Acta Math. Univ. Comenianae
Vol. XC, 4 (2021), pp. 387–402

ON THE EQUAL SUM AND PRODUCT PROBLEM

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Abstract. The paper presents the results which are connected with the following problem formulated by Andrzej Schinzel. Does the number $N(n)$ of integer solutions of the equation $x_1 + x_2 + \cdots + x_n = x_1 x_2 \cdots x_n$ satisfying $1 \le x_1 \le x_2 \le \cdots \le x_n$ tend to infinity with n? We give a general lower bound on $N(n)$. We obtain an Ω-estimate for $\frac{1}{x}\sum_{1<n\le x} N(n)$. We provide necessary conditions for n to be in the exceptional set $\{n : N(n) = 1,\, n \ge 2\}$. Using elementary methods, we show that if $N(n) = 2$, then $n-1, 2n-1 \in \{p, p^2, p^3, pq\}$, where p, q are prime numbers. We prove that the set $\{n : N(n) \le k,\, n \ge 2\}$, and the exceptional set have zero natural density. We give new bounds on sum of coordinates of not-typical solutions. We prove that the system of equations of the equal-sum-product problem has a finite number of solutions.

1. Introduction

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ denote the set of all natural numbers (i.e., positive integers). The study of natural numbers is full of problems which are simple to understand and seem to be unsolvable. An interesting example of this type of problem is the Equal-Sum-Product Problem, (see [**3**]). Some early works can be found in [**2, 5, 7**]. The sum of a sequence of integers $(3, 2, 1)$ is equal to the product of its elements. Namely, we have $3 + 2 + 1 = 3 \cdot 2 \cdot 1$. Moreover, the following equations are also true: $2+2 = 2\cdot 2$, $4+2+1+1 = 4\cdot 2\cdot 1\cdot 1$, $3+3+1+1+1 = 3\cdot 3\cdot 1\cdot 1\cdot 1$, $2+2+2+1+1 = 2\cdot 2\cdot 2\cdot 1\cdot 1$, $5+2+1+1+1 = 5\cdot 2\cdot 1\cdot 1\cdot 1$. The problem considered here consists in finding the sequences of natural numbers $(x_1, x_2, \dots, x_n)$, $n \ge 2$, satisfying the equation

$$(1) \quad x_1 + x_2 + \cdots + x_n = x_1 \cdot x_2 \cdots x_n, \text{ where } x_1 \ge x_2 \ge \cdots \ge x_{n-1} \ge x_n \ge 1.$$

We use $S(n)$, $n \ge 2$, to denote the set of all such sequences. Moreover, we denote $N(n) = |S(n)|$, in other words, $N(n)$ is the number of solutions of Equation (1). Equation (1) always has at least one *typical* solution of the form $(n, 2, \underbrace{1, 1, \dots, 1}_{n-2 \text{ times}})$.

Hence, $N(n) \ge 1$ for all natural numbers $n \ge 2$. Kurlandchik and Nowicki showed

Received January 2, 2021; revised April 16, 2021.
2020 *Mathematics Subject Classification.* Primary 11D72, 11D45; Secondary 11R45.
Key words and phrases. Equal-sum-product; exceptional set; natural density.

that for any given $n \geq 2$, the number of solutions $N(n)$ is finite (see [**4**, Theorem 3]). For $n \geq 2$, we denote $S^*(n)$ by

$$S^*(n) = S(n) \setminus \{(n, 2, \underbrace{1, 1, \ldots, 1}_{n-2 \text{ times}})\}, \text{ that is, } |S^*(n)| = N(n) - 1.$$

We analyse some basic cases (for $n = 2$ and $n = 3$) first. If $n = 2$, then in order to solve the equation $x_1 + x_2 = x_1 x_2$ in natural numbers $x_1 \geq x_2$, we rewrite it in the form $(x_1 - 1)(x_2 - 1) = 1$. Hence, $(x_1, x_2) = (2, 2)$ and $N(2) = 1$. If $n = 3$, then in order to solve the equation $x_1 + x_2 + x_3 = x_1 x_2 x_3$ in natural numbers $x_1 \geq x_2 \geq x_3$, we rewrite it in the form

$$(x_1 x_2 - 1)(x_3 - 1) + (x_1 - 1)(x_2 - 1) = 2.$$

Hence, $(x_1, x_2, x_3) = (3, 2, 1)$ and $N(3) = 1$. The following table lists values of $N(n)$ for $2 \leq n \leq 51$ (see also the sequence A033178 in [**8**]).

Table 1.

n	$N(n)$	n	$N(n)$	n	$N(n)$	n	$N(n)$	n	$N(n)$	n	$N(n)$	n	$N(n)$	n	$N(n)$	n	$N(n)$	n	$N(n)$
2	1	7	2	12	2	17	4	22	2	27	3	32	3	37	6	42	2	47	5
3	1	8	2	13	4	18	2	23	4	28	3	33	5	38	3	43	5	48	2
4	1	9	2	14	2	19	4	24	1	29	5	34	2	39	3	44	2	49	5
5	3	10	2	15	2	20	2	25	5	30	2	35	3	40	4	45	4	50	4
6	1	11	3	16	2	21	4	26	4	31	4	36	2	41	7	46	4	51	4

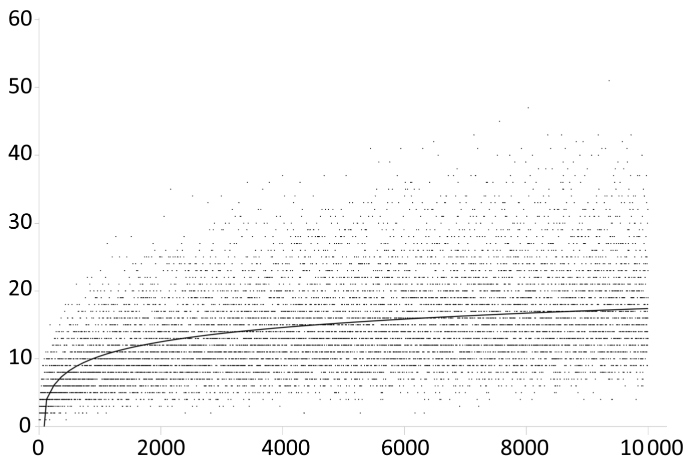

Figure 1. Figure presents the plot of the function $N : \mathbb{N}_{\geq 2} \to \mathbb{N}$ with the logarithmic regression line given by the equation $f(x) = 3.05 \log(x) - 10.65$.

Schinzel showed that $N(n) = 1$ for $n \in \{6, 24\}$ (see [**3**]). Misiurewicz showed that $n \in \{2, 3, 4, 6, 24, 114, 174, 444\}$ are the only $n < 1000$ such that $N(n) = 1$ (see [**3**]).

It is known that if $2 \leq n \leq 10^{10}$ and $N(n) = 1$, then $n \in \{2, 3, 4, 6, 24, 114, 174, 444\}$, (see A033179 in [**8**]).

If $t \in \{0, 1, \dots, \lfloor \frac{s}{2} \rfloor\}$, where s is a nonnegative integer, then

$$(2^{s-t} + 1) + (2^t + 1) + \underbrace{1 + 1 + \cdots + 1}_{2^s - 1 \text{ times}} = (2^{s-t} + 1) \cdot (2^t + 1) \cdot \underbrace{1 \cdot 1 \cdots 1}_{2^s - 1 \text{ times}},$$

and $s - t \geq t$, hence $N(2^s + 1) \geq \lfloor \frac{s}{2} \rfloor + 1$. Therefore, $\limsup_{n\to\infty} N(n) = \infty$.

Despite their apparent simplicity, there are many difficult and unanswered questions connected with the equal-sum-and-product problem which have been identified in the references already cited. In particular, it is conjectured that:

1. the set of exceptional values of the equal-sum-and-product problem, defined as $E_1 = \{n : N(n) = 1, n \geq 2\}$, is finite, see [**2**],
2. $E_1 = \{2, 3, 4, 6, 24, 114, 174, 444\}$, see [**2**],
3. the number $N(n) \to \infty$ as $n \to \infty$, see [**9**] and [**10**, Problem 2].

2. Main results

The main result of the paper runs as follows.

Theorem 2.1. *If* $n \in \mathbb{N}$, $n \geq 2$, *then*

$$N(n) \geq \left\lfloor \frac{d(n-1)+1}{2} \right\rfloor + \left\lfloor \frac{d(2n-1)+1}{2} \right\rfloor - 1, \tag{2}$$

where $d(j)$ is the number of divisors of j. Moreover,

$$\begin{aligned} N(n) \geq\ & \left\lfloor \frac{d(n-1)+1}{2} \right\rfloor + \left\lfloor \frac{d(2n-1)+1}{2} \right\rfloor - 1 \\ & + \left\lfloor \frac{d_2(3n+1)+1}{2} \right\rfloor + \left\lfloor \frac{d_3(4n+1)+1}{2} \right\rfloor + \left\lfloor \frac{d_3(4n+5)+1}{2} \right\rfloor \\ & - \delta(2|n+1) - \delta(3|n+1) - \delta(3|n+2) \\ & - \delta(5|n+2,\ n \geq 8) - \delta(7|n+3,\ n \geq 11) - \delta(11|n+4,\ n \geq 29), \end{aligned} \tag{3}$$

where $d_i(m)$ is the number of such divisors of m that are congruent to i modulo $i+1$. The function δ is Dirac delta function.

Remark. If $2 \leq n \leq 26$, then we have equality in (3).

Corollary 2.2. *If* $n \in \mathbb{N}$, $n \geq 2$, *then*

$$N(n) \geq \frac{1}{2} d(n-1) + \frac{1}{2} d(2n-1) - 1. \tag{4}$$

Proof. From the inequality $\lfloor \frac{x+1}{2} \rfloor \geq \frac{1}{2}x$, where x is an integer, we get $N(n) \geq \left\lfloor \frac{d(n-1)+1}{2} \right\rfloor + \left\lfloor \frac{d(2n-1)+1}{2} \right\rfloor - 1 \geq \frac{1}{2}d(n-1) + \frac{1}{2}d(2n-1) - 1$. Thus the corollary follows from Theorem 2.1. □

3. The sum of the numbers of solutions

The function N takes only positive values, hence the average order is the easiest non-trivial method to determine its behavior. As far as I know, there is no Ω-results for $\frac{1}{x} \sum\limits_{1<n\leq x} N(n)$ in the literature. The following figure presents the values of the $\frac{1}{x}\sum_{1<n\leq x} N(n)$ for $x \leq 10\,000$.

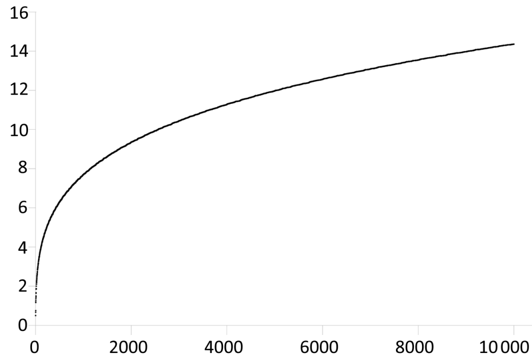

Figure 2. The plot of the function $\frac{1}{x}\sum_{1<n\leq x} N(n)$, where $x \in (2, 10\,000)$.

As an easy consequence of the Corollary 2.2, we note the following theorem.

Theorem 3.1. *For every* $\varepsilon > 0$,

$$\lim_{x\to\infty} \frac{1}{x\log^{1-\varepsilon} x} \sum_{1<n\leq x} N(n) = \infty. \tag{5}$$

Proof. By Corollary 2.2, we have $N(n) \geq \frac{1}{2}(d(n-1)+d(2n-1))-1 \geq \frac{1}{2}d(n-1)$ since $d(2n-1) \geq 2$. Hence

$$\begin{aligned}\lim_{x\to\infty} \frac{1}{x\log^{1-\varepsilon} x} \sum_{1<n\leq x} N(n) &\geq \frac{1}{2}\lim_{x\to\infty} \frac{1}{x\log^{1-\varepsilon} x} \sum_{1\leq n\leq x-1} d(n)\\ &= \frac{1}{2}\lim_{x\to\infty} \frac{1}{(x-1)\log^{1-\varepsilon}(x-1)} \sum_{1\leq n\leq x-1} d(n)\\ &= \frac{1}{2}\lim_{x\to\infty} \frac{1}{x\log^{1-\varepsilon} x} \sum_{1\leq n\leq x} d(n) = \infty\end{aligned}$$

since $\sum\limits_{1\leq n\leq x} d(n) = x\log(x) + x(2\gamma-1) + O(\sqrt{x})$, where $\gamma \approx 0,577$ is Euler's constant, see [**6**, Theorem 7.3]. □

Corollary 3.2. *For every* $\varepsilon > 0$, *the sum* $\sum_{1<n\le x} N(n)$ *dominates* $x\log^{1-\varepsilon} x$ *asymptotically. We have* $\sum_{1<n\le x} N(n) = \omega(x\log^{1-\varepsilon} x)$.

Proof. This is an immediate consequence of the preceding theorem. For all $k > 0$ and sufficiently large x, i.e., $x > C_{\varepsilon,k}$, where $C_{\varepsilon,k}$ depends only on ε and k, we have

$$\sum_{1<n\le x} N(n) > kx\log^{1-\varepsilon} x. \qquad \square$$

By a modification of the proof of Theorem 3.1, we also show next theorem

Theorem 3.3. *The following equality holds true*

$$\frac{1}{x}\sum_{1<n\le x} N(n) = \Omega(\log x). \tag{6}$$

Proof. There is

$$\exists_{c>0}\exists_{x_0\ge 2}\forall_{x>x_0} \left|\sum_{1\le n\le x} d(n) - (x\log x + x(2\gamma-1))\right| \le c\sqrt{x}$$

since $\sum_{1\le n\le x} d(n) = x\log(x) + x(2\gamma-1) + O(\sqrt{x})$, see [**6**, Theorem 7.3].

It follows that

$$\sum_{1\le n\le x} d(n) \ge x\log(x) + x(2\gamma-1) - c\sqrt{x}$$

for $x > x_0$. Hence,

$$\sum_{1\le n\le x-1} d(n) \ge (x-1)\log(x-1) + (x-1)(2\gamma-1) - c\sqrt{x-1}$$

for $x > x_0+1$. By Corollary 2.2, for $n \ge 2$, we have $N(n) \ge \frac{1}{2}d(n-1)$. Therefore,

$$\begin{aligned}\frac{1}{x}\sum_{1<n\le x} N(n) &\ge \frac{1}{2x}\sum_{1\le n\le x-1} d(n)\\ &\ge \frac{1}{2}\frac{x-1}{x}\frac{\log(x-1)}{\log x}\log x + \frac{x-1}{2x}(2\gamma-1) - c\frac{\sqrt{x-1}}{2x}\end{aligned}$$

for $x > x_0+1$. Thus, for sufficiently large x, we have

$$\frac{1}{x}\sum_{1<n\le x} N(n) \ge \frac{1}{3}\log x$$

and consequently, $\frac{1}{x}\sum_{1<n\le x} N(n) = \Omega(\log x)$, where the constant in the Ω symbol is effective. $\square$

4. The set E_1 of exceptional values

In this section, we deal with problems involving integers $n \geq 2$ such that $N(n) = 1$. It is conjectured that if $N(n) = 1$, then $n \in E_1 = \{2, 3, 4, 6, 24, 114, 174, 444\}$, see [**2**].

Lemma 4.1. *Let $n > y_1 \geq y_2 \geq \cdots \geq y_m \geq 2$ be natural numbers such that $m \geq 2$ or $m = 1$, and $y_1 \geq 3$. Let*

$$d_1 d_2 = y_1 y_2 \cdots y_m n + (y_1 + y_2 + \cdots + y_m - m - 2) y_1 y_2 \cdots y_m + 1,$$

where d_1, d_2 are natural numbers and $d_1 \equiv -1 \pmod{y_1 y_2 \cdots y_m}$. Then $N(n) > 1$.

Proof. If $d_1 \equiv -1 \pmod{y_1 y_2 \cdots y_m}$, then also $d_2 \equiv -1 \pmod{y_1 y_2 \cdots y_m}$.

Therefore, $\frac{d_1+1}{y_1 y_2 \cdots y_m}$ and $\frac{d_2+1}{y_1 y_2 \cdots y_m}$ are natural numbers. We order the elements of the sequence

$$\left(\frac{d_1+1}{y_1 y_2 \cdots y_m}, \frac{d_2+1}{y_1 y_2 \cdots y_m}, y_1, y_2, \ldots, y_m, \underbrace{1, 1, \ldots, 1}_{n-m-2 \text{ times}}\right)$$

from the greatest to the least and get the sequence $(a_1, a_2, \ldots, a_n)$, where $a_1 \geq a_2 \geq \cdots \geq a_n$ are natural numbers. We use the following equality

$$\frac{d_1+1}{y_1 y_2 \cdots y_m} + \frac{d_2+1}{y_1 y_2 \cdots y_m} + y_1 + y_2 + \cdots + y_m + \underbrace{1 + 1 + \cdots + 1}_{n-m-2 \text{ times}}$$
$$= \frac{d_1+1}{y_1 y_2 \cdots y_m} \cdot \frac{d_2+1}{y_1 y_2 \cdots y_m} \cdot y_1 \cdot y_2 \cdots y_m \cdot \underbrace{1 \cdot 1 \cdots 1}_{n-m-2 \text{ times}},$$

and so the equation above can be rewritten as

$$a_1 + a_2 + \cdots + a_n = a_1 \cdot a_2 \cdots a_n.$$

It is easy to see that from $n > y_1 \geq 3$ or $n > y_1 \geq y_2 \geq 2$, it follows that

$$(a_1, a_2, \ldots, a_n) \neq (n, 2, \underbrace{1, 1, \ldots, 1}_{n-2 \text{ times}}).$$

Since $(a_1, a_2, \ldots, a_n)$ and $(n, 2, \underbrace{1, 1, \ldots, 1}_{n-2 \text{ times}})$ are different solutions of (1), we get $N(n) > 1$. □

Theorem 4.2. *Let $n > y_1 \geq y_2 \geq \cdots \geq y_m \geq 2$ be natural numbers such that $m \geq 2$ or $m = 1$, and $y_1 \geq 3$. If $N(n) = 1$, then*

$$y_1 y_2 \cdots y_m n + (y_1 + y_2 + \cdots + y_m - m - 2) y_1 y_2 \cdots y_m + 1$$

has no divisors congruent to -1 modulo $y_1 y_2 \cdots y_m$.

Proof. If there exist natural numbers d_1, d_2 such that

$$d_1 d_2 = y_1 y_2 \cdots y_m n + (y_1 + y_2 + \cdots + y_m - m - 2) y_1 y_2 \cdots y_m + 1$$

and $d_1 \equiv -1 \pmod{y_1 y_2 \cdots y_m}$, then by Lemma 4.1, we get $N(n) > 1$.

We obtain a contradiction with assumption that $N(n) = 1$.
Hence, $y_1 y_2 \cdots y_m n + (y_1 + y_2 + \cdots + y_m - m - 2) y_1 y_2 \cdots y_m + 1$ has no divisors congruent to -1 modulo $y_1 y_2 \cdots y_m$. □

Using Theorem 4.2, we get a following structural theorem, which shows that exceptional values are rare.

Theorem 4.3. *If $N(n) = 1$ and $n > 8$, then all of the following conditions hold:*

1) $n - 1$ *is a Sophie Germain prime number,*
2) *all divisors of* $3n + 1$ *are congruent to* 1 *modulo* 3*,*
3) *all divisors of* $4n + 1$ *are congruent to* 1 *modulo* 4*,*
4) *all divisors of* $4n + 5$ *are congruent to* 1 *modulo* 4*,*
5) *all divisors of* $6n + 7$ *are congruent to* 1 *modulo* 6*,*
6) $8n + 9$ *has no divisors congruent to* 7 *modulo* 8*,*
7) $8n + 17$ *has no divisors congruent to* 7 *modulo* 8*,*
8) $8n + 41$ *has no divisors congruent to* 7 *modulo* 8*,*
9) $10n + 31$ *has no divisors congruent to* 9 *modulo* 10*,*
10) $12n + 25$ *has no divisors congruent to* 11 *modulo* 12*,*
11) $12n + 37$ *has no divisors congruent to* 11 *modulo* 12*,*
12) $12n + 49$ *has no divisors congruent to* 11 *modulo* 12*,*
13) $27n + 109$ *has no divisors congruent to* 26 *modulo* 27*,*
14) $30n + 151$ *has no divisors congruent to* 29 *modulo* 30*.*

Proof. For completeness of the proof, we prove here that $n - 1$ is a Sophie Germain prime number, for the other proofs see also [**7**]. We prove this by contradiction. Being composite, $n - 1$ has a divisor d such that $1 < d \leq \sqrt{n-1}$. Hence $2 < d + 1 \leq \frac{n-1}{d} + 1 < n$ and $d | n - 1$. Thus,

$$(n, 2, \underbrace{1, 1, \ldots, 1}_{n-2 \text{ times}}) \neq (\tfrac{n-1}{d} + 1, d + 1, \underbrace{1, 1, \ldots, 1}_{n-2 \text{ times}}).$$

We therefore conclude that $(\frac{n-1}{d} + 1, d + 1, \underbrace{1, 1, \ldots, 1}_{n-2 \text{ times}}) \in S^*(n)$, (i.e., it is not a typical solution). Therefore, $N(n) \geq 2$, a contradiction. The given condition $N(n) = 1$ is therefore satisfied only by the numbers n such that $n - 1$ is a prime number.

Similarly, if we assume that $2n - 1$ is a composite integer, then $2n - 1$ has a divisor d such that $1 < d \leq \sqrt{2n-1}$. Then d and $\frac{2n-1}{d}$ are both odd integers. Hence $\frac{1}{2}(d + 1)$ and $\frac{1}{2}(\frac{2n-1}{d} + 1)$ are integers such that $2 \leq \frac{1}{2}(d + 1) \leq \frac{1}{2}(\frac{2n-1}{d} + 1) < n$. Finally, we verify that

$$\big(\tfrac{1}{2}(\tfrac{2n-1}{d} + 1), \tfrac{1}{2}(d + 1), 2, \underbrace{1, 1, \ldots, 1}_{n-3 \text{ times}}\big) \in S^*(n).$$

Hence, $N(n) \geq 2$, which is a contradiction. Therefore, $2n - 1$ is a prime number.

1) We have shown that $n - 1$ and $2(n - 1) + 1$ are prime numbers, thus $n - 1$ is a Sophie Germain prime number.

2) In order to prove the second condition, we put $m = 1$, $y_1 = 3$ in Theorem 4.2. The assumptions of that theorem are satisfied. Hence, $3n + 1$ has no divisors congruent to -1 modulo 3. Thus, all divisors of $3n + 1$ are congruent to 1 modulo 3.

By Theorem 4.2, with:

3) $m = 2$, $y_1 = 2, y_2 = 2$, we conclude that $4n + 1$ has no divisors congruent to -1 modulo 4. Thus, all divisors of $4n + 1$ are congruent to 1 modulo 4.
4) $m = 1$, $y_1 = 4$, we conclude that $4n + 5$ has no divisors congruent to -1 modulo 4. Thus, all divisors of $4n + 5$ are congruent to 1 modulo 4.
5) $m = 2$, $y_1 = 3, y_2 = 2$, we conclude that $6n + 7$ has no divisors congruent to -1 modulo 6. Thus, all divisors of $6n + 7$ are congruent to 1 modulo 6.
6) $m = 3$, $y_1 = 2, y_2 = 2, y_3 = 2$, we conclude that $8n + 9$ has no divisors congruent to -1 modulo 8.
7) $m = 2$, $y_1 = 4, y_2 = 2$, we conclude that $8n + 17$ has no divisors congruent to -1 modulo 8.
8) $m = 1$, $y_1 = 8$, we conclude that $8n + 41$ has no divisors congruent to -1 modulo 8.
9) $m = 2$, $y_1 = 5, y_2 = 2$, we conclude that $10n + 31$ has no divisors congruent to -1 modulo 10.
10) $m = 3$, $y_1 = 3, y_2 = 2, y_3 = 2$, we conclude that $12n + 25$ has no divisors congruent to -1 modulo 12.
11) $m = 2$, $y_1 = 4, y_2 = 3$, we conclude that $12n + 37$ has no divisors congruent to -1 modulo 12.
12) $m = 2$, $y_1 = 6, y_2 = 2$, we conclude that $12n + 49$ has no divisors congruent to -1 modulo 12.
13) $m = 3$, $y_1 = 3, y_2 = 3, y_3 = 3$, we conclude that $27n + 109$ has no divisors congruent to -1 modulo 27.
14) $m = 3$, $y_1 = 5, y_2 = 3, y_3 = 2$, we conclude that $30n + 151$ has no divisors congruent to -1 modulo 30.

The proof is completed. □

Remark. If $8 < n < 10883$, then the conditions of Theorem 4.3 are also sufficient for $N(n) = 1$.

Kurlandchik and Nowicki [**4**] proved the following: if $n > 100$ and $N(n) = 1$, then the number n is congruent to $0, 24, 30, 84, 90, 114, 150$, or 174 modulo 210. This will be improved as follows.

Theorem 4.4. *If* $N(n) = 1$, $n > 8$, *then* n *is congruent to* $0, 24, 84, 90, 114, 150$ *or* 174 *modulo* 210.

Proof. If $n > 8$ is of the form either $2k+1$ or $3k+1$ or $5k+1$ or $7k+1$, then $n-1$ is not a prime number. This contradicts 1) in Theorem 4.3. If $n > 8$ is of the form either $3k+2$ or $5k+3$ or $7k+4$, then $2n-1$ is not a prime number. This contradicts 2) in Theorem 4.3. If n is of the form $30k + 12$, then $8n + 9 = 15(16k + 7)$. This contradicts 6) in Theorem 4.3. If n is of the form $7k + 5$, then $8n + 9 = 7(8k + 7)$.

This contradicts 6) in Theorem 4.3. Thus the number n has one of the forms $210k, 210k+24, 210k+30, 210k+84, 210k+90, 210k+114, 210k+150$, or $210k+174$. If n is of the form $210k+30$, then $12n+25=35(72k+11)$. This contradicts 10) in Theorem 4.3. The proof of the theorem is thus complete. □

5. The sets E_2, E_3 of exceptional values

We define the following sets of exceptional values of the equal-sum-and-product problem as

$$E_2 = \{n : N(n) = 2, n \geq 2\}, \qquad E_3 = \{n : N(n) = 3, n \geq 2\}.$$

Using a computer program (see also A033178 in [**8**]), we have shown that $|E_2 \cap [2, 10\,000]| = 49$. Namely, if $2 \leq n \leq 10^4$ and $N(n) = 2$, then

$$n \in \{7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 22, 30, 34, 36, 42, 44, 48, 54, 60, 66, 80, 84, 90, 112, 126, 142, 192, 210, 234, 252, 258, 330, 350, 354, 440, 594, 654, 714, 720, 780, 966, 1102, 2400, 2820, 4350, 4354, 5274, 6174, 6324\}.$$

We conjecture that the set $E_2 \cap [2, 10\,000]$ is in fact E_2.
It can be shown also that $|E_3 \cap [2, 10\,000]| = 74$.

Theorem 5.1. *If $N(n) = 2$, then one of the following two conditions is true:*
1. *$n-1$ is a prime and $2n-1 \in \{p, p^2, p^3, pq\}$,*
2. *$2n-1$ is a prime and $n-1 \in \{p, p^2, p^3, pq\}$,*

where p, q denote prime numbers.

Proof. By Corollary 2.2, we have two possibilities: n–1 is a prime and $d(2n-1) \leq 4$, or $2n-1$ is a prime and $d(n-1) \leq 4$. □

Theorem 5.2. *If $N(n) = 3$, one of the following three conditions is true:*
1. *$n-1$ is a prime and $d(2n-1) \leq 6$,*
2. *$d(n-1)$, $d(2n-1) \leq 4$,*
3. *$2n-1$ is a prime and $d(n-1) \leq 6$.*

Proof. This is an immediate consequence of Corollary 2.2. □

Let $E_{\leq k} = \{n : N(n) \leq k, n \geq 2\}$, $k \geq 1$. In particular, $E_{\leq 1} = E_1$, $E_{\leq 2} = E_1 \cup E_2$.

Theorem 5.3. *The set $E_{\leq k}$ has natural density 0, i.e., the ratio $\dfrac{|E_{\leq k} \cap [1, x]|}{x}$ tends to 0 as $x \to \infty$.*

Proof. Let $\Omega(m)$ counts the total number of prime factors of m (without fear of confusion with notation in (6)). An easy computation shows that $\Omega(m) \leq d(m)-1$ for every natural number m. Let $\pi_i(x) = \{m : \Omega(m) = i, 1 \leq m \leq x\}$, i.e., the number of $1 \leq m \leq x$ with i prime factors (not necessarily distinct). By

Corollary 2.2, we have $N(n) \geq \frac{1}{2}d(n-1)$. Thus, if $n \in E_{\leq k}$, then $d(n-1) \leq 2k$ and consequently $\Omega(n-1) \leq 2k-1$. Therefore,

$$|E_{\leq k} \cap [1,x]| \leq \sum_{i=0}^{2k-1} \pi_i(x-1),$$

where $x \geq 2$. Using the sieve of Eratosthenes, one can show that (see [**1**, p. 75])

$$\pi_i(x) \leq \frac{1}{i!} x \frac{(A \log\log x + B)^i}{\log x}$$

for some constants $A, B > 0$. There follows that

$$0 \leq \frac{|E_{\leq k} \cap [1,x]|}{x} \leq \frac{x-1}{x} \sum_{i=0}^{2k-1} \frac{1}{i!} \frac{(A \log\log(x-1) + B)^i}{\log(x-1)}.$$

For a fixed k, the right-hand side tends to 0 as $x \to \infty$. Thus,

$$\lim_{x \to \infty} \frac{|E_{\leq k} \cap [1,x]|}{x} = 0.$$

This completes the proof. □

The above theorem implies, that the set E_1 of exceptional values of the equal--sum-and-product problem has zero natural density. It also implies, that the set $E_k = \{n :\ N(n) = k,\, n \geq 2\}$ has zero natural density for any fixed $k \geq 1$. This observation might suggest that the set $E_k = \{n :\ N(n) = k,\, n \geq 2\}$ is finite for any fixed $k \geq 1$ and the number $N(n) \to \infty$ as an $n \to \infty$, see [**9**] and [**10**, Problem 2].

6. Bounds on not-typical solutions of equation (1).

Kurlandchik and Nowicki showed that if $(x_1, x_2, \ldots, x_n) \in S(n)$, $n \geq 3$, and $x_1 \geq x_2 \geq \cdots \geq x_n$, then $x_2 \cdots x_{n-1} x_n \leq n-1$ (see [**4**, Theorem 4]). We now give an upper bound for the sum of the coordinates of a not-typical solution of equation (1).

Theorem 6.1. *Let* $(n, 2, \underbrace{1, 1, \ldots, 1}_{n-2 \text{ times}}) \neq (x_1, x_2, \ldots, x_n) \in S(n)$ *and* $n \geq 2$. *Let* k *be the number of non-unit elements in* $(x_1, \ldots, x_n)$. *If* $k = 2$, *then*

$$n \geq 5 \quad \text{and} \quad x_1 + x_2 + \cdots + x_{n-1} + x_n \leq \frac{3}{2}(n+1).$$

If $k \geq 3$, *then*

$$n \geq 2^k - k \quad \text{and} \quad x_1 + x_2 + \cdots + x_{n-1} + x_n \leq 2n + 2k - 2^k.$$

Proof. We denote $x_1 = y_1 + 1, x_2 = y_2 + 1, \ldots, x_k = y_k + 1$, where $y_1 \geq y_2 \geq \cdots \geq y_k \geq 1$. We also have $x_{k+1} = x_{k+2} = \cdots = x_n = 1$. We rewrite equation (1) in the form

$$y_1 + y_2 + \cdots + y_k + n = (y_1 + 1)(y_2 + 1) \cdots (y_k + 1).$$

Hence, $n \geq 2^k - k$ and $k \geq 2$.

If $k = 2$, then $y_1 y_2 = n - 1$, $y_1 \geq y_2$. Note that $(y_1, y_2) \neq (n-1, 1)$ since $(n, 2, 1, \ldots, 1) \neq (x_1, x_2, \ldots, x_n)$. Thus $\frac{n-1}{2} \geq y_2 \geq 2$. Hence, by convexity of the function $f(x) = \frac{n-1}{x} + x$, $x > 0$, we have

$$y_1 + y_2 \leq \frac{n-1}{y_2} + y_2 \leq \frac{n-1}{2} + 2 = \frac{1}{2}(n+3).$$

Therefore,

$$x_1 + x_2 + \cdots + x_n \leq \frac{3}{2}(n+1).$$

The equality holds only if $y_1 = \frac{n-1}{2}, y_2 = 2$, that is, only when $(x_1, \ldots, x_n) = (\frac{n+1}{2}, 3, 1, \ldots, 1)$ and $n \geq 5$ is odd.
Let $k \geq 3$, then

$$y_k = \frac{n + y_1 + y_2 + \cdots + y_{k-1} - (y_1+1)(y_2+1)\cdots(y_{k-1}+1)}{(y_1+1)(y_2+1)\cdots(y_{k-1}+1) - 1}.$$

Denote $(y_1+1)(y_2+1)\cdots(y_{k-1}+1) = I_{k-1}$, $y_1 + y_2 + \cdots + y_{k-1} = S_{k-1}$.
We have $I_{k-1} \geq 2^{k-1}$, $S_{k-1} \geq k-1$, $I_{k-1} - S_{k-1} \geq 2^{k-1} - (k-1)$, since $y_1 \geq y_2 \geq \cdots \geq y_{k-1} \geq 1$. Note that

$$(7) \qquad 2^{k-1} - 1 \leq I_{k-1} - 1 = \frac{n + S_{k-1} - I_{k-1}}{y_k} \leq n - (2^{k-1} - (k-1)).$$

Therefore

$$\begin{aligned} y_1 + y_2 + \cdots + y_{k-1} + y_k &= S_{k-1} + \frac{n + S_{k-1} - I_{k-1}}{I_{k-1} - 1} \\ &\leq S_{k-1} + \frac{n + S_{k-1} - I_{k-1}}{I_{k-1} - 1} + \frac{(I_{k-1} - S_{k-1} - (2^{k-1} - (k-1)))I_{k-1}}{I_{k-1} - 1} \\ &= I_{k-1} - (2^{k-1} - (k-1)) + \frac{n - (2^{k-1} - (k-1))}{I_{k-1} - 1} \\ &= k - 2^{k-1} + (I_{k-1} - 1) + \frac{n - (2^{k-1} - (k-1))}{I_{k-1} - 1}. \end{aligned}$$

By convexity of the function $f(x) = x + \frac{n - (2^{k-1} - (k-1))}{x}$ and inequality (7), we get

$$\begin{aligned} &(I_{k-1} - 1) + \frac{n - (2^{k-1} - (k-1))}{I_{k-1} - 1} \\ &\leq \max\left\{2^{k-1} - 1 + \frac{n - (2^{k-1} - (k-1))}{2^{k-1} - 1}, n + k - 2^{k-1}\right\}. \end{aligned}$$

But $2^{k-1} - 1 + \frac{n - (2^{k-1} - (k-1))}{2^{k-1} - 1} \leq n + k - 2^{k-1}$ since $2^k - k \leq n$. Hence,

$$y_1 + y_2 + \cdots + y_{k-1} + y_k \leq n + 2k - 2^k,$$

thus

$$x_1 + x_2 + \cdots + x_{n-1} + x_n \leq 2n + 2k - 2^k. \qquad \square$$

The following corollary holds.

Corollary 6.2. *Let* $n > 5$ *and* $(n, 2, \underbrace{1, 1, \dots, 1}_{n-2 \text{ times}}) \neq (x_1, x_2, \dots, x_n) \in S(n)$. *Then*

$$x_1 + x_2 + \cdots + x_{n-1} + x_n \leq 2n - 2.$$

Remark. The inequality $x_1 + x_2 + \cdots + x_n \leq 2n$ when $(x_1, x_2, \dots, x_n) \in S(n)$ was among the problems of the Polish Mathematical Olympiad 1990.

Kurlandchik and Nowicki showed that $k \leq 1 + \lfloor \log_2(n) \rfloor$. The equality holds, for example, when $n = 2^s - s$, $s \geq 2$, and

$$(x_1, x_2, \dots, x_n) = (\underbrace{2, 2, \dots, 2}_{s \text{ times}}, \underbrace{1, 1, \dots, 1}_{n-s \text{ times}}) \qquad \text{(see [4, Theorem 7])}.$$

7. The system of equations of the equal-sum-product problem

We also consider the generalization of equal-sum-and-product problem (1) to the cyclic system of equations (we give examples at the end of this paper).

Theorem 7.1. *Let* $\succeq$ *denote the lexicographic order in* $\mathbb{N}^n$. *For any natural numbers* $n \geq 2$, $k \geq 1$. *the system of diophantine equations*

$$(8) \qquad \begin{cases} x_{1,1} + x_{1,2} + \cdots + x_{1,n} = x_{2,1} \cdot x_{2,2} \dots x_{2,n} \\ x_{2,1} + x_{2,2} + \cdots + x_{2,n} = x_{3,1} \cdot x_{3,2} \dots x_{3,n} \\ \dots \\ x_{k,1} + x_{k,2} + \cdots + x_{k,n} = x_{1,1} \cdot x_{1,2} \dots x_{1,n}, \end{cases}$$

for all $i \in \{1, 2, \dots, k\}$, $x_{i,1} \geq x_{i,2} \geq \cdots \geq x_{i,n}$,

$$(x_{1,1}, x_{1,2}, \dots, x_{1,n}) \succeq (x_{2,1}, x_{2,2}, \dots, x_{2,n}) \succeq \cdots \succeq (x_{k,1}, x_{k,2}, \dots, x_{k,n}),$$

has only finite number $N(n, k)$ *of solutions in positive integers* $x_{i,j}$.

Remark. Note that $N(n, k) \geq N(n, 1) = N(n)$.

In our proof of Theorem 7.1, we use the following lemma.

Lemma 7.2. *Let* n *be a natural number. If* $x_1, x_2, \dots, x_n$ *are real numbers, then the following equality holds*

$$(9) \qquad \sum_{s=1}^{n-1} \Big(\Big(\prod_{i=1}^{s} x_i - 1 \Big) (x_{s+1} - 1) \Big) = \prod_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i + n - 1.$$

Proof. We use induction on n. If $n = 1$, then the left-hand side sum in (9) is empty and the right-hand side of (9) is equal 0, hence equation (9) holds.
If $n = 2$, then equation (9) has the form

$$(x_1 - 1)(x_2 - 1) = x_1 x_2 - (x_1 + x_2) + 1,$$

hence it is true.

We now assume that $n \geq 3$ and statement (9) holds for $n-1$, i.e.,

$$\sum_{s=1}^{n-2}\Big(\Big(\prod_{i=1}^{s} x_i - 1\Big)(x_{s+1}-1)\Big) = \prod_{i=1}^{n-1} x_i - \sum_{i=1}^{n-1} x_i + n - 2.$$

Thus,

$$\begin{aligned}
&\sum_{s=1}^{n-1}\Big(\Big(\prod_{i=1}^{s} x_i - 1\Big)(x_{s+1}-1)\Big)\\
&= \Big(\prod_{i=1}^{n-1} x_i - 1\Big)(x_n - 1) + \sum_{s=1}^{n-2}\Big(\Big(\prod_{i=1}^{s} x_i - 1\Big)(x_{s+1}-1)\Big)\\
&= \Big(\prod_{i=1}^{n-1} x_i - 1\Big)(x_n - 1) + \prod_{i=1}^{n-1} x_i - \sum_{i=1}^{n-1} x_i + n - 2\\
&= \prod_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i + n - 1
\end{aligned}$$

which was to be shown. □

Now, we prove Theorem 7.1.

Proof. If we add the equations sidewise in (8), we get

$$\sum_{i=1}^{k}\sum_{j=1}^{n} x_{i,j} = \sum_{i=1}^{k}\prod_{j=1}^{n} x_{i,j}.$$

This gives

$$\sum_{i=1}^{k}\Big(\prod_{j=1}^{n} x_{i,j} - \sum_{j=1}^{n} x_{i,j} + n - 1\Big) = k(n-1).$$

Moreover, by (9),

$$\sum_{i=1}^{k}\sum_{s=1}^{n-1}\Big(\Big(\prod_{j=1}^{s} x_{i,j} - 1\Big)(x_{i,s+1}-1)\Big) = k(n-1). \tag{10}$$

For given n and k, the number of solutions of equation (10) in positive integers is bounded above. Therefore, system of equations (8) has only finite number of solutions in positive integers $x_{i,j}$. □

8. Examples

In this section, for the convenience of the reader, we present some systems of equations of the equal-sum-product problem, including their solutions. System (11) was among the problems of the XLIX Polish Mathematical Olympiad. The following statements hold:

1) If $x_{1,1} \geq x_{1,2}$, $x_{2,1} \geq x_{2,2}$ and $x_{1,1} \geq x_{2,1}$ are natural numbers such that

$$\begin{cases} x_{1,1} + x_{1,2} = x_{2,1} \cdot x_{2,2} \\ x_{2,1} + x_{2,2} = x_{1,1} \cdot x_{1,2}, \end{cases} \text{then} \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \in \left\{ \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 5 & 1 \\ 3 & 2 \end{pmatrix} \right\}.$$

2) If $x_{1,1} \geq x_{1,2}$, $x_{2,1} \geq x_{2,2}$, $x_{3,1} \geq x_{3,2}$, and $x_{1,1} \geq x_{2,1} \geq x_{3,1}$ are natural numbers such that

$$\begin{cases} x_{1,1} + x_{1,2} = x_{2,1} \cdot x_{2,2} \\ x_{2,1} + x_{2,2} = x_{3,1} \cdot x_{3,2} \\ x_{3,1} + x_{3,2} = x_{1,1} \cdot x_{1,2}, \end{cases} \text{then} \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \\ x_{3,1} & x_{3,2} \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

3) If $x_{1,1} \geq x_{1,2} \geq x_{1,3}$, $x_{2,1} \geq x_{2,2} \geq x_{2,3}$, and $x_{1,1} \geq x_{2,1}$ are natural numbers such that

$$\begin{cases} x_{1,1} + x_{1,2} + x_{1,3} = x_{2,1} \cdot x_{2,2} \cdot x_{2,3} \\ x_{2,1} + x_{2,2} + x_{2,3} = x_{1,1} \cdot x_{1,2} \cdot x_{1,3}, \end{cases} \tag{11}$$

then

$$\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \end{pmatrix} \in \left\{ \begin{pmatrix} 8 & 1 & 1 \\ 5 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 7 & 1 & 1 \\ 3 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 6 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix} \right\}.$$

4) If $x_{1,1} \geq x_{1,2} \geq x_{1,3}$, $x_{2,1} \geq x_{2,2} \geq x_{2,3}$, $x_{3,1} \geq x_{3,2} \geq x_{3,3}$ and $x_{1,1} \geq x_{2,1} \geq x_{3,1}$ are natural numbers such that

$$\begin{cases} x_{1,1} + x_{1,2} + x_{1,3} = x_{2,1} \cdot x_{2,2} \cdot x_{2,3} \\ x_{2,1} + x_{2,2} + x_{2,3} = x_{3,1} \cdot x_{3,2} \cdot x_{3,3} \\ x_{3,1} + x_{3,2} + x_{3,3} = x_{1,1} \cdot x_{1,2} \cdot x_{1,3}, \end{cases} \text{then} \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix}.$$

5) If $x_{1,1} \geq x_{1,2} \geq x_{1,3} \geq x_{1,4}$, $x_{2,1} \geq x_{2,2} \geq x_{2,3} \geq x_{2,4}$, and $x_{1,1} \geq x_{2,1}$ are natural numbers such that

$$\begin{cases} x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} = x_{2,1} \cdot x_{2,2} \cdot x_{2,3} \cdot x_{2,4} \\ x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} = x_{1,1} \cdot x_{1,2} \cdot x_{1,3} \cdot x_{1,4}, \end{cases}$$

then

$$\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \end{pmatrix} \in \left\{ \begin{pmatrix} 11 & 1 & 1 & 1 \\ 7 & 2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 9 & 1 & 1 & 1 \\ 4 & 3 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 2 & 1 & 1 \\ 4 & 2 & 1 & 1 \end{pmatrix} \right\}.$$

6) If $x_{1,1} \geq x_{1,2} \geq x_{1,3} \geq x_{1,4} \geq x_{1,5}$, $x_{2,1} \geq x_{2,2} \geq x_{2,3} \geq x_{2,4} \geq x_{2,5}$, and $x_{1,1} \geq x_{2,1}$ are natural numbers such that

$$\begin{cases} x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} + x_{1,5} = x_{2,1} \cdot x_{2,2} \cdot x_{2,3} \cdot x_{2,4} \cdot x_{2,5} \\ x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} + x_{2,5} = x_{1,1} \cdot x_{1,2} \cdot x_{1,3} \cdot x_{1,4} \cdot x_{1,5}, \end{cases}$$

then

$$\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} \end{pmatrix} \in \left\{ \begin{pmatrix} 14 & 1 & 1 & 1 & 1 \\ 9 & 2 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 11 & 1 & 1 & 1 & 1 \\ 5 & 3 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 2 & 1 & 1 & 1 \\ 5 & 2 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 3 & 1 & 1 & 1 \\ 3 & 3 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 \end{pmatrix} \right\}.$$

9. Proof of Theorem 2.1

We have the following pairwise disjoint families of pairwise different solutions of equation (1) in the set $\mathbb{N}^n$:

1) $A_1(n) = \{(\frac{n-1}{d}+1, d+1, \underbrace{1,1,\dots,1}_{n-2 \text{ times}}) : \ d|n-1,\, d \le \sqrt{n-1},\, d \in \mathbb{N}\}$.

Note that $A_1(n) \subset S(n)$ since $\frac{n-1}{d}+1 \ge d+1 \ge 2$. The cardinality of $A_1(n)$ is equal to

$$|\{d : d|n-1, 1 \le d \le \sqrt{n-1}\}| = \left\lfloor \tfrac{d(n-1)+1}{2} \right\rfloor .$$

2) $A_2(n) = \{(\frac{1}{d}(n-1+\frac{1}{2}(d+1)), \frac{1}{2}(d+1), 2, \underbrace{1,1,\dots,1}_{n-3 \text{ times}}) :$

$$d|2n-1,\, 3 \le d \le \sqrt{2n-1},\, d \in \mathbb{N}\}.$$

If $d|2n-1$, then d is an odd number such that $d|n-1+\frac{1}{2}(d+1)$. From the assumption $3 \le d \le \sqrt{2n-1}$, we also have $\frac{1}{d}(n-1+\frac{1}{2}(d+1)) \ge \frac{1}{2}(d+1) \ge 2$. Therefore, $A_2(n) \subset S^*(n)$, and $A_1(n) \cap A_2(n) = \emptyset$. The cardinality of $A_2(n)$ is equal to

$$|\{d : \ d|2n-1, 3 \le d \le \sqrt{2n-1}\}| = \left\lfloor \tfrac{d(2n-1)+1}{2} \right\rfloor - 1.$$

3) $A_3(n) = \{(\frac{1}{d}(n+\frac{1}{3}(d+1)), \frac{1}{3}(d+1), 3, \underbrace{1,1,\dots,1}_{n-3 \text{ times}}) :$

$$d|3n+1,\, d \equiv 2 \pmod 3,\, 8 \le d \le \sqrt{3n+1},\, d \in \mathbb{N}\}.$$

If $d|3n+1$, $d \equiv 2 \pmod 3$, then $3|d+1$ and $d|n+\frac{1}{3}(d+1)$. From the assumption $8 \le d \le \sqrt{3n+1}$, we also have $\frac{1}{d}(n+\frac{1}{3}(d+1)) \ge \frac{1}{3}(d+1) \ge 3$. Therefore, $A_3(n) \subset S^*(n)$, and the sets $A_1(n), A_2(n)$, and $A_3(n)$ must be pairwise disjoint. The cardinality of $A_3(n)$ is equal to

$$\begin{aligned}
&|\{d : \ d|3n+1,\, d \equiv 2 \pmod 3,\, 8 \le d \le \sqrt{3n+1}\}| \\
&= \left\lfloor \tfrac{d_2(3n+1)+1}{2} \right\rfloor - \delta(2|3n+1,\, 2 \le \sqrt{3n+1}) - \delta(5|3n+1,\, 5 \le \sqrt{3n+1}) \\
&= \left\lfloor \tfrac{d_2(3n+1)+1}{2} \right\rfloor - \delta(2|n+1) - \delta(5|n+2,\, n \ge 8).
\end{aligned}$$

4) $A_{2,2}(n) = \big\{(\frac{1}{d}(n+\frac{1}{4}(d+1)), \frac{1}{4}(d+1), 2, 2, \underbrace{1,1,\dots,1}_{n-4 \text{ times}}) :$

$$d|4n+1,\, d \equiv 3 \pmod 4,\, 7 \le d \le \sqrt{4n+1},\, d \in \mathbb{N}\big\}.$$

If $d|4n+1$, $d \equiv 3 \pmod 4$, then $4|d+1$ and $d|n+\frac{1}{4}(d+1)$. From the assumption $7 \le d \le \sqrt{4n+1}$, we also have $\frac{1}{d}(n+\frac{1}{4}(d+1)) \ge \frac{1}{4}(d+1) \ge 2$. Therefore, $A_{2,2}(n) \subset S^*(n)$, and the sets $A_1(n), A_2(n), A_3(n)$, and $A_{2,2}(n)$ must be pairwise disjoint. The cardinality of $A_{2,2}(n)$ is equal to

$$\begin{aligned}
&|\{d : \ d|4n+1,\, d \equiv 3 \pmod 4,\, 7 \le d \le \sqrt{4n+1}\}| \\
&= \left\lfloor \tfrac{d_3(4n+1)+1}{2} \right\rfloor - \delta(3|4n+1,\, 3 \le \sqrt{4n+1}) = \left\lfloor \tfrac{d_3(4n+1)+1}{2} \right\rfloor - \delta(3|n+1)
\end{aligned}$$

since $n \geq 2$.

5) $A_4(n) = \{(\frac{1}{d}(n+1+\frac{1}{4}(d+1)), \frac{1}{4}(d+1), 4, \underbrace{1, 1, \ldots, 1}_{n-3 \text{ times}}) :$

$$d|4n+5,\ d \equiv 3 \pmod 4,\ 15 \leq d \leq \sqrt{4n+5},\ d \in \mathbb{N}\}.$$

If $d|4n+5$, $d \equiv 3 \pmod 4$, then $4|d+1$ and $d|n+1+\frac{1}{4}(d+1)$. From the assumption $15 \leq d \leq \sqrt{4n+5}$, we also have $\frac{1}{d}(n+1+\frac{1}{4}(d+1)) \geq \frac{1}{4}(d+1) \geq 4$. Therefore, $A_4(n) \subset S^*(n)$,and the sets $A_1(n), A_2(n), A_3(n), A_{2,2}(n)$, and A_4 must be pairwise disjoint. The cardinality of $A_4(n)$ is equal to

$$\begin{aligned}
&|\{d :\ d|4n+5,\ d \equiv 3 \pmod 4,\ 15 \leq d \leq \sqrt{4n+5}\}| \\
&= \left\lfloor \tfrac{d_3(4n+5)+1}{2} \right\rfloor - \delta(3|4n+5,\ 3 \leq \sqrt{4n+5}) \\
&\quad - \delta(7|4n+5,\ 7 \leq \sqrt{4n+5}) - \delta(11|4n+5,\ 11 \leq \sqrt{4n+5}) \\
&= \left\lfloor \tfrac{d_3(4n+5)+1}{2} \right\rfloor - \delta(3|n+2) - \delta(7|n+3,\ n \geq 11) - \delta(11|n+4,\ n \geq 29).
\end{aligned}$$

From 1), 2), we obtain $N(n) \geq \left\lfloor \frac{d(n-1)+1}{2} \right\rfloor + \left\lfloor \frac{d(2n-1)+1}{2} \right\rfloor - 1$. Hence, the first inequality in Theorem 2.1 holds for $N(n)$. From 1), 2), 3), 4), 5), we obtain (3). Hence, the second inequality in Theorem 2.1 holds for $N(n)$.

Acknowledgment. I conclude by expressing my thanks to Piotr Miska for his kind assistance in the preparation of Theorem 5.3.

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