Acta Math. Univ. Comenianae
Vol. XCI, 3 (2022), pp. 191–204

THE NON-ISOLATED RESOLVING NUMBER OF A GRAPH CARTESIAN PRODUCT WITH A COMPLETE GRAPH

I. M. HASIBUAN, A. N. M. SALMAN and S. W. SAPUTRO

Abstract. A set of vertices W resolves a graph G if every vertex of G is uniquely determined by its vector of distances to the vertices in W. A resolving set W of G is called a non-isolated resolving set if the induced subgraph of G by W does not contain an isolated vertex. An nr-set of G is a non isolated resolving set with minimum cardinality and the non-isolated resolving number of G refers to its cardinality, denoted by $\mathrm{nr}(G)$. Let K_n be a complete graph of order n. In this paper, for any graph G of order m with $m \leq n$, we determine the sharp lower and upper bounds of the non-isolated resolving number of G Cartesian product with a complete graph, denoted by $\mathrm{nr}(G \times K_n)$. We provide the non-isolated resolving number of $G \times K_n$ for some classes of G, namely paths, complete graphs, cycles, friendship graphs, and star graphs. We also show that for any positive integers $c \leq \lfloor \frac{m}{2} \rfloor$, there exists a graph G of order m such that $\mathrm{nr}(G \times K_n)$ is equal to the upper bound minus c.

1. Introduction

Throughout this paper, all graphs are finite, simple, and undirected. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. To simplify writing, for two positive integers a and b, we define $[a,b] = \{n \in \mathbb{Z} | a \leq n \leq b\}$. We recall some definition, of certain graphs. A *path* P_n is a graph of order n with $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} | i \in [1, n-1]\}$. A *cycle* C_n is a graph of order n with $V(C_n) = \{v_1, v_2, \dots, v_n\}$ and $E(C_n) = \{v_i v_{i+1} | i \in [1, n-1]\} \cup \{v_1 v_n\}$. A *complete graph* of order n is a graph in which every two vertices are adjacent, denoted by K_n. A *complete bipartite graph* is a graph whose vertex set can be partitioned into two subsets V_1 and V_2 such that no edge has both endpoints in the same subset and every possible edge that could connect vertices in different subsets is part of the graph. In case $|V_1| = m$ and $|V_2| = n$, we denote such graph by $K_{m,n}$. In case $m = 1$ or $n = 1$, a complete bipartite graph $K_{m,n}$ is called a *star graph.* A *friendship graph* of order $2k+1$ is a graph obtained by taking k copies of a cycle C_3 with a vertex in common, denoted by F_k.

Received April 5, 2021; revised May 21, 2022.
2020 *Mathematics Subject Classification.* Primary 05C12.
Key words and phrases. Cartesian product; complete graph; metric dimension; non-isolated resolving number.

The *distance* between two vertices u and v in G, denoted by $d(u, v)$, is the length of a shortest $u - v$ path in G. For an ordered subset $W = \{w_1, w_2, \ldots, w_k\}$ of $V(G)$, the *representation* of a vertex $v \in V(G)$ with respect to W is k-vector $r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. The set W is said to be a *resolving set* of G if $r(u|W) \neq r(v|W)$ for every u and v in $V(G)$ with $u \neq v$. A resolving set containing a minimum number of vertices is called a *basis* of G. The number of elements in a basis of G is called the *metric dimension* of G and denoted by $\dim(G)$.

The concept of metric dimension was introduced independently by Harary-Melter [**11**] and Slater [**23**]. It is obvious that for every graph G of order n, $1 \leq \dim(G) \leq n - 1$. All connected graphs of order n which have metric dimension 1, $n-1$, or $n-2$ were characterized by Chartrand et al. [**8**]. Some authors also studied the metric dimension of certain classes of graph. Chartrand et al. [**8**] provided the metric dimension of cycles and paths. The metric dimension of some regular graphs was determined by Bača et al. [**4**]. Meanwhile, some authors investigated the metric dimension of certain graphs obtained by a graph operation [**7, 13, 14, 15, 16, 19, 20, 21**]. The concept of the resolving set has various applications in diverse areas including coin weighing problems [**22**], network discovery and verification [**5**], robot navigation [**17**], mastermind game [**6**], and problems of pattern recognition and image processing [**18**].

In this paper, we consider a specific resolving set W of G, where the induced subgraph of G by W does not contain an isolated vertex. A non-isolated resolving set of G with minimum cardinality is called an *nr-set* of G. The cardinality of nr-set of G is called the *non-isolated resolving number* of G, denoted by $\mathrm{nr}(G)$. Since a non-isolated resolving set of G is also a resolving set of G, it is clear that $1 \leq \dim(G) \leq \mathrm{nr}(G) \leq n - 1$. The non-isolated resolving set problem was introduced by Chitra and Arumugam [**9**]. They provided an upper bound of $\mathrm{nr}(G)$ for any graphs G, which is $\mathrm{nr}(G) \leq 2 \cdot \dim(G)$. In the same paper, they characterized all connected graphs of order n with $\mathrm{nr}(G) = n - 1$.

The non-isolated resolving numbers of graphs obtained by graph operations have been determined by some authors. Chitra and Arumugam [**9**] proved that the non-isolated resolving number of corona product graphs between any connected graph G of order n with a non connected graph of order 2 is $2n$. Abidin et al. determined the non-isolated resolving number of corona product of G with H, where G is any connected graphs and H is a complete graph, a cycle, or a path [**1**], and H is a regular graph [**2**]. Alfarisi et al. [**3**] determined the non-isolated resolving number of k-corona product graph. Dafik et al. [**10**] provided a lower bound and an upper bound of the non-isolated resolving number of edge comb product and join product of two connected graphs.

In this paper, we consider the Cartesian product of graphs G and H, denoted by $G \times H$. The Cartesian product of G and H, denoted by $G \times H$, is a graph with its vertex set $V(G) \times V(H) = \{(u, v)|u \in V(G), v \in V(H)\}$, where (u, v) is adjacent to (x, y) whenever $u = x$ and $\{v, y\} \in E(H)$, or $v = y$ and $\{u, x\} \in E(G)$. By the definition, it is clear that $G \times H$ is isomorphic to $H \times G$. Chitra and Arumugam [**9**] proved that $\mathrm{nr}(P_n \times P_n) = 4$ for any $n \geq 3$ and $\mathrm{nr}(C_n \times P_2) = 3$ for any $n \geq 4$.

They also provided an upper bound of $\mathrm{nr}(G \times P_2)$ for any non trivial connected graph G. Hasibuan et al. [**12**] determined the non-isolated resolving number of $G \times P_n$ for some classes of G.

Let K_n be a complete graph of order $n \geq 3$. In this paper, for any graph G of order m with $m \leq n$, we determine the sharp lower and upper bounds of non-isolated resolving number of $G \times K_n$. We provide $\mathrm{nr}(G \times K_n)$ for some classes of G, including paths, complete graphs, cycles, friendship graphs, and star graphs. For any positive integers $c \leq \lfloor \frac{m}{2} \rfloor$, we show that there exists a graph G of order m such that $\mathrm{nr}(G \times K_n)$ is equal to the upper bound minus c.

2. Main results

For any $u \in V(G)$ and $v \in V(H)$, we define $G(v) = \{(u, v)|u \in V(G)\}$ and $H(u) = \{(u, v)|v \in V(H)\}$. Note that an induced subgraph of $G \times H$ by $G(v)$ and $H(u)$ is isomorphic to G and H, respectively. We can say, that $G(v)$ and $H(u)$ as a column of $G \times H$ in v and a row of $G \times H$ in u, respectively. Let $V(F_k) = \{a_0, b_i, c_i|i \in [1, k]\}$ and $E(F_k) = \{a_0b_i, a_0c_i, b_ic_i|i \in [1, k]\}$ be the vertex set and the edge set of a friendship graph F_k, respectively.

We begin by presenting the following useful facts in Lemma 2.1 and Lemma 2.2. These lemmas are needed to prove some results in this paper.

Lemma 2.1. *For all integers k, m, n, p, and q at least 2, let K_n, $K_{p,q}$, F_k, and H be a complete graph of order n, a complete bipartite graph with the cardinality of its independent sets p and q, a friendship graph of order $2k+1$, and a connected graph of order m, respectively. Let G be K_n, $K_{p,q}$, or F_k, and let W be a resolving set of $G \times H$. If x and y are different vertices in G satisfying one condition below:*

(i) *x and y are different vertices in K_n,*
(ii) *x and y are different vertices in an independent set of $K_{p,q}$, or*
(iii) *x and y are different vertices in F_k with $x = b_i$ and $y = c_i$ for some $i \in [1, k]$,*

then $W \cap H(x) \neq \emptyset$ or $W \cap H(y) \neq \emptyset$.

Proof. Let t be the order of G. Let $V(G) = \{u_1, u_2, \ldots, u_t\}$ and let $V(H) = \{v_1, v_2, \ldots, v_m\}$. Let W be a resolving set of $G \times H$. We prove this lemma by contradiction. Suppose that there are two vertices x and y in G such that $W \cap H(x) = \emptyset$ and $W \cap H(y) = \emptyset$. Let $w \in W$, then $w = (u_i, v_j) \in W$ for some $i \in [1, t]$ and $j \in [1, m]$. We obtain

$$\begin{aligned} d((x, v_1), w) &= d((x, v_1), (u_i, v_j)) = d((x, v_1), (u_i, v_1)) + d((u_i, v_1), (u_i, v_j)) \\ &= d((y, v_1), (u_i, v_1)) + d((u_i, v_1), (u_i, v_j)) = d((y, v_1), (u_i, v_j)) \\ &= d((y, v_1), w). \end{aligned}$$

So, $r((x, v_1)|W) = r((y, v_1)|W)$. We get a contradiction. □

For any $a \in V(G)$, let x and y be two distinct vertices in the row of a. If z is another vertex in the same column as y, we show in the following lemma that the distance of z and y is less than the distance of z and x.

Lemma 2.2. *Let $(u, a), (u, b), (v, a)$, and (v, b) be four vertices in $V(G \times H)$ with $u \neq v$ and $a \neq b$. Then $d((u, a), (v, a)) < d((u, a), (v, b))$ and $d((u, a), (u, b)) < d((u, a), (v, b))$.*

Proof. Note that by the definition of $G \times H$, $d((u, a), (v, a)) = d((u, b), (v, b))$ and $d((u, a), (u, b)) = d((v, a), (v, b))$. We obtain

$$d((u, a), (v, a)) < d((u, a), (v, a)) + d((v, a), (v, b)) = d((u, a), (v, b))$$

and

$$d((u, a), (u, b)) < d((u, a), (u, b)) + d((u, b), (v, b)) = d((u, a), (v, b)). \qquad \square$$

2.1. General bounds

In this subsection, we provide a general bounds of $\mathrm{nr}(G \times K_n)$. The lower and upper bounds are given in Theorem 2.3. The sharpness of the lower and upper bounds can be seen in Theorems 2.4 and 2.5, respectively.

Theorem 2.3. *Let m and n be two integers with $3 \leq m \leq n$. Let G be a connected graph of order m. Let K_n be a complete graph of order n. Then*

$$n - 1 \leq \mathrm{nr}(G \times K_n) \leq \begin{cases} n - 1 & \text{if } m \leq \left\lfloor \frac{n+1}{2} \right\rfloor, \\ n & \text{if } m = \frac{n}{2} + 1 \text{ and } n \text{ is even}, \\ m + n - \left\lfloor \frac{n}{2} \right\rfloor - 2 & \text{if } m > \left\lfloor \frac{n}{2} \right\rfloor + 1. \end{cases}$$

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_m\}$ such that $d(u_1, u_i) \leq d(u_1, u_{i+1})$ for every $i \in [2, m-1]$, and let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Since a Cartesian product of two graphs is commutative, by Lemma 2.1, we get $\mathrm{nr}(G \times K_n) \geq n - 1$.

We define

$$W = \begin{cases} W_1 & \text{if } m \leq \frac{n}{2}, \\ W_2 & \text{if } m = \left\lfloor \frac{n}{2} \right\rfloor + 1, \\ W_3 & \text{if } m > \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is odd}, \\ W_4 & \text{if } m > \frac{n}{2} + 1 \text{ and } n \text{ is even}, \end{cases}$$

where

$W_1 = \{(u_i, v_{2i-1}), (u_i, v_{2i}) | i \in [1, m-1]\} \cup \{(u_{m-1}, v_j) | j \in [2m-1, n-1]\}$,

$W_2 = \{(u_i, v_{2i-1}), (u_i, v_{2i}) | i \in [1, m-1]\}$,

$W_3 = \left\{(u_i, v_{2i-1}), (u_i, v_{2i}) | i \in \left[1, \left\lfloor \frac{n}{2} \right\rfloor\right]\right\} \cup \left\{(u_j, v_{n-1}) | j \in \left[\left\lfloor \frac{n}{2} \right\rfloor + 1, m-1\right]\right\}$,

$$W_4 = \left\{(u_i, v_{2i-1}), (u_i, v_{2i}) | i \in \left[1, \left\lfloor \frac{n}{2} \right\rfloor - 1\right]\right\} \cup \left\{(u_j, v_{n-1}) | j \in \left[\left\lfloor \frac{n}{2} \right\rfloor, m-1\right]\right\}.$$

By the definition, for every vertex $x \in W$, there exists $y \in W$ which is adjacent to x. Thus, W does not contain an isolated vertex. Note that there is only one column and one row of $G \times K_n$ such that its elements are not members of W. Let (u_i, v_j) and (u_k, v_l) be two distinct vertices in $V(G \times K_n) \setminus W$. By Lemma 2.2, we only need to prove case $i \neq k$ and $j \neq l$. If $d((u_i, v_j), (u_1, v_1)) \neq d((u_k, v_l), (u_1, v_1))$, it is clear that $r((u_i, v_j)|W) \neq r((u_k, v_l)|W)$. Suppose that $d((u_i, v_j), (u_1, v_1)) = d((u_k, v_l), (u_1, v_1))$. We distinguish two cases.
Case 1: $d((u_i, v_j), (u_1, v_1)) = 1$.
Without loss of generality, let $i = 1$ and $l = 1$. We obtain

$$d((u_i, v_j), (u_1, v_2)) = 1 < 2 \leq d((u_k, v_l), (u_1, v_2)).$$

Case 2: $d((u_i, v_j), (u_1, v_1)) \neq 1$.
Note that $(u_i, v_r) \in W$ or $(u_k, v_t) \in W$ for some i and k in $[1, m-1]$, and r and t in $[1, n-1]$. Without loss of generality, let $(u_i, v_r) \in W$. We obtain

$$d((u_i, v_j), (u_i, v_r)) = 1 < 2 \leq d((u_k, v_l), (u_i, v_r)).$$

All cases imply $r((u_i, v_j)|W) \neq r((u_k, v_l)|W)$. □

Theorem 2.4. *For any two integers m and n with $3 \leq m \leq n$, let P_m be a path of order m and K_n be a complete graph of order n. Then*

$$\mathrm{nr}(P_m \times K_n) = n - 1.$$

Proof. Let $V(P_m) = \{u_i | i \in [1, m]\}$ and let $V(K_n) = \{v_i | i \in [1, n]\}$. By Theorem 2.3, we only need to prove $\mathrm{nr}(P_m \times K_n) \leq n - 1$. We define a vertex set

$$W = \{(u_1, v_j) | j \in [1, n-1]\}.$$

By the definition, for every vertex $x \in W$, there exists $y \in W$ which is adjacent to x. Thus, W does not contain an isolated vertex.

Let $x = (u_i, v_j)$ and $y = (u_k, v_l)$ be two distinct vertices in $V(P_m \times K_n) \setminus W$. Note that only one column does not contribute to W. Thus, by Lemma 2.2, we only need to prove two cases.
Case 1: $j = l$.
Without loss of generality, let $i < k$. We obtain

$$d(x, (u_1, v_1)) < d(x, (u_1, v_1)) + 1 \leq d(y, (u_1, v_1)).$$

Case 2: $i \neq k$ and $j \neq l$.
If $d(x, (u_1, v_1)) \neq d(y, (u_1, v_1))$, it is clear that $r(x|W) \neq r(y|W)$. We assume that $d(x, (u_1, v_1)) = d(y, (u_1, v_1))$. We obtain $|i - k| = 1$. Without loss of generality $x = (u_{i+1}, v_1)$ and $y = (u_i, v_l)$ for some i and l. We obtain

$$d(x, (u_1, v_2)) = i + 1 > i = d(y, (u_1, v_2)).$$

All cases imply $r(x|W) \neq r(y|W)$. So, $\mathrm{nr}(P_m \times K_n) \leq n - 1$. □

Theorem 2.5. *Let m and n be two integers with $3 \le m \le n$. Let K_n be a complete graph of order n. Then*

$$\mathrm{nr}(K_m \times K_n) = \begin{cases} n-1 & \text{if } m \le \left\lfloor \frac{n}{2} \right\rfloor \text{ or } m = \left\lfloor \frac{n}{2} \right\rfloor + 1, \text{ and } n \text{ is odd,} \\ n & \text{if } m = \frac{n}{2} + 1 \text{ and } n \text{ is even,} \\ m+n-\left\lfloor \frac{n}{2} \right\rfloor - 2 & \text{if } m > \left\lfloor \frac{n}{2} \right\rfloor + 1. \end{cases}$$

Proof. By Theorem 2.3, we only need to prove the sharp lower bound of $\mathrm{nr}(K_m \times K_n)$. By Lemma 2.1, if $m \le \lfloor \frac{n}{2} \rfloor$ or $m = \lfloor \frac{n}{2} \rfloor + 1$, and n is odd, then we have done. Now, we assume that $m = \frac{n}{2} + 1$ and n is even, or $m > \lfloor \frac{n}{2} \rfloor + 1$. Let W' be an nr-set of $K_m \times K_n$.

For case $m = \frac{n}{2} + 1$ with even n, suppose that $|W'| = n - 1$. By Lemma 2.1, we have one vertex for each column among $n-1$ columns of $K_m \times K_n$ which are contributed to W'. Let us consider a row of $K_m \times K_n$. If a row R is contributed to W', then $|R \cap W'| \ge 2$. So, there are at most $\lfloor \frac{n}{2} \rfloor - 1$ rows which are contributed to W'. It implies that at least two rows of $K_m \times K_n$ are not contributed to W', a contradiction with Lemma 2.1. Therefore, $\mathrm{nr}(K_m \times K_n) \ge n$.

Now, we prove case $m > \lfloor \frac{n}{2} \rfloor + 1$. Suppose $|W'| \le m + n - \lfloor \frac{n}{2} \rfloor - 3$. By Lemma 2.1, at least $m-1$ rows and $n-1$ columns of $K_m \times K_n$ are contributed to W', respectively. We distinguish two cases.

Case 1: $m = n$.

We can arrange all members of W' such that $d(x, y) = 2$ for every $x, y \in W'$, $x \ne y$. Note that in this case W' contains an isolated vertex. Thus, we must add at least $\lceil \frac{n-1}{2} \rceil$ vertices more to W' such that W' does not contain an isolated vertex. Therefore, we have $\mathrm{nr}(K_m \times K_n) \ge (n-1) + \lceil \frac{n-1}{2} \rceil$.

If n is odd, we obtain

$$(n-1) + \frac{n-1}{2} = \frac{3n-3}{2} > 2n - \frac{n-1}{2} - 3 = \frac{3n-5}{2}.$$

If n is even, we get

$$(n-1) + \frac{n}{2} = \frac{3n-2}{2} > 2n - \frac{n}{2} - 3 = \frac{3n-6}{2}.$$

Case 2: $m < n$.

We can arrange at most $\lfloor \frac{m}{2} \rfloor$ of W' such that $d(x, y) = 2$ for $x, y \in W'$. Therefore, we must add at least $m - \lfloor \frac{m}{2} \rfloor - 1$ vertices more to W' such that W' does not contain an isolated vertex. Therefore, we have $\mathrm{nr}(K_m \times K_n) \ge n-1+m-\lfloor \frac{m}{2} \rfloor - 1 = n + m - \lfloor \frac{m}{2} \rfloor - 2$.

If m and n are odds, we get

$$(n+m) - \frac{m-1}{2} - 2 = \frac{2n+m-3}{2} > n + m - \frac{n-1}{2} - 3 = \frac{2m+n-5}{2}.$$

If m and n are evens, we obtain

$$(n+m) - \frac{m}{2} - 2 = \frac{2n+m-4}{2} > n + m - \frac{n}{2} - 3 = \frac{2m+n-5}{2}.$$

If m is odd and n is even, we obtain

$$(n+m)-\frac{m-1}{2}-2=\frac{2n+m-3}{2}>n+m-\frac{n}{2}-3=\frac{2m+n-6}{2}.$$

If m is even and n is odd, we obtain

$$(n+m)-\frac{m}{2}-2=\frac{2n+m-4}{2}>n+m-\frac{n-1}{2}-3=\frac{2m+n-5}{2}.$$

From all cases, we obtain a contradiction. Hence, $\mathrm{nr}(K_m\times K_n)\geq m+n-\lfloor\frac{n}{2}\rfloor-2$. □

Now, we provide some properties of G such that $\mathrm{nr}(G\times K_n)=n-1$ which can be seen in Theorem 2.6 below.

Theorem 2.6. *For any two integers m and n with $3\leq m\leq n$, let G be a connected graph of order m and K_n be a complete graph of order n. Let W be an* nr-*set of G. If G satisfies one of conditions below:*

(i) $m\leq\left\lfloor\frac{n}{2}\right\rfloor$,

(ii) $m=\left\lfloor\frac{n}{2}\right\rfloor+1$ *if n is odd,*

(iii) $m>\left\lfloor\frac{n}{2}\right\rfloor$ *and* $|W|\leq\left\lfloor\frac{n}{2}\right\rfloor-1$,

(iv) $m>\left\lfloor\frac{n}{2}\right\rfloor$ *and* $|W|=\left\lfloor\frac{n}{2}\right\rfloor$ *if n is odd,*

then $\mathrm{nr}(G\times K_n)=n-1$.

Proof. Let G be a connected graph of order m and K_n be a complete graph of order n. If $m\leq\lfloor\frac{n}{2}\rfloor$ or $m=\lfloor\frac{n}{2}\rfloor+1$ when n is odd, then the theorem is completed by using Theorem 2.3. Now, we assume that $m>\lfloor\frac{n}{2}\rfloor$ and $|W|\leq\lfloor\frac{n}{2}\rfloor-1$, or $m>\lfloor\frac{n}{2}\rfloor$ and $|W|\leq\lfloor\frac{n}{2}\rfloor$ when n is odd. By Lemma 2.1, we only need to show that $\mathrm{nr}(G\times K_n)\leq n-1$. Let $V(G)=\{u_1,u_2,\dots,u_m\}$ and W be an nr-set of G. If $|W|=t$, without loss of generality, let $W=\{u_1,u_2,\dots,u_t\}$, and $V(K_n)=\{v_1,v_2,\dots,v_n\}$. Since $|W|\leq\lfloor\frac{n}{2}\rfloor-1$ or $|W|\leq\lfloor\frac{n}{2}\rfloor$ for odd n, we can define a vertex set

$$W'=\begin{cases} W_1\cup\{(u_{\lfloor\frac{n}{2}\rfloor-1},v_{n-1})\} & \text{if } n \text{ is even and } |W|\leq\left\lfloor\frac{n}{2}\right\rfloor-1,\\ W_2 & \text{if } n \text{ is odd and } |W|\leq\left\lfloor\frac{n}{2}\right\rfloor,\end{cases}$$

where

$$W_1=\left\{(u_i,v_{2i-1}),(u_i,v_{2i})|i\in\left[1,\left\lfloor\frac{n}{2}\right\rfloor-1\right]\right\},$$

$$W_2=\left\{(u_i,v_{2i-1}),(u_i,v_{2i})|i\in\left[1,\left\lfloor\frac{n}{2}\right\rfloor\right]\right\}.$$

By the definition, for every vertex $x\in W'$, there exists $y\in W'$ which is adjacent to x. Thus, W' does not contain an isolated vertex.

Let (u_i,v_j) and (u_k,v_l) be two distinct vertices in $V(G\times K_n)\smallsetminus W'$. By Lemma 2.2, we only need to prove case $j=l$, and case $i\neq k$ and $j\neq l$. If $d((u_i,v_j),(u_1,v_1))\neq d((u_k,v_l),(u_1,v_1))$, it is clear that $r((u_i,v_j)|W')\neq$

$((u_k, v_l)|W')$. Now, we assume that $d((u_i, v_j), (u_1, v_1)) = d((u_k, v_l), (u_1, v_1))$. Since W is an nr-set of G, there are u_i and u_k in $V(G)$ such that $r(u_i|W) \neq r(u_k|W)$. Therefore, there is a vertex $u \in W$ such that $d(u_i, u_p) \neq d(u_k, u_p)$ for some p in $[1, m]$.

Subcase 2a: $j = l$.

Without loss of generality, let $(u_p, v_q) \in W'$ for some q in $[1, n-1]$, then we obtain

$$\begin{aligned} d((u_i, v_j), (u_p, v_q)) &= d((u_i, v_j), (u_p, v_j)) + d((u_p, v_j), (u_p, v_q)) \\ &\neq d((u_k, v_l), (u_p, v_j)) + d((u_p, v_j), (u_p, v_q)) \\ &= d((u_k, v_l), (u_p, v_q)). \end{aligned}$$

Subcase 2b: $i \neq k$ and $j \neq l$.

We distinguish two subcases.

Subcase 2b(1): $d((u_i, v_j), (u_1, v_1)) = 1$.

Without loss of generality, let $i = 1$ and $l = 1$. We obtain

$$d((u_i, v_j), (u_1, v_2)) = 1 < 2 = d((u_k, v_l), (u_1, v_2)).$$

Subcase 2b(2): $d((u_i, v_j), (u_1, v_1)) \neq 1$.

By definition of W', $(u_i, v_p) \in W'$ or $(u_k, v_q) \in W'$ for some i and k in $[1, m-1]$, and p and q in $[1, n-1]$. Without loss of generality, let $(u_i, v_p) \in W'$. We obtain

$$d((u_i, v_j), (u_i, v_p)) < d((u_i, v_j), (u_i, v_p)) + 2 \leq d((u_k, v_l), (u_i, v_p)).$$

All cases imply that $r((u_i, v_j)|W') \neq r((u_k, v_l)|W')$. □

Note that if G satisfies a property given in Theorem 2.6, then $\mathrm{nr}(G \times K_n) = n-1$. On the other hand, we suspect that if $\mathrm{nr}(G \times K_n) = n - 1$, then G satisfies a condition in Theorem 2.6. However, we have not been able to prove it. In this paper, we present it as an open problem.

Open Problem 2.7. For any two integers m and n with $3 \leq m \leq n$, let G be a connected graph of order m, K_n be a complete graph of order n, and W be an nr-set of G. Prove (disprove) if $\mathrm{nr}(G \times K_n) = n - 1$, then G satisfies one of condition below:

(i) $m \leq \left\lfloor \frac{n}{2} \right\rfloor$,

(ii) $m = \left\lfloor \frac{n}{2} \right\rfloor + 1$, if n is odd,

(iii) $m > \left\lfloor \frac{n}{2} \right\rfloor$ and $|W| \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$,

(iv) $m > \left\lfloor \frac{n}{2} \right\rfloor$ and $|W| \leq \left\lfloor \frac{n}{2} \right\rfloor$, if n is odd.

2.2. Some exact values of $\mathrm{nr}(G \times K_n)$

In this subsection, we give an exact value of the non-isolated resolving number of $G \times K_n$ for some graphs G. We consider G is a cycle, a friendship graph, a star graph, or a complete c-partite graph. The results are as follows.

Theorem 2.8. *For any two integers m and n with $3 \leq m \leq n$, let C_m be a cycle of order m and K_n be a complete graph of order n. Then*

$$\mathrm{nr}(C_m \times K_n) = \begin{cases} n & \text{if } n \in \{3,4\}, \\ n-1 & \text{if } n \geq 5. \end{cases}$$

Proof. Let $V(C_m) = \{u_1, u_2, \ldots, u_m\}$ and let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. For the upper bound, we define

$$W = \begin{cases} \{(u_1,v_1),(u_1,v_2),(u_2,v_1)\} & \text{if } n = 3, \\ \{(u_1,v_1),(u_1,v_2),(u_1,v_3),(u_2,v_1)\} & \text{if } n = 4, \\ \{(u_1,v_1),(u_1,v_2),(u_2,v_j) | j \in [3,n-1]\} & \text{if } n \geq 5. \end{cases}$$

By the definition, for every vertex $x \in W$, there exists $y \in W$ which is adjacent to x. Thus, W does not contain an isolated vertex.

Let (u_i, v_j) and (u_k, v_l) be two distinct vertices in $V(C_m \times K_n) \setminus W$. By Lemma 2.2, we only need to prove case $j = l$, and case $i \neq k$ and $j \neq l$. Note that if $d((u_i,v_j),(u_1,v_1)) \neq d((u_k,v_l),(u_1,v_1))$, it is clear that $r((u_i,v_j)|W) \neq r((u_k,v_l)|W)$. Now, we assume that $d((u_i,v_j),(u_1,v_1)) = d((u_k,v_l),(u_1,v_1))$.

<u>Case 1</u>: $j = l$.

If $n = 3$ or $n = 4$, we obtain

$$d((u_i,v_j),(u_2,v_1)) < d((u_i,v_j),(u_2,v_1)) + 1 = d((u_k,v_l),(u_2,v_1)).$$

If $n \geq 5$, we obtain

$$d((u_i,v_j),(u_2,v_3)) < d((u_i,v_j),(u_2,v_3)) + 1 = d((u_k,v_l),(u_2,v_3)).$$

<u>Case 2</u>: $i \neq k$ and $j \neq l$.

By definition of W, note that $(u_p, v_j) \in W$ or $(u_q, v_l) \in W$ for some p and q in $[1,2]$. Without loss of generality, let $(u_p, v_j) \in W$. We obtain

$$d((u_i,v_j),(u_p,v_j)) < d((u_i,v_j),(u_p,v_j)) + 1 = d((u_k,v_l),(u_p,v_j)).$$

All cases imply $r((u_i,v_j)|W) \neq r((u_k,v_l)|W)$.

For the lower bound, we prove it by contradiction. By Theorem 2.3, we only need to prove $\mathrm{nr}(C_m \times K_n) \geq n$ for $n \in \{3,4\}$. Suppose W' is an nr-set of $C_m \times K_n$ and $|W'| \leq n-1$. By Lemma 2.1, at least $n-1$ columns are contributed to W'. Since W' does not contain an isolated vertex, all members of W' must be in the same row. For $u \in V(C_m)$, let $K_n(u) \cap W' \neq \emptyset$. Without loss of generality, let

$$W' = \begin{cases} \{(u_1,v_1),(u_1,v_2)\} & \text{if } n = 3, \\ \{(u_1,v_1),(u_1,v_2),(u_1,v_3)\} & \text{if } n = 4. \end{cases}$$

We obtain

$$r((u_2,v_1)|W') = r((u_m,v_1)|W').$$

So, we get a contradiction. Therefore, for $n \in \{3,4\}$, $\mathrm{nr}(C_m \times K_n) \geq n$. $\square$

In the next theorem, we determine the non-isolated resolving number of Cartesius product of F_k and K_n.

Theorem 2.9. *For any two integers k and n with $n \geq 5$ and $2 \leq k \leq \frac{n-1}{2}$, let F_k be a friendship graph of order $2k+1$ and K_n be a complete graph of order n. Then*

$$\mathrm{nr}(F_k \times K_n) = n - 1.$$

Proof. Note that the order of F_k is $2k+1$. Since $k \leq \frac{n-1}{2}$, we have $2k+1 \leq n$. Therefore, by Theorem 2.3, $\mathrm{nr}(F_k \times K_n) \geq n-1$. So, we only need to prove the sharp upper bound of $\mathrm{nr}(F_k \times K_n)$. Let $V(F_k) = \{u_0, u_1, u_2, \ldots, u_{2k}\}$ and $E(F_k) = \{u_0u_i, u_0u_{k+i}, u_iu_{k+i} | i \in [1,k]\}$, and let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. We define

$$W = \begin{cases} W_1 & \text{if } k = \dfrac{n-1}{2} \text{ and } n \text{ is odd,} \\ W_2 & \text{otherwise,} \end{cases}$$

where

$$W_1 = \{(u_i, v_{2i-1}), (u_i, v_{2i}) | i \in [1,k]\},$$
$$W_2 = \{(u_i, v_{2i-1}), (u_i, v_{2i}) | i \in [1,k]\} \cup \{(u_k, v_j) | j \in [2k+1, n-1]\}.$$

By the definition, for every vertex $x \in W$, there exists $y \in W$ which is adjacent to x. Thus, W does not contain an isolated vertex. Then, by definition of W, there is only one columnn of $F_k \times K_n$ which does not contribute to W. Let $x = (u_i, v_j)$ and $y = (u_l, v_m)$ be two distinct vertices in $V(F_k \times K_n) \setminus W$. By Lemma 2.2, we only need to prove $r(x|W) \neq r(y|W)$ for case $j = m$ and case $i \neq l$ and $j \neq m$. If $d(x, (u_1, v_1)) \neq d(y, (u_1, v_1))$, it is clear that $r(x|W) \neq r(y|W)$. Now, we assume that $d(x, (u_1, v_1)) = d(y, (u_1, v_1))$. For this case, we distinguish two cases.
<u>*Case 1*</u>: $(u_i, v_p) \in W$ or $(u_l, v_q) \in W$ for some p and q in $[1, n-1]$.
Without loss of generality, let $(u_i, v_p) \in W$.
If $j = m$, we obtain

$$d(x, (u_i, v_p)) < d(y, x) + d(x, (u_i, v_p)) = d(y, (u_i, v_p)).$$

If $i \neq l$ and $j \neq m$, we obtain

$$d(x, (u_i, v_p)) < d(x, (u_i, v_p)) + 1 \leq d(y, (u_i, v_p)).$$

<u>*Case 2*</u>: $(u_i, v_p) \notin W$ and $(u_l, v_q) \notin W$ for every p and q in $[1, n-1]$.
By the definition of W, there is a vertex $(u_{i-k}, v_r) \in W$ such that $u_iu_{i-k} \in E(F_k)$. Therefore, we get

$$\begin{aligned} d(x, (u_{i-k}, v_r)) &= d(x, (u_{i-k}, v_j)) + d((u_{i-k}, v_j), (u_{i-k}, v_r)) \\ &< d(y, (u_{i-k}, v_j)) + d((u_{i-k}, v_j), (u_{i-k}, v_r)) \\ &= d(y, (u_{i-k}, v_r)). \end{aligned}$$

All cases imply that $r(x|W) \neq r(y|W)$. Hence, $\mathrm{nr}(F_k \times K_n) \leq n-1$. □

Now, we determine the non-isolated resolving number of $K_{1,l} \times K_n$.

Theorem 2.10. *For any two integers l and n with $n \geq 3$ and $2 \leq l \leq n-1$, let $K_{1,l}$ be a star graph of order $l+1$ and K_n be a complete graph of order n. Then*

$$\mathrm{nr}(K_{1,l} \times K_n) = \begin{cases} n-1 & \text{if } l \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \text{ or } l = \frac{n}{2} \text{ and } n \text{ is odd}, \\ n & \text{if } l = \frac{n}{2} \text{ and } n \text{ is even}, \\ l + n - \left\lfloor \frac{n}{2} \right\rfloor - 1 & \text{if } l > \left\lfloor \frac{n}{2} \right\rfloor. \end{cases}$$

Proof. By using the similar argument with the proof of Theorem 2.5, the proof is completed. □

For certain m and n, there is a graph G such that $\mathrm{nr}(G \times K_n) = m+n-\lfloor \frac{n}{2} \rfloor - 2 - c$ for $c \leq \lfloor \frac{m}{2} \rfloor$. We recall the definition of a complete c-partite graph. A complete c-partite graph $K_{k_1,k_2,\ldots,k_c}$ is a graph, where $V(K_{k_1,k_2,\ldots,k_c})$ can be partitioned to c set $V_1, V_2, \ldots, V_c$ with $|V_i| = k_i$ for some $i \in [1, c]$ and xy is an edge whenever $x \in V_i$ and $y \in V_j$ with $i \neq j$.

Theorem 2.11. *For integers m and n with $n \geq 4$ and $\lfloor \frac{n}{2} \rfloor + 1 < m \leq n$, let K_n be a complete graph of order n. Then there is a graph G of order m such that $\mathrm{nr}(G \times K_n) = m + n - \lfloor \frac{n}{2} \rfloor - 2 - c$, for $c \leq \lfloor \frac{m}{2} \rfloor$.*

Proof. Let $V(K_n) = \{v_i | i \in [1, n]\}$ and let $G = K_{k_1,k_2,\ldots,k_{c+1}}$ with $c \geq 1$ and $k_i \geq 2$ for $i \in [1, c+1]$, where $\Sigma_{i=1}^{c+1} k_i > \lfloor \frac{n}{2} \rfloor + (c+1)$. Note that $m = |V(G)| = k_1 + k_2 + \cdots + k_{c+1}$. Let $V(G) = \{u_i | 1 \leq i \leq k_1 + k_2 + \cdots + k_{c+1}\}$. We assume $u_i \in V_i$ for some $i \in [1, c+1]$. We define a vertex set

$$W = \Big\{ (u_{c+i}, v_{2i-3}), (u_{c+i}, v_{2i-2}), (u_{c+k}, v_{n-1}) \mid i \in [2, \left\lfloor \frac{n}{2} \right\rfloor + 1],\ k \in [\left\lfloor \frac{n}{2} \right\rfloor + 2, m - c] \Big\}.$$

By the definition, for every vertex $x \in W$, there exists $y \in W$ which is adjacent to x. Thus, W does not contain an isolated vertex. Now, we show that W is a resolving set of $G \times K_n$.

Note that, only one row of $G \times K_n$ in each partition that does not contribute to W. Let $x = (u_i, v_j)$ and $y = (u_k, v_l)$ be two distinct vertices in $V(G \times K_n) \setminus W$. By Lemma 2.2, we only need to prove case $j = l$ and case $i \neq k$, and $j \neq l$.

<u>*Case 1*</u>: $j = l$.

We distinguish two subcases.

<u>*Subcase 1a*</u>: $(u_i, v_p) \in W$ or $(u_k, v_q) \in W$ for some p and q in $[1, n-1]$.

Without loss of generality, let $(u_i, v_p) \in W$. We obtain

$$d(x, (u_i, v_p)) < d(x, (u_i, v+p)) + 1 = d(y, (u_i, v_p))$$

or

$$d(x, (u_i, v_p)) < d(x, (u_i, v_p)) + 2 = d(y, (u_i, v_p)).$$

<u>*Subcase 1b*</u>: $(u_i, v_r) \notin W$ or $(u_k, v_s) \notin W$ for every r and s in $[1, n-1]$.

Note that there is u_t for some $t \in [1, k_i]$ in the same partition with u_i such that $(u_t, v_w) \in W$ for some $w \in [1, n-1]$. We obtain

$$d(x, (u_t, v_w)) = 2 + d((u_t, v_j), (u_t, v_w)) > 1 + d((u_p, v_j), (u_p, v_w)) = d(y, (u_p, v_w)).$$

Case 2: $i \neq k$ and $j \neq l$.
If $d(x,(u_{c+2},v_1)) \neq d(y,(u_{c+2},v_1))$, it is clear that $r(x|W) \neq r(y|W)$. We assume $d(x,(u_{c+2},v_1)) = d(y,(u_{c+2},v_1))$. We distinguish three subcases.
Subcase 2a: $d(x,(u_{c+2},v_1)) = 1$.
We obtain

$$d(x,(u_{c+2},v_2)) = 1 < 2 = d(y,(u_{c+2},v_2)).$$

Subcase 2b: $d(x,(u_1,v_1)) = 2$.
We obtain

$$d(x,(u_{c+2},v_2)) = 3 > 2 = d(y,(u_{c+2},v_2))$$

or

$$d(x,(u_1,v_2)) = 3 > 1 = d(y,(u_1,v_2)).$$

Subcase 2c: $d(x,(u_1,v_1)) = 3$.
Since $d(x,(u_1,v_1)) = 3$, u_i and u_k are on the same partition. Therefore, $(u_i,v_a) \in W$ or $(u_k,v_b) \in W$ for some a and b in $[1,n-1]$. Without loss of generality, let $(u_i,v_a) \in W$. We get

$$d(x,(u_i,v_a)) = 1 < 2 = d(y,(u_i,v_a))$$

or

$$d(x,(u_i,v_a)) = 1 < 3 = d(y,(u_i,v_a)).$$

All cases imply $r(x|W) \neq r(y|W)$. Thus, for $c \leq \lfloor \frac{m}{2} \rfloor$, we obtain

$$\begin{aligned}
\mathrm{nr}(G \times K_n) &\leq 2\Big(\Big\lfloor \frac{n}{2} \Big\rfloor + 1 - 2 + 1\Big) + \Big((m-c) - \Big(\Big\lfloor \frac{n}{2} \Big\rfloor + 2\Big) + 1\Big) \\
&= 2\frac{n-1}{2} + m - c - \Big\lfloor \frac{n}{2} \Big\rfloor - 1 \\
&= m + n - \Big\lfloor \frac{n}{2} \Big\rfloor - 2 - c.
\end{aligned}$$

Let W' be an nr-set of $G \times K_n$ and $|W'| \leq m + n - \lfloor \frac{n}{2} \rfloor - 3 - c$. By Lemma 2.1, at least $(n-1)$ columns and at least $(m-c)$ rows of $G \times K_n$ are contributed to W', respectively. We distinguish two cases.
Case 1: $m - c = n - 1$.
We can arrange all members of W' such that $d(x,y) = 2$ or $d(x,y) = 3$ with x and y in W'. Note that W' contains an isolated vertex. Thus, we must add at least $\lceil \frac{n-1}{2} \rceil$ vertices more to W' such that W' does not contain an isolated vertex. Therefore, we get $\mathrm{nr}(G \times K_n) \geq (n-1) + \lceil \frac{n-1}{2} \rceil$.
If n is odd, we get

$$(n-1) + \frac{n-1}{2} = \frac{3n-3}{2} > (n-1+c) + n - \frac{n-1}{2} - 3 - c = \frac{3n-7}{2}.$$

If n is even, we obtain

$$(n-1) + \frac{n}{2} = \frac{3n-2}{2} > (n-1+c) + n - \frac{n}{2} - 3 - c = \frac{3n-8}{2}.$$

Case 2: $m - c < n - 1$.
By using the same argument with the proof of Theorem 2.5, we obtain

$$\mathrm{nr}(G \times K_n) \geq (n-1) + \left\lceil \frac{n-1}{2} \right\rceil > m + n - \left\lfloor \frac{n}{2} \right\rfloor - 3 - c.$$

All cases imply that $\mathrm{nr}(G \times K_n) \geq m + n - \lfloor \frac{n}{2} \rfloor - 2 - c$. $\square$

All results in this paper are restricted to order of G, namely at most the order of K_n. For order of G larger than n, we provide the following open problem.

Open Problem 2.12. For any integer n at least 3, let G be a connected graph and K_n be a complete graph of order n. Determine $\mathrm{nr}(G \times K_n)$, where order of G is greater than n.

Acknowledgment. The authors are thankful to the anonymous referee for some valuable comments that helped to improve the presentation of the paper.

References

1. Abidin W., Salman A. N. M. and Saputro S. W., *The non-isolated resolving number of some corona graphs*, IOP Conf. Series: Journal of Physics: Conf. Series **1097** (2018), 012073.
2. Abidin W., Salman A. N. M. and Saputro S. W., *Non-isolated resolving set of corona graphs with some regular graphs*, Mathematics **10**(6) (2022), 962.
3. Alfarisi R., Dafik, Slamin, Agustin I. H. and Kristiana A. I., *The non-isolated resolving number of k-corona product of graphs*, IOP Conf. Series: Journal of Physics: Conf. Series **1008** (2018), Art. ID 012040.
4. Bača M., Baskoro E. T., Salman A. N. M., Saputro S. W. and Simanjuntak R., *The metric dimension of regular bipartite graphs*, Bull. Math. Soc. Sci. Math. Roumanie **54(102)**(1) (2011), 15–28.
5. Beerliova Z., Eberhard F., Erlebach T., Hall A., Hoffmann M., Mihaľak M. and Ram L. S., *Network dicovery and verification*, IEEE Journal On Selected Areas in Communications **24**(12) (2006), 2168–2181.
6. Caceres J., Hernando C., Mora M., Puertas M. L, Pelayo I. M. and Wood D. R., *On the metric dimension of some families graphs*, Electron. Notes Discrete Math. **22** (2005), 129–133.
7. Caceres J., Hernando C., Mora M., Pelayo I. M., Puertas M. L., Seara C. and Wood D. R., *On the metric dimension of Cartesian products of graphs*, SIAM J. Discrete Math. **21**(2) (2007), 423–441.
8. Chartrand G., Eroh L., Johnson M., and Oellermann O. R., *Resolvability in graphs and the metric dimension of a graph*, Discrete Appl. Math. **105** (2000), 99–113.
9. Chitra P. J. B. and Arumugam S., *Resolving sets without isolated vertices*, Procedia Comput. Sci. **74** (2015), 38–42.
10. Dafik D., Agustin I. H., Surahmat S., Alfarisi R. and Syafrizal Sy., *On non-isolated resolving number of special graphs and their operations*, Far East Journal of Mathematical Sciences **102**(10) (2017), 2473–2492.
11. Harary F. and Melter R. A., *On the metric dimension of a graph*, Ars Combin. **2** (1976), 191–195.
12. Hasibuan I. M., Salman A. N. M. and Saputro S. W., *Non-isolated resolving sets of certain graphs Cartesian product with a path*, IOP Conf. Series: Journal of Physics: Conf. Series **1008** (2018), 012045.
13. Iswadi H., Baskoro E. T., Salman A. N. M. and Simanjuntak R., *The metric dimension of amalgamation of cycles*, Far East Journal of Mathematical Sciences **41**(1) (2010), 19–31.

14. Iswadi H., Baskoro E. T., Salman A. N. M. and Simanjuntak R., *The resolving graph of amalgamation of cycles*, Util. Math. **83** (2010), 121–132.

15. Iswadi H., Baskoro E. T. and Simanjuntak R., *On the metric dimension of corona product of graphs*, Far East Journal of Mathematical Sciences **52**(2) (2011), 155–170.

16. Iswadi H., Baskoro E. T., Simanjuntak R., and Salman A. N. M., *The metric dimension of graphs with pendant edges*, J. Combin. Math. Combin Comput. **65** (2008), 139–145.

17. Khuller S., Raghavachari B. and Rosenfeld A., *Landmarks in graphs*, Discrete Appl. Math. **70**(3) (1996), 217–229.

18. Melter R. A. and Tomescu I., *Metric bases in digital geometry*, Computer Vision Graphics and Image Processing **25** (1984), 113–121.

19. Saputro S. W., Baskoro E. T., Salman A. N. M. and Suprijanto D., *The metric dimension of a complete n-partite and its Cartesian product with a path*, J. Combin. Math. Combin. Comput. **7** (2009), 283–293.

20. Saputro S. W., Simanjuntak R., Uttunggadewa S., Assiyatun H., Baskoro E. T., Salman A. N. M. and Bača M., *The metric dimension of the lexicographic product of graphs*, Discrete Math. **213** (2013), 1045–1051.

21. Saputro S. W., Mardiana N. and Purwasih I. A., *The metric dimension of comb product graphs*, Mat. Vesnik **69**(4) (2017), 248–258.

22. Sebo A. and Tannier E., *On metric generators of graphs*, Math. Operations Research **29**(2) (2004), 383–393.

23. Slater P. J., *Leaves of trees*, Congr. Numer. **14** (1975), 549–559.

I. M. Hasibuan, Doctoral Program of Mathematics, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Indonesia,
e-mail: `imhasibuan@uin-suska.ac.id`

A. N. M. Salman, Combinatorial Mathematics Research Group, Institut Teknologi Bandung, Indonesia,
e-mail: `msalman@math.itb.ac.id`

S. W. Saputro, Combinatorial Mathematics Research Group, Institut Teknologi Bandung, Indonesia,
e-mail: `suhadi@math.itb.ac.id`