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NEW RESULTS ON THE SEQUENCE SPACES INCLUSION EQUATIONS INVOLVING THE SPACES w_∞ AND w_0

B. DE MALAFOSSE

Abstract. Given any sequence $a = (a_n)_{n\geq 1}$ of positive real numbers and any set E of complex sequences, we write E_a for the set of all sequences $y = (y_n)_{n\geq 1}$ such that $y/a = (y_n/a_n)_{n\geq 1} \in E$, in particular, c_a denotes the set of all sequences y such that y/a converges. In this paper, we use the well known sets

$$w_\infty = \Big\{ y \in \omega : \sup_n \Big(n^{-1} \sum_{k=1}^{n} |y_k| \Big) < \infty \Big\}$$

and

$$w_0 = \Big\{ y \in \omega : \lim_{n\to\infty} \Big(n^{-1} \sum_{k=1}^{n} |y_k| \Big) = 0 \Big\}$$

called the *spaces of strongly bounded and strongly summable to zero sequences by the Cesàro method*. Then we deal with the solvability of the (SSIE) of the form $w_\infty \subset \mathcal{E} + F'_x$ with $F' = c_0$, s_1, or w_∞ and $w_0 \subset \mathcal{E} + F'_x$ with $F' = c_0$, c, s_1, or w_∞, where $\mathcal{E}$ is a linear space of sequences. We apply these results to the solvability of each of the (SSIE) $w_\infty \subset w_0 + F'_x$, $w_\infty \subset bv_p + F'_x$, $w_\infty \subset (c_0)_{R_t} + F'_x$, $w_\infty \subset (c_0)_{C(\lambda)} + F'_x$ with $F' \in \{c_0, s_1, w_\infty\}$. These results extend some of those stated in [**18, 15**].

1. Introduction

We write ω for the set of all complex sequences $y = (y_n)_{n\geq 1}$, ℓ_∞, c, and c_0 for the sets of all bounded, convergent, and null sequences, respectively, also $\ell_p = \{y \in \omega : \sum_{n=1}^\infty |y_n|^p < \infty\}$ for $1 \leq p < \infty$. If y, $z \in \omega$, then we write $yz = (y_n z_n)_{n\geq 1}$. Let $U = \{y \in \omega : y_n \neq 0\}$ and $U^+ = \{y \in \omega : y_n > 0\}$. We write $z/u = (z_n/u_n)_{n\geq 1}$ for all $z \in \omega$ and all $u \in U$, in particular, $1/u = e/u$, where e is the sequence with $e_n = 1$ for all n. Finally, if $a \in U^+$ and E is any subset of ω, then we put $E_a = (1/a)^{-1} * E = \{y \in \omega : y/a \in E\}$. Let E and F be subsets of ω. In [**5**], the sets s_a, s_a^0, and $s_a^{(c)}$ were defined for positive sequences a by $(1/a)^{-1} * E$ and $E = \ell_\infty$, c_0, c, respectively. In [**6**], the sum $E_a + F_b$ and the product $E_a * F_b$ were defined, where E, F are any of the symbols s, s^0, or $s^{(c)}$.

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Then, in [**9**], we determined the solvability of sequences spaces inclusion equations $G_b \subset E_a + F_x$ where E, F, $G \in \{s^0, s^{(c)}, s\}$ and some applications were given to sequence spaces inclusions with operators. Recall that the spaces w_∞ and w_0 of strongly bounded and summable sequences are the sets of all y such that $\left(n^{-1}\sum_{k=1}^{n}|y_k|\right)_{n\geq 1}$ is bounded and tends to zero. These spaces were studied by Maddox [**4**] and by Malkowsky, Rakočević, Başar and Altay (cf. [**1, 2, 29**]). In [**12, 22, 25**], we gave some properties of well known operators defined on the sets $W_a = (1/a)^{-1} * w_\infty$ and $W_a^0 = (1/a)^{-1} * w_0$. In this paper, we deal with special *sequence spaces inclusion equations (SSIE), (resp., sequence spaces equations (SSE)),* which are determined by an inclusion, (resp., identity) for which each term is a *sum* or *a sum of products of sets of the form* $(E_a)_T$ *and* $\left(E_{f(x)}\right)_T$, where f maps U^+ to itself, E is any linear space of sequences, and T is a triangle. Some results on (SSE) and (SSIE) were stated in [**7**]–[**20**], [**23, 24, 25, 27**]. In [**11**], we used the sets of analytic and entire sequences denoted by $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$, and defined by $\sup_{n\geq 1}\left(|y_n|^{1/n}\right) < \infty$ and $\lim_{n\to\infty}\left(|y_n|^{1/n}\right) = 0$, respectively. Then we dealt with the solvability of (SSE) of the form $E_T + F_x = F_b$, where T is either of the triangles Δ or Σ, where Δ is the operator of the first difference and Σ is the operator defined by $\Sigma_n y = \sum_{k=1}^{n} y_k$ for all sequences y. More precisely, we gave a solvability of the (SSE) $E_\Delta + F_x = F_b$, where E is any of the sets c_0, ℓ_p, $(p > 1)$, w_0, or $\mathbf{\Lambda}$, and $F = c$ or ℓ_∞. Then, there is a solvability of the (SSE) $E_\Sigma + F_x = F_b$, where E is any of the sets c_0, c, ℓ_∞, ℓ_p, $(p > 1)$, w_0, $\mathbf{\Gamma}$, $\mathbf{\Lambda}$, and $F = c$ or ℓ_∞. Finally, there is a solvability of the (SSE) $\Gamma_\Sigma + \mathbf{\Lambda}_x = \mathbf{\Lambda}_b$.

Throughout this paper, we consider the (SSIE) $F \subset E_a + F'_x$, as a perturbed inclusion equation of the elementary inclusion equation $F \subset F'_x$. In this way, it is interesting to determine the set of all positive sequences a for which the elementary and the perturbed inclusion equations have the same solutions. In [**18, 25**], writing D_r for the diagonal matrix with $(D_r)_{nn} = r^n$, $(r > 0)$, we dealt with the solvability of the (SSIE) using the operator of the first difference Δ, defined by $c \subset D_r * E_\Delta + c_x$ with $E = c_0$ or s_1. Then we considered the (SSIE) $c \subset D_r * E_{C_1} + s_x^{(c)}$ with $E = c_0$, c or s_1, and $s_1 \subset D_r * E_{C_1} + s_x$ with $E = c$ or s_1, where C_1 is the Cesàro operator defined by $(C_1)_n y = \left(\sum_{k=1}^{n} y_k\right)/n$. In this paper, we extend some results stated in [**15**], where we dealt with the class of (SSIE) of the form $F \subset E_a + F'_x$, where $F \in \{c_0, \ell_p, w_0, w_\infty\}$ and E, $F' \in \{c_0, c, \ell_\infty, \ell_p, w_0, w_\infty\}$, $(p \geq 1)$. We generalize the previous results with the study of (SSIE) of the form $w_\infty \subset \mathcal{E} + F'_x$, with $F' = c_0$, s_1, or w_∞ and $w_0 \subset \mathcal{E} + F'_x$ with $F' = c_0$, c, s_1, or w_∞, where $\mathcal{E}$ is a more general space.

This paper is organized as follows. In Section 2, we recall some well-known results on sequence spaces and matrix transformations. In Section 3, we recall some results on the multipliers and on the relation $R_{\mathcal{E}}$ associated with the identity $F_x = F_y$ for some sets F of sequences. In Section 4, we study the (SSIE) of the form $F \subset E_a + F'_x$, where E, F, and F' are linear spaces of sequences. In Section 5, we extend some results of Section 4, and we deal with the solvability of the (SSIE) of the form $w_\infty \subset \mathcal{E} + F'_x$, $w_0 \subset E + F'_x$, where $\mathcal{E}$ is a linear space. In Section 6, we apply the results of Section 5 to the study the (SSIE) of the form $w_\infty \subset \mathcal{E} + F'_x$

involving the sets w_0, bv_p, $(c_0)_{R_t}$, or $(c_0)_{C(\lambda)}$. Then we give a resolution of the (SSE) $\mathcal{E} + W_x = w_\infty$, where $\mathcal{E}$ is a linear subspace of w_0. Finally, we solve the (SSIE) $w_\infty \subset \left(s_r^0\right)_\Delta + F'_x$ for $r > 0$, where F' is any of the spaces c_0, ℓ_∞, or w_∞.

2. Preliminaries and notations

An FK space is a *complete linear metric space*, for which convergence implies *coordinatewise convergence.* A *BK space* is a Banach space of sequences that is a *FK space.* A BK space E *is said to have AK if for every sequence* $y = (y_k)_{k\geq 1} \in E$, $y = \lim_{p\to\infty} \sum_{k=1}^{p} y_k e^{(k)}$, *where* $e^{(k)} = (0,\dots,0,1,0,\dots)$, 1 *being in the k-th position.*

Let $\mathbb{R}$ be the set of all real numbers. For any given infinite matrix $A = (\mathbf{a}_{nk})_{n,k\geq 1}$, we define the operators $A_n = (\mathbf{a}_{nk})_{k\geq 1}$ for any integer $n \geq 1$, by $A_n y = \sum_{k=1}^{\infty} \mathbf{a}_{nk} y_k$, where $y = (y_k)_{k\geq 1}$, and the series are assumed convergent for all n. So, we are led to the study of the operator A defined by $Ay = (A_n y)_{n\geq 1}$ mapping between sequence spaces. When A *maps* E *into* F, *where* E *and* F *are subsets of* ω, *we write* $A \in (E,F)$, (cf. [**4, 30**]). It is well known that if E has AK, then the set $\mathcal{B}(E)$ of all *bounded linear operators* L mapping in E, with norm $\|L\| = \sup_{y\neq 0} \left(\|L(y)\|_E / \|y\|_E\right)$ satisfies the identity $\mathcal{B}(E) = (E,E)$. For any subset F of ω, we write $F_A = \{y \in \omega : Ay \in F\}$ for the matrix domain of A in F. Then, for any given sequence $u = (u_n)_{n\geq 1} \in \omega$, we define the diagonal matrix D_u by $[D_u]_{nn} = u_n$ for all n. It is interesting to rewrite the set E_u using a diagonal matrix. Let E be any subset of ω and $u \in U^+$, we have $E_u = D_u * E = \{y = (y_n)_{n\geq 1} \in \omega : y/u \in E\}$. We use the sets s_a^0, $s_a^{(c)}$, s_a and $(\ell_p)_a$ defined as follows (cf. [**5, 21**]). For given $a \in U^+$, and $p \geq 1$, we put $D_a * c_0 = s_a^0$, $D_a * c = s_a^{(c)}$, $D_a * \ell_\infty = s_a$, and $D_a * \ell_p = (\ell_p)_a$. We frequently write c_a instead of $s_a^{(c)}$ to simplify. Each of the spaces $D_a * E$, where $E \in \{c_0, c, \ell_\infty\}$ is a *BK space normed* by $\|y\|_{s_a} = \sup_n (|y_n|/a_n)$ and s_a^0 *has AK.* The set ℓ_p, $(p \geq 1)$ normed by $\|y\|_{\ell_p} = \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{1/p}$ is a BK space with AK. If $a = (R^n)_{n\geq 1}$ with $R > 0$, then we write s_R, s_R^0, $s_R^{(c)}$, (or c_R), and $(\ell_p)_R$ for the sets s_a, s_a^0, $s_a^{(c)}$, and $(\ell_p)_a$, respectively. We also write D_R for $D_{(R^n)_{n\geq 1}}$. When $R = 1$, we obtain $s_1 = \ell_\infty$, $s_1^0 = c_0$ and $s_1^{(c)} = c$. Recall that $S_1 = (s_1, s_1)$ is a Banach algebra and $(c_0, s_1) = (c, s_1) = (s_1, s_1) = S_1$. We have $A \in S_1$ if and only if $\sup_n \left(\sum_{k=1}^{\infty} |\mathbf{a}_{nk}|\right) < \infty$.

We also use the following known properties, where the infinite matrix $\mathcal{T}$ is said to be a triangle if $\mathcal{T}_{nk} = 0$ for $k > n$, and $\mathcal{T}_{nn} \neq 0$ for all n.

Lemma 1. *Let* a, $b \in U^+$, *and let* E, $F \subset \omega$ *be any linear spaces. We have* $A \in (E_a, F_b)$ *if and only if* $D_{1/b} A D_a \in (E,F)$.

Lemma 2 ([**7**, Lemma 9, p. 45])**.** *Let* $\mathcal{T}'$ *and* $\mathcal{T}''$ *be any given triangles and let* E, $F \subset \omega$. *Then for any given operator* $\mathcal{T}$ *represented by a triangle, we have* $\mathcal{T} \in (E_{\mathcal{T}'}, F_{\mathcal{T}''})$ *if and only if* $\mathcal{T}''\mathcal{T}{\mathcal{T}'}^{-1} \in (E,F)$.

3. Some results on matrix transformations and on the multipliers of special sets

In this section, we define the spaces of a-strongly bounded and a-strongly null sequences by the Cesàro method. Then, we recall some results on the multipliers of sequence spaces and consider the equivalence relation $R_{\mathcal{E}}$.

3.1. On the triangles $C(\lambda)$ and $\Delta(\lambda)$, and the sets W_a and W_a^0.

For $\lambda \in U$, the infinite matrices $C(\lambda)$ and $\Delta(\lambda)$ are triangles defined as follows. We have $[C(\lambda)]_{nk} = 1/\lambda_n$ for $k \leq n$, and the nonzero entries of $\Delta(\lambda)$ are determined by $\Delta(\lambda)]_{nn} = \lambda_n$ for all n, and $[\Delta(\lambda)]_{n,n-1} = -\lambda_{n-1}$ for all $n \geq 2$. It can be shown that the matrix $\Delta(\lambda)$ is the inverse of $C(\lambda)$, that is, $C(\lambda)(\Delta(\lambda)y) = \Delta(\lambda)(C(\lambda)y) = y$ for all $y \in \omega$. If $\lambda = e$, we obtain the well known operator of the first difference represented by $\Delta(e) = \Delta$. We then have $\Delta_n y = y_n - y_{n-1}$ for all $n \geq 1$, with the convention $y_0 = 0$. It is usually written $\Sigma = C(e)$ and then we may write $C(\lambda) = D_{1/\lambda}\Sigma$. Note that $\Delta = \Sigma^{-1}$. The Cesàro operator is defined by $C_1 = C\left((n)_{n\geq 1}\right)$. We use the sets of spaces of *a-strongly bounded and a-strongly null sequences by the Cesàro method* defined for $a \in U^+$ by

$$W_a = \left\{ y \in \omega : \sup_n \left(\frac{1}{n} \sum_{k=1}^{n} \frac{|y_k|}{a_k} \right) < \infty \right\}$$

and

$$W_a^0 = \left\{ y \in \omega : \lim_{n\to\infty} \left(\frac{1}{n} \sum_{k=1}^{n} \frac{|y_k|}{a_k} \right) = 0 \right\},$$

(cf. [**26, 22, 12**]). We have $W_a = \{y \in \omega : C_1 D_{1/a}|y| \in s_1\}$. If $a = (r^n)_{n\geq 1}$ with $r > 0$, then the sets W_a and W_a^0 are denoted by W_r and W_r^0. For $r = 1$, we obtain the well-known sets $w_\infty = \{y \in \omega : \|y\|_{w_\infty} = \sup_n (n^{-1} \sum_{k=1}^{n} |y_k|) < \infty\}$ and $w_0 = \left\{y \in \omega : \lim_{n\to\infty} \left(n^{-1} \sum_{k=1}^{n} |y_k|\right) = 0\right\}$ called the *spaces of strongly bounded and strongly null sequences by the Cesàro method* (cf. [**28**]).

3.2. On the multipliers of some sets

First, we need to recall some well known results. Let y and z be sequences and let E and F be two subsets of ω, then we write $M(E,F) = \{y \in \omega : yz \in F \text{ for all } z \in E\}$, the set $M(E,F)$ is called the *multiplier space of E and F*. We use the next lemma.

Lemma 3. *Let E, $\widetilde{E}$, F, and $\widetilde{F}$ be arbitrary subsets of ω. Then*
(i) $M(E,F) \subset M(\widetilde{E},F)$ *for all* $\widetilde{E} \subset E$.
(ii) $M(E,F) \subset M(E,\widetilde{F})$ *for all* $F \subset \widetilde{F}$.

The α-dual of a set of sequences E is defined as $E^\alpha = M(E, \ell_1)$, and the β-dual of E is defined as $E^\beta = M(E, cs)$, where $cs = c_\Sigma$ is the set of all convergent series.

Lemma 4. *Let a, $b \in U^+$ and let E and F be two subsets of ω. Then we have $D_a * E \subset D_b * F$ if and only if $a/b \in M(E,F)$.*

In the following, we use the notation $E^+ = E \cap U^+$ for any subset E of ω.

Lemma 5. *Let E, F be linear spaces of sequences and assume F satisfies the next property.*

$$z \in F \Longleftrightarrow |z| \in F \text{ for all } z \in \omega. \tag{1}$$

Then $M(E^+, F) = M(E, F)$.

Proof. Let $a \in M(E^+, F)$. Then for every $y \in E$, we have $a|y| \in F$, and by the condition in (1), this implies $|a|y|| = |ay| \in F$. Again, by the condition in (1), we have $ay \in F$ and $a \in M(E, F)$. So, we have shown $M(E^+, F) \subset M(E, F)$. Since $E^+ \subset E$, we have $M(E^+, F) \supset M(E, F)$. This concludes the proof. □

In the following, we use the results stated below.

Lemma 6 ([**15**, Lemma 6, pp. 214–215])**.** *Let $p \geq 1$. We have:*

(i) (a) $M(c, c_0) = M(\ell_\infty, c) = M(\ell_\infty, c_0) = c_0$ *and* $M(c, c) = c$.
  (b) $M(E, \ell_\infty) = M(c_0, F) = \ell_\infty$ *for E, $F = c_0$, c, or ℓ_∞.*
  (c) $M(c_0, \ell_p) = M(c, \ell_p) = M(\ell_\infty, \ell_p) = \ell_p$.
  (d) $M(\ell_p, F) = \ell_\infty$ *for* $F \in \{c_0, c, s_1, \ell_p\}$.

(ii) (a) $M(w_0, F) = s_{(1/n)_{n\geq 1}}$ *for $F = c_0$, c, or ℓ_∞.*
  (b) $M(w_\infty, c_0) = M(w_\infty, c) = s^0_{(1/n)_{n\geq 1}}$.
  (c) $M(\ell_1, w_\infty) = s_{(n)_{n\geq 1}}$ *and* $M(\ell_1, w_0) = s^0_{(n)_{n\geq 1}}$.
  (d) $M(E, w_0) = w_0$ *for $E = s_1$ or c.*
  (e) $M(E, w_\infty) = w_\infty$ *for $E = c_0$, s_1, or c.*

Remark 7. By [**24**, Remark 3.4], we have $M(w_0, w_\infty) = M(w_\infty, w_\infty) = \ell_\infty$.

3.3. The equivalence relation $R_{\mathcal{E}}$

We need to recall some results on the equivalence relation $R_{\mathcal{E}}$ which is defined using the multiplier of sequence spaces. For $b \in U^+$ and for any subset $\mathcal{E}$ of ω, we denote by $cl^{\mathcal{E}}(b)$ the equivalence class for the equivalence relation $R_{\mathcal{E}}$ defined by $xR_{\mathcal{E}}y$ if $\mathcal{E}_x = \mathcal{E}_y$ for x, $y \in U^+$. It can easily be seen that $cl^{\mathcal{E}}(b)$ is the set of all $x \in U^+$ such that $x/b \in M(\mathcal{E}, \mathcal{E})$ and $b/x \in M(\mathcal{E}, \mathcal{E})$, (cf. [**27**]). Then we have $cl^{\mathcal{E}}(b) = cl^{M(\mathcal{E},\mathcal{E})}(b)$. For instance, $cl^c(b)$ is the set of all $x \in U^+$ such that $s_x^{(c)} = s_b^{(c)}$. This is the set of all sequences $x \in U^+$ such that $x_n \sim Cb_n$ $(n \to \infty)$ for some $C > 0$. We denote by $cl^\infty(b)$ the class $cl^{\ell_\infty}(b)$. Recall that $cl^\infty(b)$ is the set of all $x \in U^+$ such that $K_1 \leq x_n/b_n \leq K_2$ for all n and for some K_1, $K_2 > 0$.

4. On the (SSIE) of the form $F \subset E_a + F'_x$, where E, F, and F' are linear spaces of sequences

In this section, we are interested in the study of the set of all positive sequences x that satisfy the inclusion $F \subset E_a + F'_x$, where E, F, and F' are linear spaces of sequences and a is a positive sequence. We may consider this problem as a *perturbation problem.* If we know the set $M(F, F')$, then the solutions of the *elementary inclusion* $F'_x \supset F$ are determined by $1/x \in M(F,F')$. Now, the question is: Let $\mathcal{E}$ be a linear space of sequences. What are the solutions of the *perturbed* inclusion $F'_x + \mathcal{E} \supset F$? An additionnal question may be the following one: What are the conditions on $\mathcal{E}$ under which the solutions of the elementary and the perturbed inclusions are the same?

4.1. Some results on the solvability of some (SSIE)

The solutions of the perturbed inclusion $F \subset E_a + F'_x$, where E, F, and F' are linear spaces of sequences cannot be obtained in the general case. So, we are led to deal with the case when $a = (r^n)_{n\geq 1}$, $r > 0$, for which most of these (SSIE) can be totally solved. In the following, we use the notation $\mathcal{I}_a(E,F,F') = \{x \in U^+ : F \subset E_a + F'_x\}$, where E, F, and F' are linear spaces of sequences and $a \in U^+$. For any set χ of sequences, we let $\overline{\chi} = \{x \in U^+ : 1/x \in \chi\}$. We use the set $\Phi = \{c_0, c, s_1, \ell_p, w_0, w_\infty\}$ with $p \geq 1$. By $c(1)$ we define the set of all sequences $\alpha \in U^+$ that satisfy $\lim_{n\to\infty} \alpha_n = 1$. Then, we consider the condition

$$G \subset G_{1/\alpha} \qquad \text{for all } \alpha \in c(1), \tag{2}$$

for any given linear space G of sequences. Notice that condition (2) is satisfied for all $G \in \Phi$. In this part, we denote by U_1^+ the set of all sequences α with $0 < \alpha_n \leq 1$ for all n. We consider the condition

$$G \subset G_{1/\alpha} \qquad \text{for all } \alpha \in U_1^+, \tag{3}$$

for any given linear space G of sequences. Then, we introduce a linear space of sequences H which contains the spaces E and F'. The proof of the next theorem is based on the fact that if H satisfies the condition in (3), then we have $H_\alpha + H_\beta = H_{\alpha+\beta}$ for all α, $\beta \in U^+$ (cf. [**24**, Proposition 5.1, pp. 599–600]). Notice that c does not satisfy this condition, but each of the sets c_0, ℓ_∞, ℓ_p, $(p \geq 1)$, w_0, and w_∞ satisfies the condition in (3). So, we have for instance, $s^0_\alpha + s^0_\beta = s^0_{\alpha+\beta}$. In the following, we write $M(F,F') = \chi$. The next result is used to determine some classes of (SSIE).

Theorem 8 ([**15**, Theorem 9, p. 216])**.** *Let $a \in U^+$ and let E, F, and F' be linear subspaces of ω. Assume*

a) *χ satisfies the condition in* (2)*.*
b) *There is a linear space of sequences H that satisfies the condition in* (3)*, and conditions (α) and (β), where*
   (α) *E, $F' \subset H$,*
   (β) *$M(F,H) = \chi$.*
   Then we have:
   (i) *$a \in \overline{M(\chi, c_0)}$ implies $\mathcal{I}_a(E,F,F') = \overline{\chi}$.*
   (ii) *$a \in \overline{M(F,E)}$ implies $\mathcal{I}_a(E,F,F') = U^+$.*

As a direct consequence of the preceding, we obtain the following result.

Corollary 9 ([**15**, Corollary 10, p. 216])**.** *Let $a \in U^+$, let E, F, and F' be linear subspaces of ω. Assume χ satisfies condition* (2) *and assume $E \subset F'$, where F' satisfies the condition in* (3)*. Then we have:*

(i) $a \in \overline{M(\chi, c_0)}$ *implies* $\mathcal{I}_a(E, F, F') = \overline{\chi}$,
(ii) $a \in \overline{M(F, E)}$ *implies* $\mathcal{I}_a(E, F, F') = U^+$.

4.2. An application to the (SSIE) of the form $w_\infty \subset E_a + F'_x$

In this part, we recall some results stated in [**15**], and we study the set $\mathcal{I}_a(E, w_\infty, s_1)$ of all the solutions of the (SSIE) $w_\infty \subset E_a + s_x$ with $E \in \{c_0, c, s_1\}$. Then we consider the (SSIE) $w_\infty \subset E_a + W_x$ with $E \in \{c_0, s_1, w_\infty\}$. We obtain the following proposition.

Proposition 10 ([**15**, Proposition 17, p. 219])**.** *Let $a \in U^+$. We have:*

(i) *Let E be any of the spaces c_0, c, or s_1. Then*
  (a) *The condition $a \in s^0_{(n)_{n\geq 1}}$ implies $\mathcal{I}_a(E, w_\infty, s_1) = \overline{s_{(1/n)_{n\geq 1}}}$.*
  (b) *The identity $\mathcal{I}_a(E, w_\infty, s_1) = U^+$ holds in the following cases:*
    (α) *$a \in \overline{s^0_{(1/n)_{n\geq 1}}}$ for $E = c_0$ or c.*
    (β) *$a \in \overline{s_{(1/n)_{n\geq 1}}}$ for $E = s_1$.*

(ii) *Let E be any of the spaces c_0, ℓ_∞, or w_∞. Then*
  (a) *The condition $a \in c_0$ implies $\mathcal{I}_a(E, w_\infty, w_\infty) = \overline{s_1}$.*
  (b) *The identity $\mathcal{I}_a(E, w_\infty, w_\infty) = U^+$ holds in the following cases:*
    (α) *$a \in \overline{s^0_{(1/n)_{n\geq 1}}}$ for $E = c_0$.*
    (β) *$a \in \overline{s_{(1/n)_{n\geq 1}}}$ for $E = s_1$. (γ) $a \in \overline{s_1}$ for $E = w_\infty$.*

5. On the (SSIE) of the form $w_\infty \subset \mathcal{E} + F'_x$ and $w_0 \subset \mathcal{E} + F'_x$

In this section, we state the main results where we deal with the solvability of the (SSIE) of the form $w_\infty \subset \mathcal{E} + F'_x$, with $F' = c_0$, s_1, or w_∞, and $w_0 \subset \mathcal{E} + F'_x$, with $F' = c_0$, c, s_1, or w_∞, where $\mathcal{E}$ is a linear space of sequences.

5.1. Solvability of the (SSIE) of the form $w_\infty \subset \mathcal{E} + F'_x$

Now we state a theorem which is an extension of Proposition 10.

Theorem 11. *Let $\mathcal{E}$ be a linear space of sequences that satisfies $\mathcal{E} \subset s^0_{(n)_{n\geq 1}}$. Then we have:*

(i) *The solutions of the (SSIE) $w_\infty \subset \mathcal{E} + s_x$ are determined by*
  $\mathcal{I}(\mathcal{E}, w_\infty, s_1) = \overline{s_{(1/n)_{n\geq 1}}}$.
(ii) *The solutions of the (SSIE) $w_\infty \subset \mathcal{E} + s^0_x$ are determined by*
  $\mathcal{I}(\mathcal{E}, w_\infty, c_0) = \overline{s^0_{(1/n)_{n\geq 1}}}$.
(ii) *The solutions of the (SSIE) $w_\infty \subset \mathcal{E} + W_x$ are determined by*
  $\mathcal{I}(\mathcal{E}, w_\infty, w_\infty) = \overline{s_1}$.

Proof. (i) Let $x \in \mathcal{I}(\mathcal{E}, w_\infty, \ell_\infty)$. Then we have $w_\infty \subset \mathcal{E} + s_x$. Now we let $\mu \in U^+$. Then the inclusion

$$s_1 \subset s^0_\mu \tag{4}$$

holds if and only if $1/\mu \in M(s_1, c_0)$, and since $M(s_1, c_0) = c_0$, the inclusion in (4) holds for all $1/\mu \in c_0^+$. As we have just seen, we have $\mathcal{E} + s_x \subset s^0_{(n)_{n\geq 1}} + D_x * s^0_\mu$, and since

$$s^0_{(n)_{n\geq 1}} + D_x * s^0_\mu = s^0_{(n)_{n\geq 1}} + s^0_{\mu x} = s^0_{(n+\mu_n x_n)_{n\geq 1}},$$

we obtain $w_\infty \subset s^0_{(n+\mu_n x_n)_{n\geq 1}}$. So, the condition

$$\left(\frac{1}{n+\mu_n x_n}\right)_{n\geq 1} \in M(w_\infty, c_0) = s^0_{(1/n)_{n\geq 1}}$$

implies

$$\frac{n}{n+\mu_n x_n} \to 0 \qquad (n \to \infty)$$

and $n/\mu_n x_n \to 0$ $(n \to \infty)$ for all $1/\mu \in c_0^+$. So, we have $(n/x_n)_{n\geq 1} \in M(c_0^+, c_0)$ and by Lemma 5, we have $M(c_0^+, c_0) = M(c_0, c_0) = s_1$ which implies $1/x \in s_{(1/n)_{n\geq 1}}$. So, we have shown $\mathcal{I}(\mathcal{E}, w_\infty, \ell_\infty) \subset \overline{s_{(1/n)_{n\geq 1}}}$. Conversely, let $x \in \overline{s_{(1/n)_{n\geq 1}}}$. Then we have $1/x \in s_{(1/n)_{n\geq 1}}$ and by the identity $s_{(1/n)_{n\geq 1}} = M(w_\infty, s_1)$, we obtain $w_\infty \subset s_x$ and $x \in \mathcal{I}(\mathcal{E}, w_\infty, \ell_\infty)$. This shows $\overline{s_{(1/n)_{n\geq 1}}} \subset \mathcal{I}(\mathcal{E}, w_\infty, \ell_\infty)$ and we conclude $\mathcal{I}(\mathcal{E}, w_\infty, \ell_\infty) = \overline{s_{(1/n)_{n\geq 1}}}$. This completes the proof of Part (i).

(ii) Let $x \in \mathcal{I}(\mathcal{E}, w_\infty, c_0)$. Then we successively obtain $w_\infty \subset \mathcal{E} + s^0_x$, $w_\infty \subset s^0_{(n)_{n\geq 1}} + s^0_x = s^0_{(n+x_n)_{n\geq 1}}$ and $(1/(n+x_n))_{n\geq 1} \in M(w_\infty, c_0)$. Then, the identity $M(w_\infty, c_0) = s^0_{(1/n)_{n\geq 1}}$ successively implies $(n/(n+x_n))_{n\geq 1} \in c_0$, $(n/x_n)_{n\geq 1} \in c_0$ and $x \in \overline{s^0_{(1/n)_{n\geq 1}}}$. So we have shown $\mathcal{I}(\mathcal{E}, w_\infty, c_0) \subset \overline{s^0_{(1/n)_{n\geq 1}}}$. Conversely, let $x \in \overline{s^0_{(1/n)_{n\geq 1}}}$. Then we have $1/x \in M(w_\infty, c_0)$ and $w_\infty \subset s^0_x$ and $x \in \mathcal{I}(\mathcal{E}, w_\infty, c_0)$. So, we have shown $\overline{s^0_{(1/n)_{n\geq 1}}} \subset \mathcal{I}(\mathcal{E}, w_\infty, c_0)$. This concludes the proof of Part (ii).

(iii) Let $x \in \mathcal{I}(\mathcal{E}, w_\infty, w_\infty)$. Then we have $w_\infty \subset \mathcal{E} + W_x$, where $\mathcal{E} \subset s^0_{(n)_{n\geq 1}}$. Now, we let $\lambda \in U^+$. Then the inclusion

$$w_\infty \subset s^0_{(n\lambda_n)_{n\geq 1}} \tag{5}$$

holds if and only if $(1/n\lambda_n)_{n\geq 1} \in M(w_\infty, c_0)$, and since $M(w_\infty, c_0) = s^0_{(1/n)_{n\geq 1}}$, the inclusion in (5) holds for all $1/\lambda \in c_0^+$. Then we have

$$w_\infty \subset s^0_{(n)_{n\geq 1}} + D_x * s^0_{(n\lambda_n)_{n\geq 1}} = s^0_{(n(1+\lambda_n x_n))_{n\geq 1}},$$

and since $M(w_\infty, c_0) = s^0_{(1/n)_{n\geq 1}}$, we obtain

$$\left(\frac{1}{n(1+\lambda_n x_n)}\right)_{n\geq 1} \in s^0_{(1/n)_{n\geq 1}} \qquad \text{for all } 1/\lambda \in c_0^+.$$

This implies $1/(1+\lambda_n x_n) \to 0$ $(n \to \infty)$ and $1/\lambda x \in c_0$ for all $1/\lambda \in c_0^+$. So, by Lemma 5, we have $1/x \in M(c_0^+, c_0)$ and $x \in \overline{s_1}$. This shows the inclusion $\mathcal{I}(\mathcal{E}, w_\infty, w_\infty) \subset \overline{s_1}$. Conversely, let $x \in \overline{s_1}$. Then, we successively obtain $1/x \in M(w_\infty, w_\infty)$ and $w_\infty \subset W_x$ and $x \in \mathcal{I}(\mathcal{E}, w_\infty, w_\infty)$. So, we have shown $\overline{s_1} \subset \mathcal{I}(\mathcal{E}, w_\infty, w_\infty)$. This concludes the proof. □

Remark 12. The condition $\mathcal{E} \subset s^0_{(n)_{n\geq 1}}$ used in Theorem 11 is stronger than the condition $\mathcal{E} \subset s_a$ where $a \in s^0_{(n)_{n\geq 1}}$ in Part (i) of Proposition 10, with $E = s_1$. Indeed, the equivalence of $\mathcal{E} = s^0_{(n)_{n\geq 1}} \subset s_a$ and $(n/a_n)_{n\geq 1} \in \ell_\infty$, does not imply $a \in s^0_{(n)_{n\geq 1}}$.

5.2. On the (SSIE) of the form $w_0 \subset \mathcal{E} + F'_x$

In this part, we deal with the (SSIE) $w_0 \subset \mathcal{E} + F'_x$, where $\mathcal{E}$ is a linear space of sequences and F' is any of the spaces c_0, c, s_1, w_0, or w_∞.

Proposition 13.

(i) *The sets of all the solutions of each of the (SSIE)* $w_0 \subset \mathcal{E} + s^{(c)}_x$ *and* $w_0 \subset \mathcal{E} + s^0_x$*, where* $\mathcal{E} \subset s^0_\lambda$ *with* $\lambda_n/n \to 0$ $(n \to \infty)$ *is a linear space of sequences, are determined by* $\mathcal{I}(\mathcal{E}, w_0, c) = \mathcal{I}(\mathcal{E}, w_0, c_0) = \overline{s_{(1/n)_{n\geq 1}}}$.

(ii) *The solutions of each of the (SSIE)* $w_0 \subset \mathcal{E} + s_x$ *and* $w_0 \subset \mathcal{E} + F'_x$*, where* $\mathcal{E} \subset s_\lambda$ *with* $\lambda_n/n \to 0$ $(n \to \infty)$ *is a linear space of sequences and* $F' = w_0$*, or* w_∞*, are determined by* $\mathcal{I}(\mathcal{E}, w_0, s_1) = \overline{s_{(1/n)_{n\geq 1}}}$ *and* $\mathcal{I}(\mathcal{E}, w_0, F') = \overline{s_1}$.

Proof. (i) Let $x \in \mathcal{I}(\mathcal{E}, w_0, c)$. Since $\mathcal{E} \subset s^0_\lambda$, we have $w_0 \subset s^0_\lambda + s^0_{\mu x}$ for all $1/\mu \in c^+_0$ and $w_0 \subset s^0_{\lambda+\mu x}$. Then we have $(\lambda + \mu x)^{-1} \in M(w_0, c_0)$, where $M(w_0, c_0) = s_{(1/n)_{n\geq 1}}$. So, we have

$$\left(\frac{n}{\lambda_n + \mu_n x_n}\right)_{n\geq 1} \in \ell_\infty \qquad \text{for all } 1/\mu \in c^+_0,$$

and there are K, $K' > 0$ such that

$$\frac{\mu_n x_n}{n} \geq K - \frac{\lambda_n}{n} > 0$$

and $\mu_n x_n / n \geq K' > 0$ for all n. We conclude

$$\left(\frac{1}{\mu_n}\frac{n}{x_n}\right)_{n\geq 1} \in \ell_\infty \qquad \text{for all } 1/\mu \in c^+_0,$$

and by Lemma 5, we have $(n/x_n)_{n\geq 1} \in M\left(c^+_0, \ell_\infty\right) = s_1$. So, we have shown $x \in \overline{s_{(1/n)_{n\geq 1}}}$ and $\mathcal{I}(\mathcal{E}, w_0, c) \subset \overline{s_{(1/n)_{n\geq 1}}}$. Conversely, let $x \in \overline{s_{(1/n)_{n\geq 1}}}$. Since $s_{(1/n)_{n\geq 1}} = M(w_0, c)$, we successively obtain $1/x \in M(w_0, c)$, $w_0 \subset s^{(c)}_x$, and $x \in \mathcal{I}(\mathcal{E}, w_0, c)$. We conclude $\mathcal{I}(\mathcal{E}, w_0, c) = \overline{s_{(1/n)_{n\geq 1}}}$. Then we have $\mathcal{I}(\mathcal{E}, w_0, c_0) \subset \mathcal{I}(\mathcal{E}, w_0, c) = \overline{s_{(1/n)_{n\geq 1}}}$ and since $s_{(1/n)_{n\geq 1}} \subset M(w_0, c_0)$, we conclude $\overline{s_{(1/n)_{n\geq 1}}} \subset \mathcal{I}(\mathcal{E}, w_0, c_0)$ and $\mathcal{I}(\mathcal{E}, w_0, c_0) = \overline{s_{(1/n)_{n\geq 1}}}$. This completes the proof of Part (i).

Part (ii). Let $x \in \mathcal{I}(\mathcal{E}, w_0, s_1)$. Since $\mathcal{E} \subset s_\lambda$, we have $w_0 \subset s_\lambda + s_x = s_{\lambda+x}$ with $\lambda_n/n \to 0$ $(n \to \infty)$. Then we have $(\lambda + x)^{-1} \in M(w_0, s_1) = s_{(1/n)_{n\geq 1}}$ and $\left(n(\lambda_n + x_n)^{-1}\right)_{n\geq 1} \in \ell_\infty$. So, there are K and $K' > 0$ such that $x_n/n \geq K - \lambda_n/n$ and $x_n/n \geq K' > 0$ for all n. We conclude $x \in \overline{s_{(1/n)_{n\geq 1}}}$, and using similar arguments as those above, we obtain $\mathcal{I}(\mathcal{E}, w_0, s_1) = \overline{s_{(1/n)_{n\geq 1}}}$.

Case of the (SSIE) $w_0 \subset \mathcal{E} + F'_x$, where $F' = w_\infty$. We have $w_0 \subset s_{(\lambda_n + nx_n)_{n\geq 1}}$ and $\left((\lambda_n + nx_n)^{-1}\right)_{n\geq 1} \in M(w_0, s_1)$, where $M(w_0, s_1) = s_{(1/n)_{n\geq 1}}$, which implies $\left(n(\lambda_n + nx_n)^{-1}\right)_{n\geq 1} \in \ell_\infty$. So, there are K and $K' > 0$ such that $x_n \geq K - \lambda_n/n$ and $x_n \geq K' > 0$ for all n, and we conclude $\mathcal{I}(\mathcal{E}, w_0, w_\infty) \subset \overline{s_1}$. The inclusion $\overline{s_1} \subset \mathcal{I}(\mathcal{E}, w_0, w_\infty)$ follows from the identity $M(w_0, w_\infty) = s_1$. The case of the (SSIE) $w_0 \subset \mathcal{E} + W_x^0$ can be studied using similar arguments as those above. This completes the proof. □

As a direct consequence of the preceding we obtain the following results.

Corollary 14. *Let $\mathcal{E}$ be a linear space of sequences that satisfy $\mathcal{E} \subset s_{(n^\alpha)_{n\geq 1}}$, where $0 \leq \alpha < 1$ and $F' = c_0$, c, s_1, w_0, or w_∞. Then the perturbed (SSIE) $w_0 \subset \mathcal{E} + F'_x$ is equivalent to $w_0 \subset F'_x$ and to $1/x \in M(w_0, F')$.*

Proof. Let $\mathcal{E} \subset s_{(n^\alpha)_{n\geq 1}}$ with $0 \leq \alpha < 1$. Then we have $\lim_{n\to\infty} n^{\alpha-\beta} = 0$ for $\beta \in]\alpha, 1[$. This implies $s_{(n^\alpha)_{n\geq 1}} \subset s^0_{(n^\beta)_{n\geq 1}}$ with $n^\beta/n \to 0$ $(n \to \infty)$, and we conclude by Proposition 13. □

Remark 15. We obtain a similar result for the (SSIE) $w_\infty \subset \mathcal{E} + F'_x$, where F' is any of the sets c_0, c, s_1, w_0, or w_∞.

In the next result, we use the set bv_p of $p-$bounded variation defined by $bv_p = (\ell_p)_\Delta$ with $p \geq 1$.

Corollary 16. *Let $p \geq 1$ and let $F' = c_0$, c, s_1, w_0, or w_∞. Then, the (SSIE) $w_0 \subset bv_p + F'_x$ is equivalent to $1/x \in M(w_0, F')$.*

Proof. Let $p > 1$ and $q = p/(p-1)$. We have $bv_p \subset s^0_{(n^\alpha)_{n\geq 1}}$ for $1/q \leq \alpha < 1$, since this inclusion is equivalent to $D_{(1/n^\alpha)_n}\Sigma \in (\ell_p, c_0)$ and by the characterization of (ℓ_p, c_0), (cf. [**29**, Theorem 1.37, p. 161]), we have $n/n^{\alpha q} = 1/n^{\alpha q - 1} \leq K$ for some $K > 0$. We conclude by Corollary 14. The case $p = 1$ is a direct consequence of the characterization of (ℓ_1, ℓ_∞) (cf. [**29**, Theorem 1.37, p. 161]). □

6. Application to some (SSIE) with operators

In this section, we apply the results of Section 5 to the solvability of the (SSIE) $w_\infty \subset w_0 + F'_x$, $w_\infty \subset bv_p + F'_x$, $w_\infty \subset (c_0)_{R_t} + F'_x$, and $w_\infty \subset (c_0)_{C(\lambda)} + F'_x$ with $F' \in \{c_0, s_1, w_\infty\}$. Then, we consider the (SSIE) of the form $w_\infty \subset \mathcal{E}_a + F'_x$ where $\mathcal{E}$, F' are any of the spaces c_0, c, ℓ_p, $(1 \leq p \leq \infty)$, w_0, or w_∞, and we solve the (SSE) $\mathcal{E} + W_x = w_\infty$ where $\mathcal{E} \subset w_0$. Finally, for $r > 0$, we solve the (SSIE) $w_\infty \subset \left(s_r^0\right)_\Delta + F'_x$, where F' is any of the sets c_0, s_1, or w_∞.

6.1. The solvability of the (SSIE) of the form $w_\infty \subset \mathcal{E} + F'_x$ involving the sets w_0, bv_p, $(c_0)_{R_t}$, and $(c_0)_{C(\lambda)}$

In this part, we use the Riesz matrix R_t with $t = (t_n)_{n\geq 1} \in U^+$, which is the triangle whose the nonzero entries are defined by $[R_t]_{nk} = t_k/T_n$ with $k \leq n$, where $T_n = \sum_{k=1}^{n} t_k$ for all n, and R_t is also called the matrix of the weighted means. From Theorem 11, we obtain the solvability of some new (SSIE) with operators.

Corollary 17. *Let $\mathcal{E}$ be any of the sets w_0, bv_p $(p \geq 1)$, $(c_0)_{R_t}$ with $T/t \in s_{(n)_{n\geq 1}}$, or $(c_0)_{C(\lambda)}$, where $\lambda_n = O(n)$ $(n \to \infty)$. Then the sets $\mathcal{I}(\mathcal{E}, w_\infty, F')$ of all positive sequences x that satisfy each of the (SSIE) $w_\infty \subset \mathcal{E} + F'_x$ with $F' \in \{c_0, s_1, w_\infty\}$ are determined by*

$$\mathcal{I}(\mathcal{E}, w_\infty, c_0) = \overline{s^0_{(1/n)_{n\geq 1}}}, \qquad \mathcal{I}(\mathcal{E}, w_\infty, s_1) = \overline{s_{(1/n)_{n\geq 1}}}, \qquad \mathcal{I}(\mathcal{E}, w_\infty, w_\infty) = \overline{s_1}.$$

Proof. We confine our study to the case when $F' = c_0$, the proof of the other cases being similar. We apply Theorem 11 to each of the spaces $\mathcal{E} = w_0$, bv_p $(p \geq 1)$, $(c_0)_{R_t}$, and $(c_0)_{C(\lambda)}$. The case $\mathcal{E} = w_0$ is a direct consequence of the inclusion $w_0 \subset s^0_{(n)_{n\geq 1}}$. Case $\mathcal{E} = bv_p$. Let $p > 1$. Then the inclusion $(\ell_p)_\Delta \subset s^0_{(n)_{n\geq 1}}$ is equivalent to $D_{(1/n)_n}\Sigma \in (\ell_p, c_0)$. By the characterization of (ℓ_p, c_0), (cf. [**29**, Theorem 1.37, p. 161]), this condition is equivalent to $n/n^q = 1/n^{q-1} = O(1)$ $(n \to \infty)$, where $q = p/(p-1)$, and is satisfied since $q > 1$. The case $p = 1$ is a direct consequence of the property $D_{(1/n)_{n\geq 1}}\Sigma \in (\ell_1, c_0)$. Case $\mathcal{E} = (c_0)_{R_t}$. The inclusion $(c_0)_{R_t} \subset s^0_{(n)_{n\geq 1}}$ is equivalent to

$$D_{(1/n)_{n\geq 1}} R_t^{-1} \in (c_0, c_0). \tag{6}$$

From the identity $R_t = D_{1/T}\Sigma D_t$, we obtain $R_t^{-1} = D_{1/t}\Delta D_T$, and the condition in (6), is equivalent to $D_{(1/nt_n)_{n\geq 1}}\Delta D_T \in (c_0, c_0)$. This condition is true since $T_n/t_n = O(n)$ $(n \to \infty)$.

The case $\mathcal{E} = (c_0)_{C(\lambda)}$ with $\lambda_n = O(n)$ $(n \to \infty)$ can be shown as above since we have $D_{(1/n)_{n\geq 1}}\Delta D_\lambda \in (c_0, c_0)$. This completes the proof. □

Remark 18. We may rewrite Corollary 17 as follows. Let $p \geq 1$. Then, by Corollary 17 with $\mathcal{E} = bv_p$, the sets of all the solutions of each of the (SSIE) $w_\infty \subset bv_p + s^0_x$, $w_\infty \subset bv_p + s_x$ and $w_\infty \subset bv_p + W_x$ are equal to $\overline{s^0_{(1/n)_{n\geq 1}}}$, $\overline{s_{(1/n)_{n\geq 1}}}$, and $\overline{s_1}$, respectively. Then, for $\mathcal{E} = w_0$, the sets of all the solutions of each of the (SSIE) $w_\infty \subset w_0 + s^0_x$, $w_\infty \subset w_0 + s_x$, and $w_\infty \subset w_0 + W_x$ are equal to $\overline{s^0_{(1/n)_{n\geq 1}}}$, $\overline{s_{(1/n)_{n\geq 1}}}$, and $\overline{s_1}$, respectively. In each case, the perturbed (SSIE) and the elementary (SSIE) have the same set of solutions.

As a direct consequence of Corollary 17, we obtain the following results.

Corollary 19.

(i) *The solutions of the (SSIE) $w_\infty \subset (c_0)_{C_1} + s^0_x$ are determined by* $\mathcal{I}((c_0)_{c_1}, w_\infty, c_0) = \overline{s^0_{(1/n)_{n\geq 1}}}$.

(ii) *The solutions of the (SSIE) $w_\infty \subset (c_0)_{C_1} + s_x$ are determined by* $\mathcal{I}((c_0)_{C_1}, w_\infty, \ell_\infty) = \overline{s_{(1/n)_{n\geq 1}}}$.

(iii) *The solutions of the (SSIE) $w_\infty \subset (c_0)_{C_1} + W_x$ are determined by* $\mathcal{I}((c_0)_{C_1}, w_\infty, w_\infty) = \overline{s_1}$.

6.2. On the solvability of the (SSIE) $w_\infty \subset E_a + F'_x$, where $E = c_0$, c, ℓ_p $1 \le p \le \infty$, w_0, or w_∞

We easily deduce the next corollaries that are direct consequences of Theorem 11 and Lemma 6.

Corollary 20. *Let $a \in s_1^+$. Then we have:*

(i) *The solutions of the (SSIE) $w_\infty \subset W_a^0 + s_x^0$ are determined by $\mathcal{I}_a(w_0, w_\infty, c_0) = \overline{s^0_{(1/n)_{n\ge1}}}$.*

(ii) *The solutions of the (SSIE) $w_\infty \subset W_a^0 + s_x$ are determined by $\mathcal{I}_a(w_0, w_\infty, s_1) = \overline{s_{(1/n)_{n\ge1}}}$.*

(iii) *The solutions of the (SSIE) $w_\infty \subset W_a^0 + W_x$ are determined by $\mathcal{I}_a(w_0, w_\infty, w_\infty) = \overline{s_1}$.*

Proof. These results follow from Theorem 11, where we have

$$W_a^0 \subset s^0_{(n)_{n\ge1}} \tag{7}$$

if and only if $(a_n/n)_{n\ge1} \in M(w_0, c_0)$, where $M(w_0, c_0) = s_{(1/n)_{n\ge1}}$. So, the condition $a \in s_1$ implies the condition in (7), and we conclude by Part (i) (b) of Theorem 11. Part (iii) can be shown in a similar way. This completes the proof. □

Using similar arguments as those above, we obtain the next corollaries.

Corollary 21. *Let $a \in c_0$. Then we have:*

(i) *The solutions of the (SSIE) $w_\infty \subset W_a + s_x^0$ are determined by $\mathcal{I}_a(w_\infty, w_\infty, c_0) = \overline{s^0_{(1/n)_{n\ge1}}}$.*

(ii) *The solutions of the (SSIE) $w_\infty \subset W_a + s_x$ are determined by $\mathcal{I}_a(w_\infty, w_\infty, \ell_\infty) = \overline{s_{(1/n)_{n\ge1}}}$.*

(iii) *The solutions of the (SSIE) $w_\infty \subset W_a + W_x$ are determined by $\mathcal{I}_a(w_\infty) = \overline{s_1}$.*

Corollary 22. *Let $a \in s^0_{(n)_{n\ge1}} \cap U^+$. Then we have:*

(i) *The solutions of the (SSIE) $w_\infty \subset E_a + s_x^0$ with $E \in \{c, s_1\}$ are determined by $\mathcal{I}_a(E, w_\infty, c_0) = \overline{s^0_{(1/n)_{n\ge1}}}$.*

(ii) *The solutions of the (SSIE) $w_\infty \subset E_a + s_x$ with $E \in \{c, s_1\}$ are determined by $\mathcal{I}_a(E, w_\infty, \ell_\infty) = \overline{s_{(1/n)_{n\ge1}}}$.*

(iii) *The solutions of the (SSIE) $w_\infty \subset E_a + W_x$ with $E \in \{c, s_1\}$ are determined by $\mathcal{I}_a(E, w_\infty, w_\infty) = \overline{s_1}$.*

Corollary 23. *Let $a \in s^+_{(n)_{n\ge1}}$. Then we have:*

(i) *The solutions of the (SSIE) $w_\infty \subset E_a + s_x^0$ with $E \in \{c_0, \ell_p\}$, $(p \ge 1)$ are determined by $\mathcal{I}_a(E, w_\infty, c_0) = \overline{s^0_{(1/n)_{n\ge1}}}$.*

(ii) *(ii) The solutions of the (SSIE) $w_\infty \subset E_a + s_x$ with $E \in \{c_0, \ell_p\}$ are determined by $\mathcal{I}_a(E, w_\infty, \ell_\infty) = \overline{s_{(1/n)_{n\ge1}}}$.*

(iii) *The solutions of the (SSIE)* $w_\infty \subset E_a + W_x$ *with* $E \in \{c_0, \ell_p\}$, $(p \geq 1)$ *are determined by* $\mathcal{I}_a(E, w_\infty, c_0) = \overline{s_1}$.

Using Proposition 13, we obtain similar results on the (SSIE) $w_0 \subset E_a + F'_x$, where $E = c_0$, c, or ℓ_p with $1 \leq p \leq \infty$.

6.3. On the solvability of the (SSIE) of the form $F \subset \mathcal{E} + W_x$, where F is either w_0 or w_∞, and $\mathcal{E} \in \{c_0, c, \ell_p, cs, cs_0, bs\}$

Now, we recall the definitions of the sets $cs = c_\Sigma$, $bs = (\ell_\infty)_\Sigma$, and $cs_0 = (c_0)_\Sigma$ that are called the sets of all convergent, bounded, and convergent to zero series. More precisely, we have $cs = \{y \in \omega : \sum_{k=1}^\infty y_k \text{ is convergent}\}$, $bs = \left\{y : \left(\sum_{k=1}^n y_k\right)_{n\geq 1} \in \ell_\infty\right\}$, and $cs_0 = \left\{y : \left(\sum_{k=1}^n y_k\right)_n \in c_0\right\}$. We write $\Psi = \{c_0, c, \ell_p, cs, cs_0, bs\}$. Since we have $\mathcal{E} \subset \ell_\infty$ for all $\mathcal{E} \in \Psi$, by Theorem 11 and Proposition 13, we obtain the following corollary.

Corollary 24. *Let F be either w_0 or w_∞, and let $\mathcal{E} \in \Psi$, $(1 \leq p \leq \infty)$. Then we have:*

(i) *The solutions of each of the (SSIE)* $F \subset \mathcal{E} + W_x$ *are determined by* $\mathcal{I}^w = \overline{s_1}$.

(ii) *The solutions of each of the (SSIE)* $F \subset \mathcal{E} + s_x$ *are determined by* $\mathcal{I}^\infty = \overline{s_{(1/n)_{n\geq 1}}}$.

(iii) *The solutions of the (SSIE)* $w_\infty \subset \mathcal{E} + s_x^0$ *are determined by* $\mathcal{I}_1^0 = \overline{s^0_{(1/n)_{n\geq 1}}}$,
and
the solutions of the (SSIE) $w_0 \subset \mathcal{E} + s_x^0$ *are determined by* $\mathcal{I}_2^0 = \mathcal{I}^\infty = \overline{s_{(1/n)_{n\geq 1}}}$.

(iv) *The solutions of the (SSIE)* $w_0 \subset \mathcal{E} + s_x^{(c)}$, *are determined by* $\mathcal{I}^c = \mathcal{I}^\infty = \overline{s_{(1/n)_{n\geq 1}}}$.

6.4. On the solvability of the (SSE) $\mathcal{E} + W_x = w_\infty$

In this part, we consider the (SSE) $\mathcal{E} + W_x = w_\infty$ with $\mathcal{E} \subset w_\infty \cap s^0_{(n)_{n\geq 1}}$. For instance, the identity $w_0 + W_x = w_\infty$ is equivalent to the next statement. The condition $\sup_n \left(n^{-1} \sum_{k=1}^n |y_k|\right) < \infty$ holds if and only if there are u, $v \in \omega$ with $y = u + v$ and $\lim_{n\to\infty} \left(n^{-1} \sum_{k=1}^n |u_k|\right) = 0$ and $\sup_n \left(n^{-1} \sum_{k=1}^n |v_k| / x_k\right) < \infty$ for all y. We obtain the following result.

Theorem 25. *The set $S(\mathcal{E}, w_\infty)$ is the set of all positive sequences x such that $\mathcal{E} + W_x = w_\infty$, where $\mathcal{E} \subset w_\infty \cap s^0_{(n)_{n\geq 1}}$ is a linear space, is determined by*

$$S(\mathcal{E}, w_\infty) = cl^\infty(e).$$

Proof. Let $x \in S(\mathcal{E}, w_\infty)$. Then we have

$$\mathcal{E} + W_x \subset w_\infty \tag{8}$$

and

$$w_\infty \subset \mathcal{E} + W_x. \tag{9}$$

By the hypothesis, we have $\mathcal{E} \subset w_\infty$, and the inclusion in (8) implies $W_x \subset w_\infty$. This means, $x \in M(w_\infty, w_\infty)$, and by Remark 7, we conclude $x \in s_1^+$. Then we have $\mathcal{E} \subset w_\infty \cap s^0_{(n)_{n\geq 1}} \subset s^0_{(n)_{n\geq 1}}$, and by Theorem 11, the (SSIE) in (9) is equivalent to $x \in \overline{s_1}$. We conclude $S(\mathcal{E}, w_\infty) \subset cl^\infty(e)$. Conversely, assume $x \in cl^\infty(e)$. Then we have $W_x = w_\infty$, and since $\mathcal{E} \subset w_\infty$, we obtain $\mathcal{E} + W_x = \mathcal{E} + w_\infty = w_\infty$. This completes the proof. □

From Theorem 25, we deduce that each of the equations $cs + W_x = w_\infty$, $c_0 + W_x = w_\infty$, $\ell_p + W_x = w_\infty$, $cs_0 + W_x = w_\infty$, and $w_0 + W_x = w_\infty$ is equivalent to $x \in cl^\infty(e)$.

6.5. On the (SSIE) of the form $w_\infty \subset \left(s_r^0\right)_\Delta + F'_x$

In this subsection, for $r > 0$, we solve the (SSIE) $w_\infty \subset \left(s_r^0\right)_\Delta + F'_x$, where F' is any of the sets c_0, s_1, or w_∞. From Theorem 11, we obtain the next results.

Proposition 26. *Let $r > 0$. Then we have:*

(i) *Let $\mathcal{I}^0_{r,\delta} = \mathcal{I}\left(\left(s_r^0\right)_\Delta, w_\infty, c_0\right)$ be the set of all positive sequences x that satisfy the (SSIE) $w_\infty \subset \left(s_r^0\right)_\Delta + s_x^0$ determined by*

$$\mathcal{I}^0_{r,\delta} = \begin{cases} \overline{s^0_{(1/n)_{n\geq 1}}} & \text{if } r \leq 1, \\ U^+ & \text{if } r > 1. \end{cases}$$

(ii) *Let $\mathcal{I}^1_{r,\delta} = \mathcal{I}\left(\left(s_r^0\right)_\Delta, w_\infty, s_1\right)$ be the set of all positive sequences x that satisfy the (SSIE) $w_\infty \subset \left(s_r^0\right)_\Delta + s_x$ determined by*

$$\mathcal{I}^1_{r,\delta} = \begin{cases} \overline{s_{(1/n)_{n\geq 1}}} & \text{if } r \leq 1, \\ U^+ & \text{if } r > 1. \end{cases}$$

(iii) *Let $\mathcal{I}^w_{r,\delta} = \mathcal{I}\left(\left(s_r^0\right)_\Delta, w_\infty, w_\infty\right)$ be the set of all positive sequences x that satisfy the (SSIE) $w_\infty \subset \left(s_r^0\right)_\Delta + W_x$ determined by*

$$\mathcal{I}^w_{r,\delta} = \begin{cases} \overline{s_1} & \text{if } r \leq 1, \\ U^+ & \text{if } r > 1. \end{cases}$$

Proof. (i) Consider the case $r \leq 1$. We have $n^{-1}\sum_{k=1}^n r^k = O(1)$ $(n \to \infty)$, and by Theorem 11 with $\mathcal{E} = \left(s_r^0\right)_\Delta$, this implies $\mathcal{E} \subset s^0_{(n)_{n\geq 1}}$. If $r > 1$, then we have $\left(s_r^0\right)_\Delta = s_r^0$ by [**5**, Theorem 2.6, p. 1789]. Then we have $w_\infty \subset s_r^0$ since the condition $\lim_{n\to\infty} n/r^n = 0$ implies $(1/r^n)_{n\geq 1} \in M(w_\infty, c_0)$. The statements in Parts (ii) and (iii) may be shown in a similar way. This concludes the proof. □

Example 27. The perturbed (SSIE) $w_\infty \subset (c_0)_\Delta + s_x^0$ and $w_\infty \subset s_x^0$ are equivalent and the set of all positive sequences x that satisfy each of these (SSIE) is determined by $\mathcal{I}^0_{1,\delta} = \overline{s^0_{(1/n)_{n\geq 1}}}$.

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B. de Malafosse, Université du Havre, France,
e-mail: `bdemalaf@wanadoo.fr`