Acta Math. Univ. Comenianae
Vol. XC, 4 (2021), pp. 377–386

ON THE COMPLEMENT CONNECTED STEINER NUMBER OF A GRAPH

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Abstract. For a connected graph $G = (V, E)$ of order $p \geq 3$, a Steiner set $W \subseteq V(G)$ is said to be a complement connected Steiner set if $W = V(G)$ or the subgraph $\langle V(G) - W\rangle$ is connected. The minimum cardinality of a complement connected Steiner set of G is the complement connected Steiner number of G and is denoted by $s_{cc}(G)$. It is shown that for every triplet a, b, c of integers with $3 \leq a \leq b \leq c$, there exists a connected graph G with $m_{cc}(G) = a, g_{cc}(G) = b$, and $s_{cc}(G) = c$, where $m_{cc}(G)$ and $g_{cc}(G)$ are the complement connected monophonic number and the complement connected geodetic number of the graph G, respectively. It is proved that for any two integers a and b with $3 \leq a \leq b$, there exists a connected graph G such that $m_{cc}(G) = s_{cc}(G) = a$ and $g_{cc}(G) = b$. Also, we have shown that, for every triplet a, b, c of integers with $3 < a < b < c$ and $b > a + 1$, there exists a connected graph G with $m_{cc}(G) = a, s_{cc}(G) = b$, and $g_{cc}(G) = c$.

1. Introduction

We consider finite and simple graphs. The set of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Also $p = |V(G)|$ and $q = |E(G)|$ denote the order and the size of G, correspondingly. If $uv \in E(G)$, we say that u is a neighbor of v and the set of neighbors of v, denote by $N(v)$. The *degree* of a vertex $v \in V(G)$ is $\deg(v) = |N(v)|$. For basic graph theoretic terminology we refer to [**5**]. The minimum and maximum degrees of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Let $U \subseteq V(G)$. Then $G - U$ denotes the graph obtained from G by deleting all the vertices of U together with all the edges with at least one vertex in U. A set U is called a cut-set of G if $G - U$ has more components than G. The subgraph induced by a set S of vertices of a graph G is denoted by $\langle S\rangle$ with $V(\langle S\rangle) = S$ and $E(\langle S\rangle) = \{uv \in E(G) : u, v \in S\}$. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the radius, $\operatorname{rad} G$, and the maximum eccentricity is its diameter, $\operatorname{diam} G$, of G. Two vertices x and y are *antipodal* if $d(x, y) = \operatorname{diam} G$. A vertex v in G is a *peripheral vertex* if

Received November 5, 2020; revised May 25, 2021.
2020 *Mathematics Subject Classification.* Primary 05C12, 05C38, 05C05.
Key words and phrases. Complement connected Steiner number; complement connected geodetic number; complement connected monophonic number; Steiner number; geodetic number; monophonic number.

$e(v) = \text{diam}(G)$. A vertex v is an *extreme vertex* of a graph G if the subgraph induced by its neighbors is complete.

The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. An u-v path of length $d(u, v)$ is called an u-v geodesic. It is known that the distance is a metric on the vertex set $V(G)$. The distance in graphs is studied in [**5**]. The interval $I[u, v]$ consists of all vertices lying on some u-v geodesic of G. For $S \subseteq V(G)$, $I[S] = \bigcup_{u,v \in S} I[u, v]$. A set S of vertices is a geodetic set if $I[S] = V(G)$, and the minimum cardinality of a geodetic set is the geodetic number $g(G)$. A geodetic set of cardinality $g(G)$ is called a g-set. The geodetic number of a graph was introduced in [**5**] and further studied in [**1, 2, 3, 6, 16, 17, 18**]. A geodetic set S of a connected graph G is said to be a *complement connected geodetic set* if $S = V(G)$ or the subgraph $\langle V(G) - S\rangle$ is connected. The minimum cardinality of a complement connected geodetic set of G is the *complement connected geodetic number* of G and is denoted by $g_{cc}(G)$. A complement connected geodetic set of minimum cardinality is called a g_{cc}-*set* of G [**19**]. Geodetic concepts have applications in game theory, telephone switching centres, facility location, distributed computing, information retrieval, and communication networks.

A chord of a path P is an edge joining two non-adjacent vertices of P. A path P is called a monophonic path if it is a chordless path. The interval $J[u, v]$ consists of all vertices lying on some u-v monophonic of G. For $M \subseteq V(G)$, $J[M] = \bigcup_{u,v \in M} J[u, v]$. A set M of vertices is a *monophonic set* if $J[M] = V(G)$, and the minimum cardinality of a monophonic set is the *monophonic number* $m(G)$. The monophonic number of a graph was studied in [**8, 11, 20**]. A monophonic set M of a connected graph G is said to be a *complement connected monophonic set* if $M = V(G)$ or the subgraph $\langle V(G) - M\rangle$ is connected. The minimum cardinality of a complement connected monophonic set is the *complement connected monophonic number* of G and is denoted by $m_{cc}(G)$. A complement connected monophonic set of minimum cardinality is called the m_{cc}-*set* of G [**25**]. A graph G is *distance-hereditary* if it is connected and every induced path is isometric; that is, if the distance function in every induced subgraph of G is the same as in G itself. Also, a *distance-hereditary graph* is a graph in which every monophonic path is a geodesic [**10**].

Let $W \subseteq V(G)$. A *Steiner tree* for W (*Steiner W-tree*) is a connected subgraph of G with a minimum number of edges that contains all vertices of W. The number of edges in a Steiner W-tree is the *Steiner distance* $d(W)$ of W in G. The *Steiner interval* $S(W)$ contains all the vertices that lie on some Steiner W-tree. If $S(W) = V(G)$, we call W a Steiner set of G. A Steiner set of minimum cardinality is a *minimum Steiner set* or simply an *s-set* and its cardinality is the *Steiner number* $s(G)$ of G. The Steiner number of a graph was introduced in [**7**] and further studied in [**4, 9, 10, 12, 13, 14, 18, 21, 22, 23, 24**]. A Steiner set W of a connected graph G is said to be a *complement connected Steiner set* if $W = V(G)$ or the subgraph $\langle V(G) - W\rangle$ is connected. The minimum cardinality of a complement connected Steiner set of G is the *complement connected Steiner number* of G and is denoted by $s_{cc}(G)$. A complement connected Steiner set of minimum cardinality is called the s_{cc}-*set* of G [**19**]. Steiner tree problem is a combinatorial problem in finding the shortest interconnection possible, given a set of objects. It has applications in circuit layout, network design, and transportation. Consider a communication sub-network as a graph model (model of the graph) and each processor as a vertex. Then the minimum cardinality of a set of leading processors is a minimum complement connected Steiner set for the graph representing communication sub-network.

Throughout the following, G denotes a connected graph with at least two vertices. The following theorems are used in the sequel.

Theorem 1.1 ([**10**]). *Every Steiner (geodetic) set of a connected graph G is a monophonic set of G.*

Theorem 1.2 ([**19**]). *For a connected graph G, $g_{cc}(G) = 2$ if and only if there exist peripheral vertices u and v such that any one of the following holds:*

(i) *every vertex of G is on a diametral path joining u and v,*
(ii) *u or v, or both are extreme vertices,*
(iii) *degree of at least one vertex in $\langle V(G) - \{u, v\}\rangle$ is more than 3 in G.*

Theorem 1.3 ([**15**]). *Let G be a connected graph of order $p \geq 2$. Then $s_{cc}(G) = 2$ if and only if there exists a pair of antipodal vertices u and v such that every vertex of G lies in a u-v geodesic and $W = \{u, v\}$ is not a cut-set of G.*

2. On the complement connected Steiner number of a graph

In general, there is no relation between a complement connected Steiner set and a complement connected geodetic set. In the following, we proved that in a distance-hereditary graph G, every complement connected Steiner set of G is a complement connected geodetic set of G.

Observation 2.1. *Every complement connected geodetic set of a connected graph G is a complement connected monophonic set of G.*

Proof. Let S be a complement connected geodetic set of G. Then S is a geodetic set of G and $\langle V(G) - S\rangle$ is connected. By Theorem 1.1, S is a monophonic set of G. Since $\langle V(G) - S\rangle$ is connected, S is a complement connected monophonic set of G. □

Corollary 2.2. *For any connected graph G of order p, $2 \leq m_{cc}(G) \leq g_{cc}(G) \leq p$.*

Observation 2.3. *Every complement connected Steiner set of a connected graph G is a complement connected monophonic set of G.*

Proof. Let W be a complement connected Steiner set of G. Then W is a Steiner set of G and $\langle V(G) - W\rangle$ is connected. By Theorem 1.1, W is a monophonic set of G. Since $\langle V(G) - W\rangle$ is connected, W is a complement connected monophonic set of G. □

Corollary 2.4. *For any connected graph G of order p, $2 \leq m_{cc}(G) \leq s_{cc}(G) \leq p$.*

Remark 2.5. From Corollaries 2.2 and 2.4, we have $m_{cc}(G) \leq g_{cc}(G)$ and $m_{cc}(G) \leq s_{cc}(G)$. For the graph $G = K_{1,a-1}, (a \geq 3)$, $m_{cc}(G) = g_{cc}(G) = s_{cc}(G) = a - 1$. From the following examples we observe that there is no relation between $g_{cc}(G)$ and $s_{cc}(G)$.

Example 2.6. For the graph G given in Figure 2.1, $S = \{v_1, v_4, v_6\}$ is an m_{cc}-set as well as a g_{cc}-set of G. Also $W = \{v_1, v_4, v_6, v_7\}$ is an s_{cc}-set of G. Hence $m_{cc}(G) = g_{cc}(G) = 3 < s_{cc}(G) = 4$. Thus $m_{cc}(G) = g_{cc}(G) < s_{cc}(G)$.

Example 2.7. For the graph G given in Figure 2.2, $W = \{v_8, v_9, v_{10}\}$ is an m_{cc}-set as well as an s_{cc}-set of G. Also $S = \{v_4, v_8, v_9, v_{10}\}$ is a g_{cc}-set of G. Hence $m_{cc}(G) = s_{cc}(G) = 3 < g_{cc}(G) = 4$. Thus $m_{cc}(G) = s_{cc}(G) < g_{cc}(G)$.

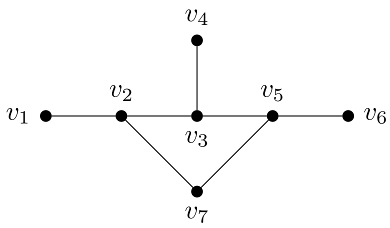

Figure 2.1. A graph G with $m_{cc}(G) = g_{cc}(G) < s_{cc}(G)$.

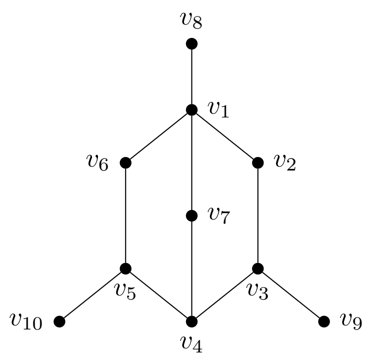

Figure 2.2. A graph G with $m_{cc}(G) = s_{cc}(G) < g_{cc}(G)$.

Proposition 2.8. *In a distance-hereditary graph G, every complement connected Steiner set of G is a complement connected geodetic set of G.*

Proof. Let G be a distance-hereditary graph and let W be a complement connected Steiner set of G. Then W is a Steiner set of G and $\langle V(G)-W\rangle$ is connected. By Theorem 1.1, W is a monophonic set of G. Since G is a distance-hereditary graph, every monophonic path of G is a geodesic. Therefore, W is a geodetic set of G. Since $\langle V(G) - W\rangle$ is connected, W is a complement connected geodetic set of G. □

Corollary 2.9. *In a distance-hereditary graph G, $g_{cc}(G) \leq s_{cc}(G)$.*

Theorem 2.10. *Let G be a connected graph of order $p \geq 2$. Then the following conditions are equivalent:*

(i) $g_{cc}(G) = 2$,
(ii) $s_{cc}(G) = 2$,
(iii) *There exists a pair of antipodal vertices u and v such that every vertex of G lies in a u-v geodesic and $W = \{u, v\}$ is not a cut-set of G.*

Proof. (i) $\Longrightarrow$ (ii) This follows from Theorem 1.2.
(ii) $\Longrightarrow$ (iii) This follows from Theorem 1.3.
(iii) $\Longrightarrow$ (i) The result is trivial. □

Theorem 2.11. *There is no connected graph G of order $p \geq 3$ with $m_{cc}(G) = g_{cc}(G) = 2$ and $s_{cc}(G) \geq 3$.*

Proof. On the contrary, suppose that there exists a connected graph G with $s_{cc}(G) \geq 3$. Since $g_{cc}(G) = 2$, by Theorem 2.10, we have $s_{cc}(G) = 2$, which is a contradiction to our assumption. Therefore, there is no connected graph G of order $p \geq 3$ with $m_{cc}(G) = g_{cc}(G) = 2$ and $s_{cc}(G) \geq 3$. □

Theorem 2.12. *There is no connected graph G of order $p \geq 3$ with $m_{cc}(G) = s_{cc}(G) = 2$ and $g_{cc}(G) \geq 3$.*

Proof. The proof is similar to the proof of Theorem 2.11. □

3. Realization results

Since every complement connected geodetic (monophonic, Steiner) set is a geodetic (monophonic, Steiner) set, we have the following observation.

Observation 3.1. *Each extreme vertex of a graph G belongs to every complement connected geodetic (monophonic, Steiner) set of G.*

In view of Corollaries 2.2, 2.4, and Remark 2.5, we have the following realization results. For this purpose, we present some graphs from which various graphs arising in Theorems 3.2, 3.3, and 3.4 are generated using identification. Let $P\colon x, y, z$ be a path on three vertices. Let $H_a \cong K_{2,a+1}$ be the graph given in Figure 3.1, obtained from P by adding new vertices $h_1, h_2, \ldots, h_a$ and joining each $h_i (1 \leq i \leq a)$ with x and z.

Let $P\colon p, q, r, s, t$ be a path on five vertices. Let R_c be the graph given in Figure 3.2 obtained from P by adding $2c$ new vertices $u_1, u_2, \ldots, u_c, v_1, v_2, \ldots, v_c$ and introducing the edges pu_i, tu_i, rv_i, u_iv_i for $(1 \leq i \leq c)$ and qv_1.

Let $P\colon u, v, w$ be a path on three vertices. Let $P_i\colon w_i, x_i, y_i$ $(1 \leq i \leq b)$ be a copy of path on three vertices. Let Q_b be the graph given in Figure 3.3, obtained from P and $P_i (1 \leq i \leq b)$ by joining each $w_i (1 \leq i \leq b)$ with u and each $y_i (1 \leq i \leq b)$ with w.

Theorem 3.2. *For any three integers a, b and c with $3 \leq a \leq b \leq c$, there exists a connected graph G with $m_{cc}(G) = a$, $g_{cc}(G) = b$, and $s_{cc}(G) = c$.*

Proof. Case 1. $3 \leq a = b = c$. Let $G = K_{1,a}$. Then it is clear that $m_{cc}(G) = g_{cc}(G) = s_{cc}(G) = a$.

Case 2. $3 \leq a = b < c$. Let G be the graph obtained from H_{c-a} by introducing new vertices $x_0, z_0, z_1, z_2, \ldots, z_{a-2}$ and the edges $xx_0, zz_0, yz_1, yz_2, \ldots, yz_{a-2}$. Let $Z = \{x_0, z_0, z_1, z_2, \ldots, z_{a-2}\}$ be the set of all extreme vertices of G. Then by Observation 3.1, Z is a subset of every complement connected geodetic (monophonic) set of G and so $m_{cc}(G) \geq a$, and $g_{cc}(G) \geq a$. Since $I[Z] = V(G)$, $J[Z] = V(G)$,

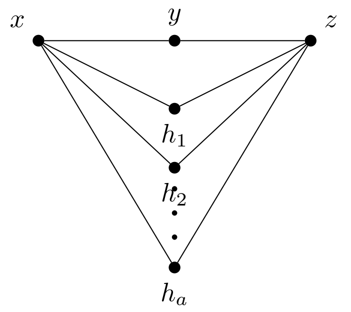

Figure 3.1. H_a.

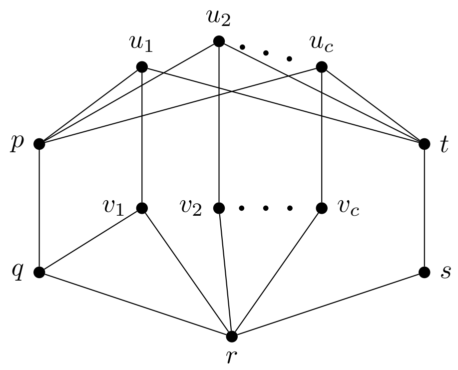

Figure 3.2. R_c.

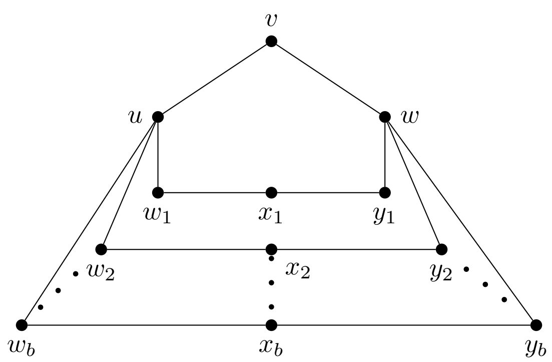

Figure 3.3. Q_b.

and $\langle V(G) - Z \rangle$ are connected, Z is a complement connected geodetic (monophonic) set of G, so that $m_{cc}(G) = g_{cc}(G) = a$.

Next we prove that $s_{cc}(G) = c$. By Observation 3.1, Z is a subset of every complement connected Steiner set of G. Now, the vertex h_i $(1 \le i \le c - a)$ does not lie on any Steiner Z-tree of G. We observe that every complement connected Steiner set of G contains each h_i $(1 \le i \le c - a)$, and so $s_{cc}(G) \ge c - a + a = c$. Now, $W = Z \cup \{h_1, h_2, \ldots, h_{c-a}\}$ is a complement connected Steiner set of G so that $s_{cc}(G) = c$.

Case 3. $3 \le a < b = c$. Let G be the graph obtained from Q_{b-a} by introducing new vertices $s_1, z_1, z_2, \ldots, z_{a-1}$ and new edges $us_1, wz_1, wz_2, \ldots, wz_{a-1}$. First we show that $m_{cc}(G) = a$. Let $Z = \{s_1, z_1, z_2, \ldots, z_{a-1}\}$ be the set of extreme vertices of G. Then by Observation 3.1, Z is a subset of every complement connected monophonic set of G, and so $m_{cc}(G) \ge a$. Now $J[Z] = V(G)$ and $\langle V(G) - Z \rangle$ are connected. Therefore, Z is a complement connected monophonic set of G, so that $m_{cc}(G) = a$.

Next we prove that $g_{cc}(G) = b$. By Observation 3.1, Z is a subset of every complement connected geodetic set of G. Since the vertices w_i, x_i, y_i $(1 \leq i \leq b - a)$ do not lie on any geodesic joining some pair of vertices of Z, Z is not a geodetic set of G. Let $H_i = \{w_i, x_i, y_i\}$ $(1 \leq i \leq b - a)$. It is seen that every complement connected geodetic set of G must contain at least one vertex from each H_i $(1 \leq i \leq b-a)$, and so $g_{cc}(G) \geq a+b-a = b$. Let $S_1 = Z \cup \{x_1, x_2, \ldots, x_{b-a}\}$. Then $I[S_1] = V(G)$ and $\langle V(G) - S_1 \rangle$ is connected and so S_1 is a complement connected geodetic set of G, so that $g_{cc}(G) = b$.

Next we prove that $s_{cc}(G) = b$. By Observation 3.1, Z is a subset of every complement connected Steiner set of G. Since the vertices w_i, x_i, y_i $(1 \leq i \leq b-a)$ do not lie on any Steiner Z-tree, Z is not a Steiner set of G. We notice that every complement connected Steiner set contains only the vertex x_i $(1 \leq i \leq b - a)$ from each H_i $(1 \leq i \leq b - a)$ and so $s_{cc}(G) \geq b - a + a = b$. Now $W = Z \cup \{x_1, x_2, \ldots, x_{b-a}\}$ is a Steiner set of G. Since $\langle V(G) - W \rangle$ is connected, W is a complement connected Steiner set of G so that $s_{cc}(G) = b$.

<u>*Case 4.*</u> $3 \leq a < b < c$. Let H be the graph obtained from Q_{b-a} and H_{c-b} by identifying the vertex w of Q_{b-a} and x of H_{c-b}. Let G be the graph obtained from H by adding vertices $s_1, t_1, z_1, z_2, \ldots, z_{a-2}$ and the edges $us_1, zt_1, yz_1, yz_2, \ldots, yz_{a-2}$. First we show that $m_{cc}(G) = a$. Let $Z = \{s_1, t_1, z_1, z_2, \ldots, z_{a-2}\}$ be the set of all end-vertices of G. Then by Observation 3.1, Z is a subset of every complement connected monophonic set of G, and so $m_{cc}(G) \geq a$. Since $J[Z] = V(G)$, Z is a monophonic set of G. Also, since $\langle V(G) - Z \rangle$ is connected, Z is a complement connected monophonic set of G, so that $m_{cc}(G) = a$.

Next we prove that $g_{cc}(G) = b$. Since the vertices w_i, x_i, y_i $(1 \leq i \leq b - a)$ do not lie on any geodesic joining some pair of vertices of Z, Z is not a geodetic set of G. Let $H_i = \{w_i, x_i, y_i\}$ $(1 \leq i \leq b - a)$. It is easily observed that every complement connected geodetic set of G contains at least one vertex from each H_i $(1 \leq i \leq b - a)$ and so $g_{cc}(G) \geq b - a + a = b$. Now $S = Z \cup \{x_1, x_2, \ldots, x_{b-a}\}$ is a geodetic set of G. Since $\langle V(G) - S \rangle$ is connected, S is a complement connected geodetic set of G so that $g_{cc}(G) = b$.

Next we prove that $s_{cc}(G) = c$. Since the vertices w_i, x_i, y_i $(1 \leq i \leq b - a)$ and h_i $(1 \leq i \leq c - b)$ do not lie on the Steiner Z-tree of G, Z is not a complement connected Steiner set of G. We notice that every complement connected Steiner set of G contains only the vertex $x_i (1 \leq i \leq b - a)$ from each H_i $(1 \leq i \leq b - a)$ and the vertex h_i $(1 \leq i \leq c - b)$, and so $s_{cc}(G) \geq b - a + a + c - b = c$. Let $W = Z \cup \{x_1, x_2, \ldots, x_{b-a}, h_1, h_2, \ldots, h_{c-b}\}$. Then W is a Steiner set of G. Since $\langle V(G) - W \rangle$ is connected, W is a complement connected Steiner set of G, so that $s_{cc}(G) = c$. □

Theorem 3.3. *For any two integers a, b with $3 \leq a \leq b$, there exists a connected graph G such that $m_{cc}(G) = s_{cc}(G) = a$ and $g_{cc}(G) = b$.*

Proof. <u>*Case 1.*</u> For $3 \leq a = b$, the graph constructed in <u>*Case 1*</u> of Theorem 3.2 satisfies the requirements of this theorem.

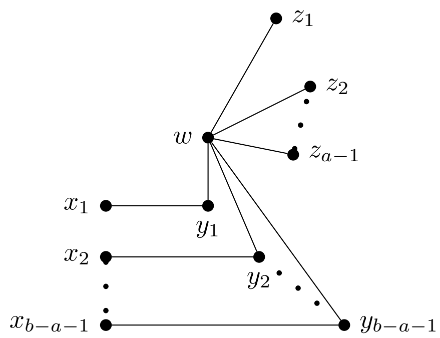

Figure 3.4. The Steiner W-tree of G.

Case 2. $3 \leq a < b$. Let H be the graph obtained from R_{b-a} by removing the edge qv_1. Let G be the graph obtained from H by adding the vertices $x, y, z_1, z_2, \ldots, z_{a-2}$ and introducing the edges xp, yr, and tz_i $(1 \leq i \leq a-2)$.

We show that $m_{cc}(G) = s_{cc}(G) = a$. Let $Z = \{x, y, z_1, z_2, \ldots, z_{a-2}\}$ be the set of all extreme vertices of G. By Observation 3.1, Z is a subset of every complement connected monophonic (Steiner) set of G. Since $J[Z] = V(G)$ and $S(Z) = V(G)$, and also $\langle V(G) - Z\rangle$ are connected, Z is a complement connected monophonic (Steiner) set of G, so that $m_{cc}(G) = s_{cc}(G) = a$. Next, we show that $g_{cc}(G) = b$. Let S be a complement connected geodetic set of G. Then by Observation 3.1, $Z \subseteq S$. Since the vertices $v_1, v_2, \ldots v_{b-a}$ do not lie on any geodesic joining a pair of vertices of Z, Z is not a complement connected geodetic set of G. Now, $S = Z \cup \{v_1, v_2, \ldots, v_{b-a}\}$ is a geodetic set of G. Since the subgraph $\langle V(G) - S\rangle$ is connected, S is a complement connected geodetic set of G, so that $g_{cc}(G) = b$. □

Theorem 3.4. *For any three integers a, b and c with $3 < a < b < c$, and $b > a + 1$, there exists a connected graph G such that $m_{cc}(G) = a, s_{cc}(G) = b$ and $g_{cc}(G) = c$.*

Proof. Let $3 < a < b < c$ and $b > a + 1$. Let G be the graph obtained from Q_{b-a-1} and R_{c-b+2} by identifying the vertex u of Q_{b-a-1} and t of R_{c-b+2}, and adding new vertices $z_1, z_2, \ldots, z_{a-1}$ and the edges wz_i: $(1 \leq i \leq a-1)$. Let $Z = \{z_1, z_2, \ldots, z_{a-1}\}$ be the set of all end-vertices of G. First we show that $m_{cc}(G) = a$. By Observation 3.1, Z is a subset of every complement connected monophonic set of G and so, $m_{cc}(G) \geq a - 1$. Since $J[Z] \neq V(G)$, Z is not a complement connected monophonic set of G, and so $m_{cc}(G) \geq a$. Let $Z_1 = Z \cup \{q\}$. Then $J[Z_1] = V(G)$ and $\langle V(G) - Z_1\rangle$ are connected. Therefore, Z_1 is a complement connected monophonic set of G, so that $m_{cc}(G) = a$. Next we show that $s_{cc}(G) = b$. By Observation 3.1, Z is a subset of every complement connected Steiner set of G. Since $S(Z) \neq V(G)$, Z is not Steiner set of G. Let $H_i = \{w_i, x_i, y_i\}$ $(1 \leq i \leq b-a-1)$. We notice that every complement connected Steiner set contains only the vertex $x_i (1 \leq i \leq b-a-1)$ from each H_i $(1 \leq i \leq b-a-1)$ and so, $s_{cc}(G) \geq b - a - 1 + a - 1 = b - 2$. Let $W = Z \cup \{x_1, x_2, \ldots, x_{b-a-1}\}$. Then the Steiner W-tree of G is given in Figure 3.4. From Figure 3.4, we observe that $S(W) = \{w, x_1, x_2, \ldots x_{b-a-1}, y_1, y_2, \ldots y_{b-a-1}, z_1, z_2, \ldots z_{a-1}\}$, so that $S(W) \neq V(G)$, and so $s_{cc}(G) \geq b-1$.

It is verified that $W \cup \{x\}$, where $x \notin W$ is not a complement connected Steiner set of G, and so $s_{cc}(G) \geq b$. If $x \in \{w, y_1, y_2, \ldots, y_{b-a-1}\}$, then the Steiner $W \cup \{x\}$-tree is as shown in Figure 3.4, so that $S(W \cup \{x\}) \neq V(G)$. If $x = p$, then the vertices $q, r, s, v_1, v_2, \ldots, v_{c-b+2}$ do not belong to $S(W \cup \{x\})$. If $x = q$, then the vertices $v_2, v_3, \ldots, v_{c-b+2}$ do not belong to $S(W \cup \{x\})$. If $x = r$, then the vertices $p, q, u_1, u_2, \ldots, u_{c-b+2}, v_1, v_2, \ldots, v_{c-b+2}$ do not belong to $S(W \cup \{x\})$. If $x = s$, then the vertices $p, q, r, u_1, u_2, \ldots, u_{c-b+2}, v_1, v_2, \ldots, v_{c-b+2}$ do not belong to $S(W \cup \{x\})$. If $x = t$, then the vertices $p, q, r, s, u_1, u_2, \ldots, u_{c-b+2}, v_1, v_2, \ldots, v_{c-b+2}$ do not belong to $S(W \cup \{x\})$. If $x = v$, then the vertices $p, q, r, s, t, w_1, w_2, \ldots, w_{b-a+1}, u_1, u_2, \ldots, u_{c-b+2}, v_1, v_2, \ldots, v_{c-b+2}$ do not belong to $S(W \cup \{x\})$. If $x = u_1$, then the vertices $p, q, r, s, u_2, u_3, \ldots, u_{c-b+2}, v_1, v_2, \ldots, v_{c-b+2}$ do not belong to $S(W \cup \{x\})$. If $x = u_2$, then the vertices $p, q, r, s, u_1, u_3, \ldots, u_{c-b+2}, v_1, v_2, \ldots, v_{c-b+2}$ do not belong to $S(W \cup \{x\})$. $\ldots$ If $x = u_{c-b+2}$, then the vertices $p, q, r, s, u_1, u_2, \ldots, u_{c-b+1}, v_1, v_2, \ldots, v_{c-b+2}$ do not belong to $S(W \cup \{x\})$. Similarly, if $x = v_1$, then the vertices $p, q, r, s, u_2, u_3, \ldots, u_{c-b+2}, v_2, v_3, \ldots, v_{c-b+2}$ do not belong to $S(W \cup \{x\})$. If $x = v_2$, then the vertices $p, q, r, s, u_1, u_3, \ldots, u_{c-b+2}, v_1, v_3, \ldots, v_{c-b+2}$ do not belong to $S(W \cup \{x\})$. $\ldots$ If $x = v_{c-b+2}$, then the vertices $p, q, r, s, u_1, u_2, \ldots, u_{c-b+1}, v_1, v_2, \ldots, v_{c-b+1}$ do not belong to $S(W \cup \{x\})$. Therefore, $W \cup \{x\}, x \notin W$ is not a complement connected Steiner set. Now let $W_1 = W \cup \{p, r\}$. Then $S(W_1) = V(G)$ and $\langle V(G) - W_1 \rangle$ are connected and so, W_1 is a complement connected Steiner set of G, so that $s_{cc}(G) = b$.

Next we prove that $g_{cc}(G) = c$. By Observation 3.1, Z is a subset of every complement connected geodetic set of G. Since $I[Z] \neq V(G)$, Z is not a complement connected geodetic set of G. We observe that every complement connected geodetic set contains at least one vertex from each H_i $(1 \leq i \leq b-a-1)$ and the vertex v_i $(2 \leq i \leq c-b+2)$, and so $g_{cc}(G) \geq a-1+b-a-1+c-b+1 = c-1$. Let $S = W \cup \{v_2, v_3, \ldots, v_{c-b+2}\}$. Then the vertices p, q, r, s, u_1, v_1 do not belong to $I[W]$. Therefore, $I[S] \neq V(G)$, and so S is not a complement connected geodetic set of G and so $g_{cc}(G) \geq c$. Let $S_1 = S \cup \{q\}$. Then $I[S_1] = V(G)$ and $\langle V(G) - S_1 \rangle$ is connected, and so S_1 is a complement connected geodetic set of G, so that $g_{cc}(G) = c$. □

Acknowledgment. We are thankful to the referees for their constructive and detailed comments and suggestions which improved the paper overall.

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