is contraction mapping.*

Then there exists $z_1\in\mathfrak{K}$ *such that* $z_1=\Xi_1z_1+\Xi_2z_1$.

Lemma 2.9. *Let* $2<\alpha_1\le 3$ *and* $\vartheta,\chi,\iota\in\mathbb{R}$*. Then* $y\in\mathfrak{PC}([0,T],\mathbb{R})$*,* x *is a nonlinear solution of fractional differential equations with Katugampola type:*

$$(6)\qquad {}^{c}\mathcal{D}^{\alpha_1}x(t)=h(t),\qquad t\in[0,T],$$

$$(7)\qquad x(t_k^{+})=x(t_k^{-})+y_k,\qquad y_k\in\mathbb{R},\quad k=1,\ldots,m,$$

$$(8)\qquad x(T)=\vartheta\,{}^{\varrho}I^{q}x(\tau),\qquad 0<\tau<T,$$

$$x^{'}(T) = \chi^{\varrho} I^{q} x^{'}(\nu), \qquad 0 < \nu < T, \tag{9}$$

$$x^{''}(T) = \iota^{\varrho} I^{q} x^{''}(\zeta), \qquad 0 < \zeta < T, \tag{10}$$

if and only if

$$x(t) = \begin{cases} \mathcal{J}^{\alpha_1} h(t) + \frac{1}{\varpi_1(\vartheta,\tau)} (\vartheta^{\varrho} I^q \mathcal{J}^{\alpha_1} h(\tau) - \mathcal{J}^{\alpha_1} h(T)) \\ -\frac{1}{\varpi_1(\vartheta,\tau)} \left(\frac{\varpi_2(\vartheta,\tau)}{\varpi_1(\vartheta,\tau)} - t \right) (\chi^{\varrho} I^q \mathcal{J}^{\alpha_1 - 1} h(\nu) - \mathcal{J}^{\alpha_1 - 1} h(T)) \\ +\frac{1}{\varpi_1(\iota,\zeta)} \left(\frac{\varpi_3(\vartheta,\tau)}{2\varpi_1(\vartheta,\tau)} - \frac{\varpi_2(\vartheta,\tau)\varpi_2(\chi,\nu)}{\varpi_1(\vartheta,\tau)\varpi_1(\chi,\nu)} + \frac{\varpi_2(\chi,\nu)t}{\varpi_1(\chi,\nu)} - \frac{t^2}{2} \right) \\ \times (\mathcal{J}^{\alpha_1 - 2} h(T) - \iota^{\varrho} I^q \mathcal{J}^{\alpha_1 - 2} h(\zeta)) \qquad \textit{for} \quad t \in [0, t_1), \\ \\ y_1 + \mathcal{J}^{\alpha_1} h(t) + \frac{1}{\varpi_1(\vartheta,\tau)} (\vartheta^{\varrho} I^q \mathcal{J}^{\alpha_1} h(\tau) - \mathcal{J}^{\alpha_1} h(T)) \\ -\frac{1}{\varpi_1(\vartheta,\tau)} \left(\frac{\varpi_2(\vartheta,\tau)}{\varpi_1(\vartheta,\tau)} - t \right) (\chi^{\varrho} I^q \mathcal{J}^{\alpha_1 - 1} h(\nu) - \mathcal{J}^{\alpha_1 - 1} h(T)) \\ +\frac{1}{\varpi_1(\iota,\zeta)} \left(\frac{\varpi_3(\vartheta,\tau)}{2\varpi_1(\vartheta,\tau)} - \frac{\varpi_2(\vartheta,\tau)\varpi_2(\chi,\nu)}{\varpi_1(\vartheta,\tau)\varpi_1(\chi,\nu)} + \frac{\varpi_2(\chi,\nu)t}{\varpi_1(\chi,\nu)} - \frac{t^2}{2} \right) \\ \times (\mathcal{J}^{\alpha_1 - 2} h(T) - \iota^{\varrho} I^q \mathcal{J}^{\alpha_1 - 2} h(\zeta)) \qquad \textit{for} \quad t \in (t_1, t_2), \\ \\ y_1 + y_2 + \mathcal{J}^{\alpha_1} h(t) + \frac{1}{\varpi_1(\vartheta,\tau)} (\vartheta^{\varrho} I^q \mathcal{J}^{\alpha_1} h(\tau) - \mathcal{J}^{\alpha_1} h(T)) \\ -\frac{1}{\varpi_1(\vartheta,\tau)} \left(\frac{\varpi_2(\vartheta,\tau)}{\varpi_1(\vartheta,\tau)} - t \right) (\chi^{\varrho} I^q \mathcal{J}^{\alpha_1 - 1} h(\nu) - \mathcal{J}^{\alpha_1 - 1} h(T)) \\ +\frac{1}{\varpi_1(\iota,\zeta)} \left(\frac{\varpi_3(\vartheta,\tau)}{2\varpi_1(\vartheta,\tau)} - \frac{\varpi_2(\vartheta,\tau)\varpi_2(\chi,\nu)}{\varpi_1(\vartheta,\tau)\varpi_1(\chi,\nu)} + \frac{\varpi_2(\chi,\nu)t}{\varpi_1(\chi,\nu)} - \frac{t^2}{2} \right) \\ \times (\mathcal{J}^{\alpha_1 - 2} h(T) - \iota^{\varrho} I^q \mathcal{J}^{\alpha_1 - 2} h(\zeta)) \qquad \textit{for} \quad t \in (t_2, t_3), \\ \\ \vdots \\ \\ \sum_{k=1}^{m} y_i + \mathcal{J}^{\alpha_1} h(t) + \frac{1}{\varpi_1(\vartheta,\tau)} (\vartheta^{\varrho} I^q \mathcal{J}^{\alpha_1} h(\tau) - \mathcal{J}^{\alpha_1} h(T)) \\ -\frac{1}{\varpi_1(\vartheta,\tau)} \left(\frac{\varpi_2(\vartheta,\tau)}{\varpi_1(\vartheta,\tau)} - t \right) (\chi^{\varrho} I^q \mathcal{J}^{\alpha_1 - 1} h(\nu) - \mathcal{J}^{\alpha_1 - 1} h(T)) \\ +\frac{1}{\varpi_1(\iota,\zeta)} \left(\frac{\varpi_3(\vartheta,\tau)}{2\varpi_1(\vartheta,\tau)} - \frac{\varpi_2(\vartheta,\tau)\varpi_2(\chi,\nu)}{\varpi_1(\vartheta,\tau)\varpi_1(\chi,\nu)} + \frac{\varpi_2(\chi,\nu)t}{\varpi_1(\chi,\nu)} - \frac{t^2}{2} \right) \\ \times (\mathcal{J}^{\alpha_1 - 2} h(T) - \iota^{\varrho} I^q \mathcal{J}^{\alpha_1 - 2} h(\zeta)) \qquad \textit{for} \quad t \in (t_m, T], \end{cases}$$

where

$$\varpi_1(\alpha_1, \xi) = \left(1 - \alpha_1 \frac{\xi^{\varrho q}}{\varrho^q} \frac{1}{\Gamma(q+1)} \right) \neq 0, \tag{11}$$

$$\varpi_2(\alpha_1, \xi) = \left(T - \alpha_1 \frac{\xi^{\varrho q} + 1}{\varrho^q} \frac{\Gamma\left(\frac{\varrho}{1+\varrho}\right)}{\Gamma\left(\frac{1+\varrho q+\varrho}{\varrho}\right)} \right), \tag{12}$$

$$\varpi_3(\alpha_1, \xi) = \left(T^2 - \alpha_1 \frac{\xi^{\varrho q} + 2}{\varrho^q} \frac{\Gamma\left(\frac{\varrho}{2+\varrho}\right)}{\Gamma\left(\frac{2+\varrho q+\varrho}{\varrho}\right)} \right). \tag{13}$$

Proof. Assume that x satisfies (6) and (8)–(10). If $t \in [0, t_1)$, then

$$^{c}\mathcal{D}^{\alpha_1} x(t) = h(t), \quad t \in [0, T], \qquad x(T) = \vartheta^{\varrho} I^q x(\tau),$$
$$x'(T) = \chi^{\varrho} I^q x'(\nu), \qquad x''(T) = \iota^{\varrho} I^q x''(\zeta).$$

We can obtain

$$\begin{aligned}
x(t) = {} & \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}h(s)\mathrm{d}s + \frac{1}{\varpi_1(\vartheta,\tau)}(\vartheta^{\varrho}I^q\mathcal{J}^{\alpha_1}h(\tau)-\mathcal{J}^{\alpha_1}h(T)) \\
& - \frac{1}{\varpi_1(\vartheta,\tau)}\left(\frac{\varpi_2(\vartheta,\tau)}{\varpi_1(\vartheta,\tau)}-t\right)(\chi^{\varrho}I^q\mathcal{J}^{\alpha_1-1}h(\nu)-\mathcal{J}^{\alpha_1-1}h(T)) \\
& + \frac{1}{\varpi_1(\iota,\zeta)}\left(\frac{\varpi_3(\vartheta,\tau)}{2\varpi_1(\vartheta,\tau)}-\frac{\varpi_2(\vartheta,\tau)\varpi_2(\chi,\nu)}{\varpi_1(\vartheta,\tau)\varpi_1(\chi,\nu)}+\frac{\varpi_2(\chi,\nu)t}{\varpi_1(\chi,\nu)}-\frac{t^2}{2}\right) \\
& \times(\mathcal{J}^{\alpha_1-2}h(T)-\iota^{\varrho}I^q\mathcal{J}^{\alpha_1-2}h(\zeta)).
\end{aligned}$$

If $t\in(t_1,t_2)$, then

$$^{c}\mathcal{D}^{\alpha_1}x(t)=h(t), \qquad x(t_k^+)=x(t_k^-)+y_k.$$

We have

$$\begin{aligned}
x(t) = {} & y(t_1^+) - \frac{1}{\Gamma(\alpha_1)}\int_0^{t_1}(t-s)^{\alpha_1-1}h(s)\mathrm{d}s + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}h(s)\mathrm{d}s \\
& + \frac{1}{\varpi_1(\vartheta,\tau)}(\vartheta^{\varrho}I^q\mathcal{J}^{\alpha_1}h(\tau)-\mathcal{J}^{\alpha_1}h(T)) \\
& - \frac{1}{\varpi_1(\vartheta,\tau)}\left(\frac{\varpi_2(\vartheta,\tau)}{\varpi_1(\vartheta,\tau)}-t\right)(\chi^{\varrho}I^q\mathcal{J}^{\alpha_1-1}h(\nu)-\mathcal{J}^{\alpha_1-1}h(T)) \\
& + \frac{1}{\varpi_1(\iota,\zeta)}\left(\frac{\varpi_3(\vartheta,\tau)}{2\varpi_1(\vartheta,\tau)}-\frac{\varpi_2(\vartheta,\tau)\varpi_2(\chi,\nu)}{\varpi_1(\vartheta,\tau)\varpi_1(\chi,\nu)}+\frac{\varpi_2(\chi,\nu)t}{\varpi_1(\chi,\nu)}-\frac{t^2}{2}\right) \\
& \times(\mathcal{J}^{\alpha_1-2}h(T)-\iota^{\varrho}I^q\mathcal{J}^{\alpha_1-2}h(\zeta)) \\
= {} & y(t_1^+) + y_1 - \frac{1}{\Gamma(\alpha_1)}\int_0^{t_1}(t-s)^{\alpha_1-1}h(s)\lceil s + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}h(s)\mathrm{d}s \\
& + \frac{1}{\varpi_1(\vartheta,\tau)}(\vartheta^{\varrho}I^q\mathcal{J}^{\alpha_1}h(\tau)-\mathcal{J}^{\alpha_1}h(T)) \\
& - \frac{1}{\varpi_1(\vartheta,\tau)}\left(\frac{\varpi_2(\vartheta,\tau)}{\varpi_1(\vartheta,\tau)}-t\right)(\chi^{\varrho}I^q\mathcal{J}^{\alpha_1-1}h(\nu)-\mathcal{J}^{\alpha_1-1}h(T)) \\
& + \frac{1}{\varpi_1(\iota,\zeta)}\left(\frac{\varpi_3(\vartheta,\tau)}{2\varpi_1(\vartheta,\tau)}-\frac{\varpi_2(\vartheta,\tau)\varpi_2(\chi,\nu)}{\varpi_1(\vartheta,\tau)\varpi_1(\chi,\nu)}+\frac{\varpi_2(\chi,\nu)t}{\varpi_1(\chi,\nu)}-\frac{t^2}{2}\right) \\
& \times(\mathcal{J}^{\alpha_1-2}h(T)-\iota^{\varrho}I^q\mathcal{J}^{\alpha_1-2}h(\zeta)) \\
= {} & y_1 + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}h(s)\mathrm{d}s + \frac{1}{\varpi_1(\vartheta,\tau)}(\vartheta^{\varrho}I^q\mathcal{J}^{\alpha_1}h(\tau)-\mathcal{J}^{\alpha_1}h(T)) \\
& - \frac{1}{\varpi_1(\vartheta,\tau)}\left(\frac{\varpi_2(\vartheta,\tau)}{\varpi_1(\vartheta,\tau)}-t\right)(\chi^{\varrho}I^q\mathcal{J}^{\alpha_1-1}h(\nu)-\mathcal{J}^{\alpha_1-1}h(T)) \\
& + \frac{1}{\varpi_1(\iota,\zeta)}\left(\frac{\varpi_3(\vartheta,\tau)}{2\varpi_1(\vartheta,\tau)}-\frac{\varpi_2(\vartheta,\tau)\varpi_2(\chi,\nu)}{\varpi_1(\vartheta,\tau)\varpi_1(\chi,\nu)}+\frac{\varpi_2(\chi,\nu)t}{\varpi_1(\chi\nu)}-\frac{t^2}{2}\right) \\
& \times(\mathcal{J}^{\alpha_1-2}h(T)-\iota^{\varrho}I^q\mathcal{J}^{\alpha_1-2}h(\zeta)).
\end{aligned}$$

If $t \in (t_2, t_3)$, then

$$
\begin{aligned}
x(t) &= y(t_2^+) - \frac{1}{\Gamma(\alpha_1)}\int_0^{t_2}(t-s)^{\alpha_1-1}h(s)\mathrm{d}s + \frac{1}{\Gamma(\alpha_1)}\int_0^{t}(t-s)^{\alpha_1-1}h(s)\mathrm{d}s \\
&\quad + \frac{1}{\varpi_1(\vartheta,\tau)}(\vartheta^{\varrho} I^{q}\mathcal{J}^{\alpha_1}h(\tau) - \mathcal{J}^{\alpha_1}h(T)) \\
&\quad - \frac{1}{\varpi_1(\vartheta,\tau)}\left(\frac{\varpi_2(\vartheta,\tau)}{\varpi_1(\vartheta,\tau)} - t\right)(\chi^{\varrho} I^{q}\mathcal{J}^{\alpha_1-1}h(\nu) - \mathcal{J}^{\alpha_1-1}h(T)) \\
&\quad + \frac{1}{\varpi_1(\iota,\zeta)}\left(\frac{\varpi_3(\vartheta,\tau)}{2\varpi_1(\vartheta,\tau)} - \frac{\varpi_2(\vartheta,\tau)\varpi_2(\chi,\nu)}{\varpi_1(\vartheta,\tau)\varpi_1(\chi,\nu)} + \frac{\varpi_2(\chi,\nu)t}{\varpi_1(\chi,\nu)} - \frac{t^2}{2}\right) \\
&\quad \times (\mathcal{J}^{\alpha_1-2}h(T) - \iota^{\varrho} I^{q}\mathcal{J}^{\alpha_1-2}h(\zeta)) \\
&= y(t_2^+) + y_2 - \frac{1}{\Gamma(\alpha_1)}\int_0^{t_2}(t-s)^{\alpha_1-1}h(s)\mathrm{d}s + \frac{1}{\Gamma(\alpha_1)}\int_0^{t}(t-s)^{\alpha_1-1}h(s)\mathrm{d}s \\
&\quad + \frac{1}{\varpi_1(\vartheta,\tau)}(\vartheta^{\varrho} I^{q}\mathcal{J}^{\alpha_1}h(\tau) - \mathcal{J}^{\alpha_1}h(T)) \\
&\quad - \frac{1}{\varpi_1(\vartheta,\tau)}\left(\frac{\varpi_2(\vartheta,\tau)}{\varpi_1(\vartheta,\tau)} - t\right)(\chi^{\varrho} I^{q}\mathcal{J}^{\alpha_1-1}h(\nu) - \mathcal{J}^{\alpha_1-1}h(T)) \\
&\quad + \frac{1}{\varpi_1(\iota,\zeta)}\left(\frac{\varpi_3(\vartheta,\tau)}{2\varpi_1(\vartheta,\tau)} - \frac{\varpi_2(\vartheta,\tau)\varpi_2(\chi,\nu)}{\varpi_1(\vartheta,\tau)\varpi_1(\chi\nu)} + \frac{\varpi_2(\chi,\nu)t}{\varpi_1(\chi,\nu)} - \frac{t^2}{2}\right) \\
&\quad \times (\mathcal{J}^{\alpha_1-2}h(T) - \iota^{\varrho} I^{q}\mathcal{J}^{\alpha_1-2}h(\zeta)) \\
&= y_1 + y_2 + \frac{1}{\Gamma(\alpha_1)}\int_0^{t}(t-s)^{\alpha_1-1}h(s)\mathrm{d}s + \frac{1}{\varpi_1(\vartheta,\tau)}(\vartheta^{\varrho} I^{q}\mathcal{J}^{\alpha_1}h(\tau) - \mathcal{J}^{\alpha_1}h(T)) \\
&\quad - \frac{1}{\varpi_1(\vartheta,\tau)}\left(\frac{\varpi_2(\vartheta,\tau)}{\varpi_1(\vartheta,\tau)} - t\right)(\chi^{\varrho} I^{q}\mathcal{J}^{\alpha_1-1}h(\nu) - \mathcal{J}^{\alpha_1-1}h(T)) \\
&\quad + \frac{1}{\varpi_1(\iota,\zeta)}\left(\frac{\varpi_3(\vartheta,\tau)}{2\varpi_1(\vartheta,\tau)} - \frac{\varpi_2(\vartheta,\tau)\varpi_2(\chi,\nu)}{\varpi_1(\vartheta,\tau)\varpi_1(\chi,\nu)} + \frac{\varpi_2(\chi,\nu)t}{\varpi_1(\chi,\nu)} - \frac{t^2}{2}\right) \\
&\quad \times (\mathcal{J}^{\alpha_1-2}h(T) - \iota^{\varrho} I^{q}\mathcal{J}^{\alpha_1-2}h(\zeta)).
\end{aligned}
$$

If $t \in (t_m, T]$, then

$$
\begin{aligned}
x(t) = {} & \sum_{k=1}^{m} y_i + \mathcal{J}^{\alpha_1}h(t) + \frac{1}{\varpi_1(\vartheta,\tau)}(\vartheta^{\varrho} I|^{q}\mathcal{J}^{\alpha_1}h(\tau) - \mathcal{J}^{\alpha_1}h(T)) \\
& - \frac{1}{\varpi_1(\vartheta,\tau)}\left(\frac{\varpi_2(\vartheta,\tau)}{\varpi_1(\vartheta,\tau)} - t\right)(\chi^{\varrho} I^{q}\mathcal{J}^{\alpha_1-1}h(\nu) - \mathcal{J}^{\alpha_1-1}h(T)) \\
& + \frac{1}{\varpi_1(\iota,\zeta)}\left(\frac{\varpi_3(\vartheta,\tau)}{2\varpi_1(\vartheta,\tau)} - \frac{\varpi_2(\vartheta,\tau)\varpi_2(\chi,\nu)}{\varpi_1(\vartheta,\tau)\varpi_1(\chi,\nu)} + \frac{\varpi_2(\chi,\nu)t}{\varpi_1(\chi,\nu)} - \frac{t^2}{2}\right) \\
& \times (\mathcal{J}^{\alpha_1-2}h(T) - \iota^{\varrho} I^{q}\mathcal{J}^{\alpha_1-2}h(\zeta)) \quad \text{for } t \in (t_m, T].
\end{aligned} \tag{14}
$$

Conversely, assume that x satisfies the impulsive fractional integral equation (14). That completes of this proof. □

3. Main results

The following results are to be required for main results:

(H$_1$) f is continuous and there exists a constant $l > 0$ such that

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \le l\,[|u_1-u_2| + |v_1-v_2|]$$

for any $u_1, v_1, u_2, v_2 \in \mathbb{R}$ and $t \in \mathcal{J}$.

(H$_2$) There exists a constant $l_1 > 0$ such that

$$|k(t,s,u) - k(t,s,v)| \le l_1\,|u-v|$$

for any $u, v \in \mathbb{R}$ and $t, s \in \Delta$.

(H$_3$) Let $f\colon \mathcal{J}\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ be a function, and there exists a function $\mu \in \mathfrak{PC}([0,T],\mathbb{R})$ such that

$$|f(t,x,y)| \le \mu(t) \qquad \text{for any } (t,x,y) \in [0,T]\times\mathbb{R}.$$

(H$_4$) There exists a constant $M^* > 0$ such that $\sum_{i=1}^{m} |y_i| \le M^*$.

We define an operator $H\colon \mathfrak{PC}([0,T],\mathbb{R} \to \mathfrak{PC}([0,T],\mathbb{R})$ on problem (1)–(5) as

$$\begin{aligned}(Hx)(t) =& \sum_{i=1}^{m} y_i + \mathcal{J}^{\alpha_1} f(s,x(s),Bx(s))(t) + \frac{1}{\varpi_1(\vartheta,\tau)}(\vartheta^{\varrho} I^q \mathcal{J}^{\alpha_1}(\tau) f(s,x(s),Bx(s))\\ & - \mathcal{J}^{\alpha_1} f(s,x(s),Bx(s))(T)) - \frac{1}{\varpi_1(\chi,\nu)}\left(\frac{\varpi_2(\vartheta,\tau)}{\varpi_1(\vartheta,\tau)} - t\right)\\ & \times (\chi^{\varrho} I^q \mathcal{J}^{\alpha_1-1} f(s,x(s),Bx(s))(\nu) - \mathcal{J}^{\alpha_1-1} f(s,x(s),Bx(s))(T))\\ & + \frac{1}{\varpi_1(\iota,\zeta)}\left(\frac{\varpi_3(\vartheta\tau)}{2\varpi_1(\vartheta,\tau)} - \frac{\varpi_2(\vartheta\tau)\varpi_2(\chi,\nu)}{\varpi_1(\vartheta,\tau)\varpi_1(\chi,\nu)} + \frac{\varpi_2(\chi,\nu)t}{\varpi_1(\chi,\nu)} - \frac{t^2}{2}\right)\\ & \times (\mathcal{J}^{\alpha_1-2} f(s,x(s),Bx(s))(T) - \iota^{\varrho} I^q \mathcal{J}^{\alpha_1-2} f(s,x(s),Bx(s))(\zeta)).\end{aligned} \tag{15}$$

Also

$$\begin{aligned}\Theta =& M^* + \frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{1}{|\varpi_1(\vartheta,\tau)|\,\Gamma(\alpha_1+1)} \times \left(|\vartheta|\,\frac{\Gamma(\frac{\alpha_1+\varrho}{\varrho})}{\Gamma(\frac{\alpha_1+\varrho q+\varrho}{\varrho})}\frac{\tau^{\alpha_1+\varrho q}}{\varrho^q} + T^{\alpha_1}\right)\\ & \times \frac{1}{|\varpi_1(\chi,\nu)|\,\Gamma(\alpha_1)}\left(\frac{|\varpi_2(\chi,\nu)|}{|\varpi_1(\chi,\nu)|} + T\right)\left(|\chi|\,\frac{\Gamma(\frac{\alpha_1-1+\varrho}{\varrho})}{\Gamma(\frac{\alpha_1-1+\varrho q+\varrho}{\varrho})}\frac{\nu^{\alpha_1-1+\varrho q}}{\varrho^q} + T^{\alpha_1-1}\right)\\ & + \frac{1}{|\varpi_1(\iota,\zeta)|\,\Gamma(\alpha_1-1)}\\ & \times \left(\frac{|\varpi_3(\chi,\nu)|}{2\,|\varpi_1(\chi,\nu)|} + \frac{|\varpi_2(\vartheta,\tau)|\,|\varpi_2(\chi,\nu)|}{|\varpi_1(\vartheta,\tau)|\,|\varpi_1(\chi,\nu)|} + \frac{|\varpi_2(\chi,\nu)|\,T}{|\varpi_1(\chi,\nu)|} + \frac{T^2}{2}\right)\\ & \times \left(|\iota|\,\frac{\Gamma(\frac{\alpha_1-2+\varrho}{\varrho})}{\Gamma(\frac{\alpha_1-2+\varrho q+\varrho}{\varrho})}\frac{\nu^{\alpha_1-2+\varrho q}}{\varrho^q} + T^{\alpha_1-2}\right).\end{aligned} \tag{16}$$

and

$$\begin{aligned}\Theta_1 = &\frac{|\vartheta|}{|\varpi_1(\vartheta,\tau)|\,\Gamma(\alpha_1+1)}\frac{\Gamma(\frac{\alpha_1+\varrho}{\varrho})}{\Gamma(\frac{\alpha_1+\varrho q+\varrho}{\varrho})}\frac{\tau^{\alpha_1+\varrho q}}{\varrho^q}\\ &+\frac{|\chi|}{|\varpi_1(\chi,\nu)|\,\Gamma(\alpha_1)}\left(\frac{|\varpi_2(\vartheta,\tau)|}{|\varpi_1(\vartheta,\tau)|}+T\right)\frac{\Gamma(\frac{\alpha_1-1+\varrho}{\varrho})}{\Gamma(\frac{\alpha_1-1+\varrho q+\varrho}{\varrho})}\frac{\nu^{\alpha_1-1+\varrho q}}{\varrho^q}\\ &+\frac{|\iota|}{|\varpi_1(\iota,\zeta)|\,\Gamma(\alpha_1-1)}\Big(\frac{|\varpi_3(\vartheta,\tau)|}{2\,|\varpi_1(\vartheta,\tau)|}+\frac{|\varpi_2(\vartheta,\tau)|\,|\varpi_2(\chi,\nu)|}{|\varpi_1(\vartheta,\tau)|\,|\varpi_1(\chi,\nu)|}\\ &+\frac{|\varpi_2|\,T}{|\varpi_1(\chi,\nu)|}+\frac{T^2}{2}\Big)\frac{\Gamma(\frac{\alpha_1-2+\varrho}{\varrho})}{\Gamma(\frac{\alpha_1-2+\varrho q+\varrho}{\varrho})}\frac{\nu^{\alpha_1-2+\varrho q}}{\varrho^q}.\end{aligned} \tag{17}$$

Theorem 3.1. *Let f be a continuous function. Assume that the hypothesis* (H_1) *and* (H_2) *is satisfied. If*

$$l(1+l_1)\Theta < 1$$

then the problems (1)–(5) *has a unique solution on* $[0,T]$.

Proof. We define the operator $H\colon \mathfrak{PC}(\mathcal{J},\mathbb{R}) \to \mathfrak{PC}(\mathcal{J},\mathbb{R})$ defined in (15), we get

$$\begin{aligned}&|(Hx)(t)-(Hy)(t)|\\ &= \mathcal{J}^{\alpha_1}\left|f(s,x(s),Bx(s))-f(s,y(s),Bx(s))\right|(T)\\ &\quad+\frac{|\vartheta|}{|\varpi_1(\vartheta,\tau)|}\,{}^{\varrho}I^q\mathcal{J}^{\alpha_1}|f(s,x(s),Bx(s))-f(s,x(s),Bx(s))|(\tau)\\ &\quad+\frac{1}{|\varpi_1(\chi,\tau)|}\mathcal{J}^{\alpha_1}\left|f(s,x(s),Bx(s))-f(s,y(s),Bx(s))\right|(T)\\ &\quad+\frac{|\chi|}{|\varpi_1(\chi,\nu)|}\Big(\frac{|\varpi_2(\vartheta,\tau)|}{|\varpi_1(\vartheta,\tau)|}+T\Big)\\ &\quad\times{}^{\varrho}I^q\mathcal{J}^{\alpha_1-1}\left|f(s,x(s),Bx(s))-f(s,y(s),Bx(s))\right|(T)\\ &\quad+\frac{1}{|\varpi_1(\iota,\zeta)|}\Big(\frac{|\varpi_3(\vartheta,\tau)|}{2\,|\varpi_1(\vartheta,\tau)|}+\frac{|\varpi_2(\vartheta,\tau)|\,|\varpi_2(\chi,\nu)|}{|\varpi_1(\vartheta,\tau)|\,|\varpi_1(\chi,\nu)|}\\ &\quad+\frac{\varpi_2(\chi,\nu)T}{|\varpi_1(\chi,\nu)|}+\frac{T^2}{2}\Big){}^{\varrho}I^q\mathcal{J}^{\alpha_1-2}\left|f(s,x(s),Bx(s))-f(s,x(s),Bx(s))\right|(\zeta)\\ &\le l(1+l_1)\Bigg\{\frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)}+\frac{1}{|\varpi_1(\vartheta,\tau)|\,\Gamma(\alpha_1+1)}\left(|\vartheta|\frac{\Gamma(\frac{\alpha_1+\varrho}{\varrho})}{\Gamma(\frac{\alpha_1+\varrho q+\varrho}{\varrho})}\frac{\tau^{\alpha_1+\varrho q}}{\varrho^q}+T^{\alpha_1}\right)\\ &\quad+\frac{1}{|\varpi_1(\chi,\nu)|\,\Gamma(\alpha_1)}\left(\frac{|\varpi_2(\vartheta,\tau)|}{|\varpi_1(\vartheta,\tau)|}+T\right)\left(|\chi|\frac{\Gamma\left(\frac{\alpha_1-1+\varrho}{\varrho}\right)}{\Gamma(\frac{\alpha_1-1+\varrho q+\varrho}{\varrho})}\frac{\nu^{\alpha_1-1+\varrho q}}{\varrho^q}+T^{\alpha_1-1}\right)\\ &\quad+\frac{1}{|\varpi_1(\iota,\zeta)|\,\Gamma(\alpha_1-1)}\Big(\frac{|\varpi_3(\vartheta,\tau)|}{2\,|\varpi_1(\vartheta,\tau)|}+\frac{|\varpi_2(\vartheta,\tau)|\,|\varpi_2(\chi,\nu)|}{|\varpi_1(\vartheta,\tau)|\,|\varpi_1(\chi,\nu)|}\\ &\quad+\frac{|\varpi_2(\chi,\nu)|\,T}{|\,|\varpi_1(\chi,\nu)|}+\frac{T^2}{2}\Big)\times\left(|\iota|\frac{\Gamma\left(\frac{\alpha_1-2+\varrho}{\varrho}\right)}{\Gamma\left(\frac{\alpha_1-2+\varrho q+\varrho}{\varrho}\right)}\frac{\zeta^{\alpha_1-2+\varrho q}}{\varrho^q}+T^{\alpha_1-2}\right)\Bigg\}\|x-y\|\\ &\le l(1+l_1)\Theta\,\|x-y\|.\end{aligned}$$

The above equation is less than one, therefore, H is a contraction. The problem stated in (1)–(5) has a unique solution on $[0,T]$. □

Theorem 3.2. *Assume that the hypothese* (H_3) *and* (H_4) *are fulfilled, then* (1)–(5) *problem has at least one solution on* $[0,T]$.

Proof. Define the new operators $\Xi_1$and Ξ_2 as

$$\begin{aligned}(\Xi_1 x)(t) &= \mathcal{J}^{\alpha_1} f(s,x(s),Bx(s))(T)\\ &+\frac{1}{\varpi_1(\chi,\nu)}\Big(\frac{\varpi_2(\vartheta,\tau)}{\varpi_2(\vartheta,\tau)}-t\Big)\mathcal{J}^{\alpha_1-1} f(s,x(s),Bx(s))(T)\\ &+\frac{1}{\varpi_1(\iota,\zeta)}\Big(\frac{\varpi_3(\vartheta,\tau)}{2\varpi_1(\vartheta,\tau)}-\frac{\varpi_2(\vartheta,\tau)\varpi_2(\chi,\nu)}{\varpi_1(\vartheta,\tau)\varpi_1(\chi,\nu)}+\frac{\varpi_2(\chi,\nu)t}{\varpi_1(\chi,\nu)}-\frac{t^2}{2}\Big)\\ &\times \mathcal{J}^{\alpha_1-2} f(s,x(s),Bx(s))(T)\end{aligned} \tag{18}$$

and

$$\begin{aligned}(\Xi_2 x)(t)\ &\sum_{i=1}^{m} y_i+\frac{\vartheta}{\varpi_1(\iota,\zeta)}{}^{\varrho}I^{q}\mathcal{J}^{\alpha_1} f(x,x(s),Bx(s))(\tau)\\ &-\frac{\chi}{\varpi_1(\chi,\nu)}\Big(\frac{\varpi_2(\vartheta,\tau)}{\varpi_1(\vartheta,\tau)}-t\Big){}^{\varrho}I^{q}\mathcal{J}^{\alpha_1} f(x,x(s),Bx(s))(\nu)\\ &-\frac{\iota}{\varpi_1(\iota,\zeta)}\Big(\frac{\varpi_3(\vartheta,\tau)}{2\varpi_1(\vartheta,\tau)}-\frac{\varpi_2(\vartheta,\tau)\varpi_2(\chi,\nu)}{\varpi_1(\vartheta,\tau)\varpi_1(\chi,\nu)}+\frac{\varpi_2(\chi,\nu)t}{\varpi_1(\chi,\nu)}-\frac{t^2}{2}\Big)\\ &\times \mathcal{J}^{\alpha_1-2}{}^{\varrho}I^{q}\mathcal{J}^{\alpha_1} f(s,x(s),Bx(s))(\zeta).\end{aligned} \tag{19}$$

Consider

$$B_d=\{x\in\mathfrak{PC}:\|x\|<d\}\,.$$

For any $x,y\in B_d$, $\Xi_1 x+\Xi_2 y\in B_d$, where Ξ_1 and Ξ_2 are denoted by (18) and (19), respectively.

$$\begin{aligned}&\|\Xi_1 x+\Xi_2 y\|\\ &\le\ \mathcal{J}^{\alpha_1}\left|f(s,x(s),Bx(s))\right|\mathrm{d}s(T)+\frac{1}{|\varpi_1(\vartheta,\tau)|}\\ &\quad\times(\mathcal{J}^{\alpha_1}\left|f(s,x(s),Bx(s))\right|\mathrm{d}s(T)+|\vartheta|\,{}^{\varrho}I^{q}\mathcal{J}^{\alpha_1}\left|f(s,x(s),Bx(s))\right|\mathrm{d}s(\tau))\\ &\quad+\frac{1}{\varpi_1(\chi,\nu)}\Big(\frac{\varpi_2(\vartheta,\tau)}{\varpi_1(\vartheta,\tau)}+T\Big)\\ &\quad\times(\mathcal{J}^{\alpha_1-1}\left|f(s,x(s),Bx(s))\right|\mathrm{d}s(T)+\chi^{\varrho}I^{q}\mathcal{J}^{\alpha_1-1}\left|f(s,x(s),Bx(s))\right|\mathrm{d}s(\nu))\\ &\quad+\frac{1}{\varpi_1(\iota,\zeta)}\Big(\frac{\varpi_3(\vartheta,\tau)}{2\varpi_1(\vartheta,\tau)}\qquad+\frac{\varpi_2(\vartheta,\tau)\varpi_2(\chi,\nu)}{\varpi_1(\vartheta,\tau)\varpi_1(\chi,\nu)}+\frac{\varpi_2(\chi,\nu)t}{\varpi_1(\chi,\nu)}+\frac{T^2}{2}\Big)\\ &\quad\times(\mathcal{J}^{\alpha_1-2}\left|f(s,x(s),Bx(s))\right|\mathrm{d}s(T)\\ &\quad+|\iota|^{\varrho}\,I^{q}\mathcal{J}^{\alpha_1-2}\left|f(s,x(s),Bx(s))\right|\mathrm{d}s(\zeta))+\sum_{i=1}^{m} y_i\end{aligned}$$

$$
\begin{aligned}
&\leq \mathcal{J}^{\alpha_1}\mu(t)(T) + \frac{1}{|\varpi_1(\vartheta,\tau)|}(\mathcal{J}^{\alpha_1}\mu(t)\mathrm{d}s(T) + |\vartheta|\,{}^{\varrho}I^{q}\mathcal{J}^{\alpha_1}\mu(t)(\tau)) \\
&\quad + \frac{1}{\varpi_1(\chi,\nu)}\Big(\frac{\varpi_2(\vartheta,\tau)}{\varpi_1(\vartheta,\tau)} + T\Big)(\mathcal{J}^{\alpha_1-1}\mu(t)(T) + \chi^{\varrho}I^{q}\mathcal{J}^{\alpha_1-1}\mu(t)(\nu)) \\
&\quad + \frac{1}{\varpi_1(\iota,\zeta)}\Big(\frac{\varpi_3(\vartheta,\tau)}{2\varpi_1(\vartheta,\tau)} + \frac{\varpi_2(\vartheta,\tau)\varpi_2(\chi,\nu)}{\varpi_1(\vartheta,\tau)\varpi_1(\chi,\nu)} + \frac{\varpi_2(\chi,\nu)t}{\varpi_1(\chi,\nu)} + \frac{T^2}{2}\Big) \\
&\quad \times (\mathcal{J}^{\alpha_1-2}\mu(s)(T) + |\iota|^{\varrho}\,I^{q}\mathcal{J}^{\alpha_1-2}\mu(t)(\zeta)) + \sum_{i=1}^{m} y_i \\
&\leq \mu(t)\Big\{\frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{1}{|\varpi_1(\vartheta,\tau)|\,\Gamma(\alpha_1+1)} \times \Big(|\vartheta|\,\frac{\Gamma(\frac{\alpha_1+\varrho}{\varrho})}{\Gamma(\frac{\alpha_1+\varrho q+\varrho}{\varrho})}\frac{\tau^{\alpha_1+\varrho q}}{\varrho^q} + T^{\alpha_1}\Big) \\
&\quad \times \frac{1}{|\varpi_1(\chi,\nu)|\,\Gamma(\alpha_1)}\Big(\frac{|\varpi_2(\chi,\nu)|}{|\varpi_1(\chi,\nu)|} + T\Big) \\
&\quad \times \Bigg(|\chi|\,\frac{\Gamma(\frac{\alpha_1+\varrho}{\varrho})}{\Gamma\Big(\frac{\alpha_1+\varrho q+\varrho}{\varrho}\Big)}\Gamma(\frac{\alpha_1-1+\varrho q+\varrho}{\varrho})\frac{\nu^{\alpha_1-1+\varrho q}}{\varrho^q} + T^{\alpha_1-1}\Bigg) \\
&\quad + \frac{1}{|\varpi_1(\iota,\zeta)|\,\Gamma(\alpha_1-1)}\Bigg(\frac{|\varpi_3(\chi,\nu)|}{2\,|\varpi_1(\chi,\nu)|} + \frac{|\varpi_2(\chi,\nu)|\,|\varpi_2(\chi,\nu)|}{|\varpi_2(\chi,\nu)|\,|\varpi_1(\chi,\nu)|} + \frac{|\varpi_2(\chi,\nu)|\,T}{|\varpi_1(\chi,\nu)|} + \frac{T^2}{2}\Bigg) \\
&\quad \times \Bigg(|\iota|\,\frac{\Gamma(\frac{\alpha_1+\varrho}{\varrho})}{\Gamma(\frac{\alpha_1+\varrho q+\varrho}{\varrho})}\Gamma(\frac{\alpha_1-1+\varrho q+\varrho}{\varrho})\frac{\nu^{\alpha_1-1+\varrho q}}{\varrho^q} + T^{\alpha_1-2}\Bigg) + \sum_{i=1}^{m} y_i\Big\} \\
&\leq \mu(t)\Big\{\frac{T^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{1}{|\varpi_1(\vartheta,\tau)|\,\Gamma(\alpha_1+1)} \times \Big(|\vartheta|\,\frac{\Gamma(\frac{\alpha_1+\varrho}{\varrho})}{\Gamma(\frac{\alpha_1+\varrho q+\varrho}{\varrho})}\frac{\tau^{\alpha_1+\varrho q}}{\varrho^q} + T^{\alpha_1}\Big) \\
&\quad \times \frac{1}{|\varpi_1(\chi,\nu)|\,\chi(\alpha_1)}\Big(\frac{|\varpi_2(\chi,\nu)|}{|\varpi_1(\chi,\nu)|} + T\Big) \\
&\quad \times \Bigg(|\chi|\,\frac{\Gamma(\frac{\alpha_1+\varrho}{\varrho})}{\Gamma(\frac{\alpha_1+\varrho q+\varrho}{\varrho})}\Gamma(\frac{\alpha_1-1+\varrho q+\varrho}{\varrho})\frac{\nu^{\alpha_1-1+\varrho q}}{\varrho^q} + T^{\alpha_1-1}\Bigg) \\
&\quad + \frac{1}{|\varpi_1(\iota,\zeta)|\,\Gamma(\alpha_1-1)}\Bigg(\frac{|\varpi_3(\chi,\nu)|}{2\,|\varpi_1(\chi,\nu)|} + \frac{|\varpi_2(\chi,\nu)|\,|\varpi_2(\chi,\nu)|}{|\varpi_2(\chi,\nu)|\,|\varpi_1(\chi,\nu)|} + \frac{|\varpi_2(\chi,\nu)|\,T}{|\varpi_1(\chi,\nu)|} + \frac{T^2}{2}\Bigg) \\
&\quad \times \Bigg(|\iota|\,\frac{\Gamma(\frac{\alpha_1+\varrho}{\varrho})}{\Gamma(\frac{\alpha_1+\varrho q+\varrho}{\varrho})}\Gamma(\frac{\alpha_1-1+\varrho q+\varrho}{\varrho})\frac{\nu^{\alpha_1-1+\varrho q}}{\varrho^q} + T^{\alpha_1-2}\Bigg) + M^*\Big\} \\
&\leq \mu(t)\Phi \leq d.
\end{aligned}
$$

Thus $\Xi_1 x + \Xi_2 y \in B_d$.

By (H_3), Ξ_2 is a contraction, and by (H_1), the operator $\Xi_1 x$ is continuous. Also, Hence $\Xi_1 y$ is uniformly bounded on B_d.

Now let us prove that $(\Xi_1)(t)$ is equicontinuous.

Let $t_1, t_2 \in \mathcal{J}$, $t_2 \leq t_1$ and $x \in B_r$. Using the fact that f is bounded on the compact set,

$$\sup_{(t,x,y)\in[0,1]\times B_d} \|f(t,x,y)\| = \bar{f},$$

we get

$$
\begin{aligned}
&|(Hx)(t_2) - (Hx)(t_1)| \\
\leq\ & |\mathcal{J}^{\alpha_1} f(s, x(s), Bx(s))(t_2) - \mathcal{J}^{\alpha_1} f(s, x(s), Bx(s))(t_1)| \\
&+ \frac{|t_2 - t_1|}{|\varpi_1(\chi,\nu)|}\Big(|\chi|\, {}^{q}\mathcal{J}^{\alpha_1-1} |f(s, x(s), Bx(s))| (\nu) \\
&+ \mathcal{J}^{\alpha_1-1} |f(s, x(s), Bx(s))| (T)\Big) \\
&+ \frac{1}{|\varpi_1(\iota,\tau)|}\Big(\frac{|\varpi_2(\chi,\nu)|}{|\varpi_2(\chi,\nu)|} |t_2 - t_1| + \frac{|t_2^2 - t_1^2|}{2}\Big) \\
&\times \big(\mathcal{J}^{\alpha_1-2} |f(s, x(s), Bx(s)| (T) + |\iota|\, {}^{\varrho}I^q \mathcal{J}^{\alpha_1-2} |f(s, x(s), Bx(s))| (\zeta)\big) \\
\leq\ & \frac{\bar{f}}{\Gamma(\alpha_1)}\Big[\int_0^{t_1} \Big[(t_2 - s)^{\alpha_1-1} - (t_1 - s)^{\alpha_1-1}\Big] \mathrm{d}s + \int_{t_1}^{t_2} (t_2 - s)^{\alpha_1-1} \mathrm{d}s\Big] \\
&+ |t_2 - t_1| \frac{\bar{f}}{|\varpi_1(\chi,\nu)|}\Big(|\chi|\, \mathcal{J}^{\alpha_1-1}(\nu) + \mathcal{J}^{\alpha_1-1}(T) \\
&+ \Big(\frac{|\varpi_2(\chi,\nu)|}{|\varpi_1(\iota,\zeta)|} + \frac{|t_2 - t_1|}{2} \frac{|\varpi_1(\chi,\nu)|}{|\varpi_1(\iota\zeta)|}\Big) \big(\mathcal{J}^{\alpha_1-2}(T) + |\iota|\, {}^{\varrho}I^q \mathcal{J}^{\alpha_1-2}(\zeta)\big) \Big) \\
\leq\ & \frac{\bar{f}}{\Gamma(\alpha_1)}\Big[\int_0^{t_1} \big[(t_2 - s)^{\alpha_1-1} - (t_1 - s)^{\alpha_1-1}\big] \mathrm{d}s + \int_{t_1}^{t_2} (t_2 - s)^{\alpha_1-1} \mathrm{d}s\Big] \\
&+ |t_2 - t_1| \frac{\bar{f}}{|\varpi_1(\chi,\nu)|}\Big\{\frac{1}{\chi(\alpha_1)}\Big(T^{\alpha_1-1} + |\chi| \frac{\Gamma(\frac{\alpha_1-1+\varrho}{\varrho})}{\Gamma(\frac{\alpha_1-1+\varrho q+\varrho}{\varrho})} \frac{\nu^{\alpha_1-1+\varrho q}}{\varrho^q}\Big) \\
&+ \frac{1}{\Gamma(\alpha_1-1)}\left(\frac{|\varpi_2(\chi,\nu)|}{|\varpi_1(\iota,\zeta)|} + \frac{|t_2 - t_1|}{2} \frac{|\varpi_1(\chi,\nu)|}{|\varpi_1(\iota,\zeta)|}\right) \\
&\times \big(T^{\alpha_1-2}\big) + |\iota| \frac{\Gamma(\frac{\alpha_1-1+\varrho}{\varrho})}{\Gamma(\frac{\alpha_1-1+\varrho q+\varrho}{\varrho})} \frac{\nu^{\alpha_1-1+\varrho q}}{\varrho^q} \frac{\zeta^{\alpha_1-2+\varrho q}}{\varrho^q}\Big\}.
\end{aligned}
$$

It is clear that the right-hand side of the above inequality $\to 0$ is independent of $u, v \in B_d$ as $t_2 - t_1 \to 0$. As H satisfies the above assumptions, therefore, by the Arzela-Ascoli theorem, it follows that $H\colon \mathfrak{PC}([0,T],\mathbb{R}) \to \mathfrak{PC}([0,T],\mathbb{R})$ which is completely continuous.

We show that the operator $(\Xi_2 x)$ is a contraction.

$$
\begin{aligned}
&\|\Xi_2 x - \Xi_2 y\| \\
=\ & \frac{|\vartheta|}{|\varpi_1(\iota,\zeta)|} {}^{\varrho}I^q \mathcal{J}^{\alpha_1} \|f(x, x(s), Bx(s)) - f(x, y(s), By(s))\|(\tau) \\
&+ \frac{|\chi|}{|\varpi_1(\chi,\nu)|}\Big(\frac{|\varpi_2(\vartheta,\tau)|}{|\varpi_1(\vartheta,\tau)|} + T\Big) {}^{\varrho}I^q \mathcal{J}^{\alpha_1} \|f(x, x(s), Bx(s)) - f(x, y(s), By(s))\|(\nu) \\
&+ \frac{|\iota|}{|\varpi_1(\iota,\zeta)|}\Big(\frac{|\varpi_3(\vartheta,\tau)|}{2|\varpi_1(\vartheta,\tau)|} - \frac{|\varpi_2(\vartheta,\tau)\varpi_2(\chi,\nu)|}{|\varpi_1(\vartheta\tau)\varpi_1(\chi,\nu)|} + \frac{|\varpi_2(\chi,\nu)|T}{|\varpi_1(\chi,\nu)|} - \frac{T^2}{2}\Big) \\
&\times \mathcal{J}^{\alpha_1-2} {}^{\varrho}I^q \mathcal{J}^{\alpha_1} \|f(s, x(s), Bx(t)) - f(x, y(s), By(s))\|(\zeta)
\end{aligned}
$$

$$\leq\ l(1+l_1)\,\|x-y\|\Big\{\frac{|\vartheta|}{|\varpi_1(\iota\zeta)|}\,{}^{\varrho}I^{q}\mathcal{J}^{\alpha_1}(\tau)+\frac{|\chi|}{|\varpi_1(\chi,\nu)|}\Big(\frac{|\varpi_2(\vartheta,\tau)|}{|\varpi_1(\vartheta,\tau)|}+T\Big){}^{\varrho}I^{q}\mathcal{J}^{\alpha_1}(\nu)$$
$$+\frac{|\iota|}{|\varpi_1(\iota,\zeta)|}\Big(\frac{|\varpi_3(\vartheta,\tau)|}{2|\varpi_1(\vartheta,\tau)|}-\frac{|\varpi_2(\vartheta,\tau)\varpi_2(\chi,\nu)|}{|\varpi_1(\vartheta,\tau)\varpi_1(\chi,\nu)|}+\frac{|\varpi_2(\chi,\nu)|T}{|\varpi_1(\chi,\nu)|}+\frac{T^2}{2}\Big)$$
$$\times\,\mathcal{J}^{\alpha_1-2\varrho}I^{q}\mathcal{J}^{\alpha_1}(\zeta)\Big\}$$
$$\leq\ l(1+l_1)\Big\{\frac{|\vartheta|}{|\varpi_1(\vartheta,\tau)|\,\Gamma(\alpha_1+1)}\frac{\Gamma(\frac{\alpha_1+\varrho}{\varrho})}{\Gamma(\frac{\alpha_1+\varrho q+\varrho}{\varrho})}\frac{\tau^{\alpha_1+\varrho q}}{\varrho^{q}}$$
$$+\frac{|\chi|}{|\varpi_1(\chi,\nu)|\,\Gamma(\alpha_1)}\Big(\frac{|\varpi_2(\vartheta,\tau)|}{|\varpi_1(\vartheta,\tau)|}+T\Big)\frac{\Gamma(\frac{\alpha_1-1+\varrho}{\varrho})}{\Gamma(\frac{\alpha_1-1+\varrho q+\varrho}{\varrho})}\frac{\nu^{\alpha_1-1+\varrho q}}{\varrho^{q}}$$
$$+\frac{|\iota|}{|\varpi_1(\iota,\zeta)|\,\Gamma(\alpha_1-1)}\Big(\frac{|\varpi_3(\vartheta,\tau)|}{2\,|\varpi_1(\vartheta,\tau)|}+\frac{|\varpi_2(\vartheta,\tau)|\,|\varpi_2(\chi,\nu)|}{|\varpi_1(\vartheta,\tau)|\,|\varpi_1(\chi,\nu)|}$$
$$+\frac{|\varpi_2(\chi,\nu)|\,T}{|\,|\varpi_1(\chi,\nu)|}+\frac{T^2}{2}\Big)\frac{\Gamma(\frac{\alpha_1-2+\varrho}{\varrho})}{\Gamma(\frac{\alpha_1-2+\varrho q+\varrho}{\varrho})}\frac{\nu^{\alpha_1-2+\varrho q}}{\varrho^{q}}\Big\}\,\|x-y\|$$
$$\leq\ l(1+l_1)\Theta_1\,\|x-y\|\,.$$

So, $\|\Xi_1x-\Xi_2y\|\leq l(1+l_1)\Theta_1\,\|x-y\|$. As $l(1+l_1)\Theta_1<1$, Ξ_2 the operator is a contraction. Therefore, the problem (1)–(5) has at least one solution on $[0,T]$. □

4. Example

Consider the impulsive Katugampola integral boundary value problem as follows:

$$(20)\qquad {}^{C}\mathcal{D}^{7/3}x(t)=\frac{|x|}{2(\mathrm{e}^{t}+1)^2(1+|x|)}+\frac{1}{2}\int_0^t \mathrm{e}^{\frac{-1}{4}}x(s)\mathrm{d}s\qquad t\in\Big[0,\frac{1}{2}\Big],$$
$$x(t_k^{+})=x(t_k^{-})+\frac{1}{6},$$
$$x\Big(\frac{1}{2}\Big)=\frac{1}{2}{}^{3}I^{\frac{1}{2}}x(3/8),$$
$$x'\Big(\frac{1}{2}\Big)=\frac{1}{2}{}^{3}I^{\frac{1}{2}}x'(1/3),$$
$$(21)\qquad x''\Big(\frac{1}{2}\Big)=\frac{1}{2}{}^{3}I^{\frac{1}{2}}x''(2/5).$$

Set

$$f(t,u,Bu)=\frac{|x|}{2(\mathrm{e}^{t}+1)^2(1+|x|)}+Bu(t),\qquad (t,u)\in\mathcal{J}\times[0,\infty),$$

and

$$Bu(t)=\frac{1}{2}\int_0^t \mathrm{e}^{\frac{-1}{4}}u(s)\mathrm{d}s.$$

Let $x,y\in\mathbb{R}$ and $t\in\mathcal{J}$, we have

$$|Bx(t)-By(t)|=\frac{1}{2}\left|\int_0^t \mathrm{e}^{-\frac{1}{2}}x(s)\mathrm{d}s-\int_0^t \mathrm{e}^{-\frac{1}{4}}y(s)\mathrm{d}s\right|\leq\frac{1}{8}\,|x-y|$$

and

$$\begin{aligned}|f(t,x,Bx(t))-f(t,y,By(t))| &= \frac{1}{2(\mathrm{e}^t+1)^2}\left|\frac{x}{1+x}-\frac{y}{1+y}+\frac{1}{8}(Bx(t)-By(t))\right| \\ &\leq \frac{|x-y|}{2(\mathrm{e}^t+1)^2}+\frac{1}{8}\left|(Bx(t)-By(t))\right| \\ &\leq \frac{1}{8}|x-y|+|(Bx(t)-By(t))|.\end{aligned}$$

Here $\alpha_1 = 7/3$, $T = 1/2$, $\vartheta = 1/2$, $\chi = 1/2$, $\iota = 1/2$, $\tau = 3/8$, $\nu = 1/3$, $\zeta = 2/5$, $\varrho = 2$, $q = 1$, $\varpi_1(\frac{1}{2},\frac{3}{8}) = 0.9361$, $\varpi_1(\frac{1}{2},\frac{1}{3}) = 0.9475$, $\varpi_1(\frac{1}{2},\frac{2}{5}) = 0.9289$, $\varpi_2(\frac{1}{2},\frac{3}{8}) = 0.4779$, $\varpi_2(\frac{1}{2},\frac{1}{3}) = 0.4838$, $\varpi_3(\frac{1}{2},\frac{3}{8}) = 0.4922$, Hence, the hypothesis H_1 holds with $l = \frac{1}{8}$, $l_1 = \frac{1}{2}$, and we check that

$$l(1+l_1)\Theta < 1 \approx 0.312260625.$$

Thus, the hypothesis H_1 and Theorem 3.1 are satisfied, and show that the problem (20)–(21) has a unique solution on $\left[0,\frac{1}{2}\right]$.

5. Conclusion

In this work, we discuss the existence results for impulsive fractional integro-differential equations with Katugampola fractional integral boundary conditions. Our results guarantee the existence of integral solution via fractional calculus theory and Krasnoselkii's fixed point theorem. Example shows the efficience of solution and effectiveness of the theoretical outcome. One can also extend the problem with more advanced delays.

Acknowledgement. We are thankful to the reviewers for their comments and suggestions which helped us to improve the manuscript.

References

1. Almeida R., Malinowska A. B. and Odzijewicz T., *Fractional differential equations with dependence on the Caputo-Katugampola derivative*, J. Comput. Nonlinear Dyn. **11** (2016), 1–11.

2. Anguraj A., Karthikeyan P., Rivero M. and Trujillo J. J., *On new existence results for fractional integro-differential equations with impulsive and integral conditions*, Comput. Math. Appl. **66** (2014), 2587–2594.

3. Ahmad B., Alghanmi M., Ntouyas S. K. and Alsaedi A., *Fractional differential equations involving generalized derivative with Stieltjes and fractional integral boundary conditions*, App. Math. Lett. **84** (2018), 111–117.

4. Ahmad B., Ntouyas S. K. and Tariboon J., *Nonlocal fractional-order boundary value problems with generalized Riemann-Liouville integral boundary conditions*, J. Comput. Anal. Appl. **23** (2017), 480–497.

5. Ahmad B., Karthikeyan P. and Buvaneswari K., *Fractional differential equations with coupled slit-strips type integral boundary conditions*, AIMS Math. **4** (2019), 1596–1609.

6. Ahmad B., Alghanmi M., Ntouyas S. K. and Alsaedi A., *A study of fractional differential equations and inclusions involving generalized Caputo-type derivative equipped with generalized fractional integral boundary conditions*, AIMS Math. **4** (2018), 26–42.

7. Ahmad B. and Wang G., *A study of an impulsive four-point nonlocal boundary value problem of nonlinear fractional differential equations*, Comput. Math. Appl. **62** (2011), 1341–1349.

8. Benchohra M., Henderson J. and Ntouyas S. K., *Impulsive Differential Equations and Inclusions*, Hindawi Publishing Corporation, New York, 2006.

9. Benchohra M., Bouriah S. and Nieto J. J., *Terminal Value Problem for Differential Equations with Hilfer-Katugampola Fractional Derivative*, Symmetry **11** (2019), 1-14.

10. Benchohra M., Bouriah S. and Graef J. R., *Boundary value problems for nonlinear implicit Caputo-Hadamard type fractional differential equations with impulses*, Mediterr. J. Math. **14** (2017), 1–21.

11. Chang Y. K., Anguraj A. and Karthikeyan P., *Existence results for initial value problems with integral condition for impulsive fractional differential equations*, J. Fract. Cal. Appl. **2** (2012), 1–10.

12. Graef J. R., Henderson J. and Ouahab A., *Impulsive Differential Inclusions. A Fixed Point Approach*, De Gruyter, Berlin/Boston, 2013.

13. Granas A. and Dugundji J., *Fixed Point Theory*, Springer-Verlag, New work, 2003.

14. Janaki M., Kanagarajan K. and Vivek D., *Note on impulsive fractional differential equations via Katugampola fractional derivative*, Glob. J. Math. **12** (2018), 829–838.

15. Katugampola U. N., *New approach to a generalized fractional integral*, Appl. Math. Comput. **218** (2015), 860–865.

16. Katugampola U. N., *A New approach to generalized fractional derivatives*, Bul. Math. Anal. Appl. **6** (2014), 1–15.

17. Katugampola U. N., *Existence and uniqueness results for a class of generalized fractional differential equations*, Class. Anal. ODEs (2014), 1-9.

18. Karthikeyan P. and Arul R., *Uniqueness and stability results for non-linear impulsive implicit Hadamard fractional differential integral boundary value problems*, J. Comput. Nonlinear Dyn. **9** (2020), 23–29.

19. Kilbas A. A., Srivastava H. M. and Trujillo J. J., *Theory And Applications Of Fractional Differential Equations*, North-Holland Mathematics Studies, 2006.

20. Lakshmikantham V., Bainov D. D. and Simeonov P. S., *Theory Of Impulsive Differential Equations*, Worlds Scientific, Singapore, 1989.

21. Mahmudov N. and Emin S., *Fractional-order boundary value problems with Katugampola fractional integral conditions*, Adv. Difference Equ. **2018** (2018), 1–17.

22. Mahmudov N., Awadalla M. and Abuassba K., *Nonlinear sequential fractional differential equations with nonlocal boundary conditions*, Adv. Difference Equ. **2017** (2017), 1–17.

23. Nandhini A., Vivek D. and Elsayed E. M., *Nonlocal initial value problems for Katugampola-Caputo type fractional differential equations on time scales*, Open J. Math. Sci. **3** (2019), 7–10.

24. Samko S. G., Kilbas A. A. and Marichev O. I., *Fractional Integrals And Derivatives, Theory and Applications*, Gordon and Breach, Yverdon, 1993.

25. Samoilenko A. M. and Perestyuk N. A., *Impulsive Differential Equations*, World Scientific, Singapore, 1995.

26. Thongsalee N., Ntouyas S. K. and Tariboon J., *Nonlinear Riemann-Liouville fractional differential equations with nonlocal Erdelyi-Kober fractional integral conditions*, Fract. Cal. Appl. Anal. **19** (2016), 480–497.

27. Vanterler da C. Sousa J. and Capelas de Oliveira E., *On the Ψ-Hilfer fractional derivative*, Commun. Nonlinear Sci. Numer. Simul. **60** (2018), 72–91.

28. Vanterler da C. Sousa J., Oliveira D. S. and Capelas de Oliveira E., *On the existence and stability for noninstantaneous impulsive fractional integro-differential equation*, Math. Methods Appl. Sci. **42** (2019), 1249–1261.

29. Vanterler da C. Sousa J. and Capelas de Oliveira E., *A Gronwall inequality and the Cauchy-type problem by means of Ψ-Hilfer operator*, Differ. Equ. Appl. **11** (2019), 87–106

30. Wang Y., Liang S. and Wang Q., *Existence results for fractional differential equations with integral and multipoint boundary conditions*, Bound. Value Probl **4** (2018), 2–11.

P. Karthikeyan, Department of Mathematics, Sri Vasavi College, Erode, TN, 638316, India,
e-mail: pkarthisvc@gmail.com

K. Venkatachalam, Department of Mathematics, Sri Vasavi College, Erode, TN, 638316, India,
e-mail: arunsujith52@gmail.com

S. Abbas, School of Basic Science, Indian Institute of Technology Mandi, H.P., 175005, India,
e-mail: abbas@iitmandi.ac.in