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A NOTE ON DG-GORENSTEIN INJECTIVE COVERS

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Abstract. We consider a ring R such that the class of Gorenstein injective modules is closed under direct limits. We prove that the class of dg-Gorenstein injective complexes is covering in $Ch(R)$ if and only if every complex of Gorenstein injective modules is dg-Gorenstein injective. In particular, when R is commutative noetherian with a dualizing complex, we obtain the following result: the class of dg-Gorenstein injective complexes is covering if and only if R is Gorenstein.

1. Introduction

The process of extending homological algebra from modules to complexes started with the last chapter of Cartan and Eilenberg's book "Homological Algebra". For a while, a major obstacle in constructing a satisfactory theory of homological dimensions of complexes was the fact that there was no sufficiently general result for the existence of resolutions of unbounded complexes. This changed with Spaltenstein's work ([**21**]) and with Avramov and Halperin's work ([**1**]). They introduced the DG-resolutions (in the more general context of DG-modules over DG-rings). Avramov and Foxby used these resolutions ([**2**]) to define projective, injective, and flat dimensions for arbitrary complexes.

Enochs, Jenda, and Xu also considered the classes of dg-projective and dg-injective complexes in their work on orthogonality in the category of complexes ([**8**]). They showed the existence of various covers and envelopes with respect to the classes of dg-injective, dg-projective, and exact complexes.

More recently, Gillespie introduced ([**14**]) some generalizations of the dg-injective and dg-projective complexes. He considers a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in the category of R-modules, and constructs four classes of complexes in $\mathrm{Ch}(R)$ that are associated with the cotorsion pair $(\mathcal{A}, \mathcal{B})$: the class $\widetilde{\mathcal{A}}$ of all acyclic $\mathcal{A}$-complexes, the class $\widetilde{\mathcal{B}}$ of all acyclic $\mathcal{B}$-complexes, the class of all dg-$\mathcal{A}$ complexes, $\mathrm{dg}(\mathcal{A})$, and the class of all dg-$\mathcal{B}$ complexes, $\mathrm{dg}(\mathcal{B})$. We refer the reader to the preliminaries section for the undefined notions.

Yang and Liu showed in [**23**, Theorem 3.5] that when $(\mathcal{A}, \mathcal{B})$ is a complete hereditary cotorsion pair in R-Mod, the pairs $(\mathrm{dg}(\mathcal{A}), \widetilde{\mathcal{B}})$ and $(\widetilde{\mathcal{A}}, \mathrm{dg}(\mathcal{B}))$ are complete

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and hereditary cotorsion pairs in the category of complexes Ch(R). Moreover, by Gillespie [**14**], we have that $\widetilde{\mathcal{A}} = \mathrm{dg}(\mathcal{A}) \bigcap \mathcal{E}$ and $\widetilde{\mathcal{B}} = \mathrm{dg}(\mathcal{B}) \bigcap \mathcal{E}$ (where $\mathcal{E}$ is the class of all acyclic complexes). For example, from the (complete and hereditary) cotorsion pairs $(Proj, R\text{-Mod})$ and $(R\text{-Mod}, Inj)$, one obtains the standard (complete and hereditary) cotorsion pairs $(\mathcal{E}, \mathrm{dg}(Inj))$ and $(\mathrm{dg}(Proj), \mathcal{E})$.

Very recently, Saroch and Stovicek proved ([**20**]) that over any ring R, the pair $({}^{\perp}\mathcal{GI}, \mathcal{GI})$ is a complete hereditary cotorsion pair. Therefore, $(\mathrm{dg}({}^{\perp}\mathcal{GI}), \widetilde{\mathcal{GI}})$ and $(\widetilde{{}^{\perp}\mathcal{GI}}, \mathrm{dg}(\mathcal{GI}))$ are complete and hereditary cotorsion pairs in Ch(R).

The dg-injective complexes correspond to the injective modules in the sense that the injective dimension of complexes is defined by means of dg-injective resolutions. Since the class of injective modules is covering over noetherian rings, it seems natural to expect that the class of dg-injective complexes is covering in Ch(R) whenever R is a noetherian ring. But this is not the case; in [**17**], we proved that the dg-injectives form a covering class if and only if the ring R is regular (that is, R is Noetherian and every finitely generated R-module has finite projective dimension.)

We consider here the question of the existence of dg-Gorenstein injective covers. It is known that if the class of Gorenstein injective modules, $\mathcal{GI}$, is closed under direct limits, then it is covering ([**11**, Proposition 3]). So it is a natural question to consider whether or not the class of dg-Gorenstein injective complexes is also covering in this case. We prove that this is not the case. More precisely, we show (Lemma 3.4) that if the class of dg-Gorenstein injective complexes is covering, then every complex of Gorenstein injective modules is dg-Gorenstein injective (that is, the class of dg-Gorenstein injective complexes coincides with the class of Gorenstein injective complexes). Using this, we show (Theorem 3.5) that when the class of Gorenstein injective modules is closed under direct limits, the following statements are equivalent:

(1) the class of dg-Gorenstein injective complexes, $\mathrm{dg}(\mathcal{GI})$, is covering.
(2) the class of dg-Gorenstein injective complexes, $\mathrm{dg}(\mathcal{GI})$, coincides with that of complexes of Gorenstein injective modules, $\mathrm{dw}(\mathcal{GI})$ (i.e., with the class of Gorenstein injective complexes).

It is known that the class of Gorenstein injective modules is closed under direct limits over a commutative noetherian ring with a dualizing complex. Using Theorem 3.5, we prove (Theorem 3.6) that over such a ring, the following statements are equivalent:

(1) the class of dg-Gorenstein injective complexes, $\mathrm{dg}(\mathcal{GI})$, is covering.
(2) R is a Gorenstein ring.

2. Preliminaries

Throughout the paper, R denotes an associative ring with unity. Unless otherwise specified, by R-module we mean a left R-module. We use Inj to denote the class of injective modules. We use $\mathcal{GI}$ to denote the class of Gorenstein injective modules.

Given a class of modules $\mathcal{C}$, we denote by $\mathcal{C}^{\perp}$ its right orthogonal class, i.e., the class of modules X such that $\mathrm{Ext}^1(C, X) = 0$ for any $C \in \mathcal{C}$. The left orthogonal class of $\mathcal{C}$ is defined dually. We recall that a pair of classes of R-modules $(\mathcal{C}, \mathcal{L})$ is a *cotorsion pair* if $\mathcal{C}^{\perp} = \mathcal{L}$ and ${}^{\perp}\mathcal{L} = \mathcal{C}$. A cotorsion pair is *complete* if for any ${}_RM$, there are exact sequences $0 \to L \to C \to M \to 0$ and $0 \to M \to L' \to C' \to 0$, respectively, with $C, C' \in \mathcal{C}$ and $L, L' \in \mathcal{L}$. We also recall that a cotorsion pair is said to be *hereditary* if $\mathrm{Ext}^i(C, L) = 0$ for all $i \geq 1$, all $C \in \mathcal{C}$, all $L \in \mathcal{L}$. It is known that this is equivalent with the class $\mathcal{C}$ being closed under kernels of epimorphisms, and it is also equivalent with the condition that $\mathcal{L}$ is closed under cokernels of monomorphisms.

As already mentioned, there are four classes of complexes in $\mathrm{Ch}(R)$ that are associated with a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in R-Mod:

1. An acyclic complex X is an $\mathcal{A}$-complex if $Z_j(X) \in \mathcal{A}$ for all integers j. We denote by $\widetilde{\mathcal{A}}$ the class of all acyclic $\mathcal{A}$-complexes.
2. An acyclic complex U is a $\mathcal{B}$-complex if $Z_j(X) \in \mathcal{B}$ for all integers j. We denote by $\widetilde{\mathcal{B}}$ the class of all acyclic $\mathcal{B}$-complexes.
3. A complex Y is a dg-$\mathcal{A}$ complex if each $Y_n \in \mathcal{A}$ and each map $Y \to U$ is null-homotopic for each complex $U \in \widetilde{\mathcal{B}}$. We denote by $\mathrm{dg}(\mathcal{A})$ the class of all dg-$\mathcal{A}$ complexes.
4. A complex W is a dg-$\mathcal{B}$ complex if each $W_n \in \mathcal{B}$ and each map $V \to W$ is null-homotopic for each complex $V \in \widetilde{\mathcal{A}}$. We denote by $\mathrm{dg}(\mathcal{B})$, the class of all dg-$\mathcal{B}$ complexes.

The Gorenstein injective modules were introduced by Enochs and Jenda ([**9**]). They are the cycles of the exact complexes of injective modules that also remain exact when applying the functor $\mathrm{Hom}(A, -)$ for any injective module A. We use $\mathcal{GI}$ to denote the class of Gorenstein injective modules. Very recently, it was proved ([**20**]) that $({}^{\perp}\mathcal{GI}, \mathcal{GI})$ is a complete hereditary cotorsion pair over any ring R. So in particular, for this cotorsion pair, we obtain the following definition.

Definition 2.1. A complex I is dg-Gorenstein injective if each I_n is a Gorenstein injective left R-module and if every morphism $f \colon E \to I$ from any complex E in $\widetilde{{}^{\perp}\mathcal{GI}}$ to I, is homotopic to zero.

By [**23**], $(\mathrm{dg}({}^{\perp}\mathcal{GI}), \widetilde{\mathcal{GI}})$ and $(\widetilde{{}^{\perp}\mathcal{GI}}, (\mathrm{dg}\,\mathcal{GI}))$ are both complete and hereditary cotorsion pairs in $\mathrm{Ch}(R)$.

We also recall that a dw-Gorenstein injective complex ("degree-wise" Gorenstein injective) is a complex of Gorenstein injective modules. The Gorenstein injective complexes are defined in a similar manner with the Gorenstein injective modules: they are the cycles of exact complexes of injective complexes that stay exact when applying the functor $\mathrm{Hom}(I, -)$ with I any injective complex. It is known ([**19**]) that these are precisely the complexes of Gorenstein injective modules, $\mathrm{dw}(\mathcal{GI})$.

3. RESULTS

We start by proving the following lemma.

Lemma 3.1. *Every left bounded complex of Gorenstein injective modules is a dg-Gorenstein injective complex.*

Proof. Let $G = 0 \to G_0 \to G_{-1} \to G_{-2} \to \dots$ be a left bounded complex of Gorenstein injective modules. We show that if $A \in {}^{\perp}\widetilde{\mathcal{GI}}$, then any morphism $f\colon A \to G$ is homotopic to zero.

Let $A \in {}^{\perp}\widetilde{\mathcal{GI}}$, $A = \dots \to A_1 \to A_0 \to A_{-1} \to \dots$, and consider a map $f\colon A \to X$.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A_1 & \xrightarrow{d_1} & A_0 & \xrightarrow{d_0} & A_{-1} & \xrightarrow{d_{-1}} & A_{-2} & \longrightarrow & \cdots \\
 & & & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_{-1}} & & \downarrow{\scriptstyle f_{-2}} & & \\
 & & 0 & \longrightarrow & G_0 & \xrightarrow[g_0]{} & G_{-1} & \xrightarrow[g_{-1}]{} & G_{-2} & \longrightarrow & \cdots
\end{array}$$

Since $f_0 d_1 = 0$, we have that $\operatorname{Im} d_1 = \operatorname{Ker} d_0 \subseteq \operatorname{Ker} f_0$. This allows defining $u_0\colon \operatorname{Im} d_0 \to G_0$, by $u_0(d_0(x)) = f_0(x)$.

The exact sequence $0 \to \operatorname{Im} d_0 \to A_{-1} \xrightarrow{d_{-1}} \operatorname{Im} d_{-1} \to 0$ with $\operatorname{Im} d_{-1}$ in ${}^{\perp}\mathcal{GI}$ gives the exact sequence $\operatorname{Hom}(A_{-1}, G_0) \to \operatorname{Hom}(\operatorname{Im} d_0, G_0) \to \operatorname{Ext}^1(\operatorname{Im} d_{-1}, G_0) = 0$. So there is $s_0 \in \operatorname{Hom}(A_{-1}, G_0)$ such that $s_0|\operatorname{Im} d_0 = u_0$. Thus $s_0 d_0 = f_0$.

$$\begin{array}{ccc}
\operatorname{Im} d_0 & \hookrightarrow & A_{-1} \\
\downarrow{\scriptstyle u_0} & \swarrow{\scriptstyle s_0} & \\
G_0 & &
\end{array}$$

We have

$$(f_{-1} - g_0 s_0) d_0 = f_{-1} d_0 - g_0 s_0 d_0 = g_0 f_0 - g_0 f_0 = 0.$$

So

$$\operatorname{Im} d_0 = \operatorname{Ker} d_{-1} \subseteq \operatorname{Ker}(f_{-1} - g_0 s_0).$$

This allows defining $u_{-1}\colon \operatorname{Im} d_{-1} \to G_{-1}$ by $u_{-1}(d_{-1}(x)) = (f_{-1} - g_0 s_0)(x)$.

The exact sequence $0 \to \operatorname{Im} d_{-1} \to A_{-2} \xrightarrow{d_{-2}} \operatorname{Im} d_{-2} \to 0$ gives the exact sequence $\operatorname{Hom}(A_{-2}, G_{-1}) \to \operatorname{Hom}(\operatorname{Im} d_0, G_{-1}) \to \operatorname{Ext}^1(\operatorname{Im} d_{-1}, G_{-1}) = 0$. So there is $s_{-1}\colon A_{-2} \to G_{-1}$ such that $s_{-1}|\operatorname{Im} d_{-1} = u_{-1}$. Thus $s_{-1} d_{-1} = u_{-1} d_{-1} = f_{-1} - g_0 s_0$.

Continuing, we obtain that for each $j \leq -1$, there is $s_j\colon A_j \to G_{j+1}$ such that $f_j = s_j d_j + g_{j+1} s_{j+1}$. So f is homotopic to zero. $\square$

Corollary 3.2. *If the class of dg-Gorenstein injective complexes is covering in* $\mathrm{Ch}(R)$*, then* R *is a noetherian ring.*

Proof. In this case, the class of dg-Gorenstein injective complexes is closed under direct sums.

Let (G_i) be a family of Gorenstein injective modules. Then for each i, $0 \to G_i \to 0$ (with G_i in the zeroth place) is a dg-Gorenstein injective complex, so the direct sum is also dg-Gorenstein injective. Thus the class $\mathcal{GI}$ is closed under direct sums. By [**5**], R is a noetherian ring. □

Proposition 3.3 ([**11**, Proposition 5])**.** *If the class of Gorenstein injective modules* $\mathcal{GI}$ *is closed under direct limits, then the class of Gorenstein injective complexes is covering in* $\mathrm{Ch}(R)$.

We recall **Wakamatsu's Lemma** ([**22**, Lemma 2.1.1]): *If* $\mathcal{F}$ *is a class closed under extensions and if* $\varphi\colon F \to M$ *is an* $\mathcal{F}$*-cover, then* $\operatorname{Ker}\varphi \in \mathcal{F}^{\perp}$.

Lemma 3.4. *If every complex has a dg-Gorenstein injective cover, then every complex of Gorenstein injective modules is dg-Gorenstein injective, (i.e.,* $\mathrm{dw}(\mathcal{GI}) = \mathrm{dg}(\mathcal{GI})$*).*

Proof. Let $E = \ldots \to E_2 \to E_1 \to E_0 \to \ldots$ be a complex of Gorenstein injective modules. For each $n \geq 0$, let $X_n = \ldots \to 0 \to 0 \to E_n \to E_{n-1} \to \ldots$ with E_n in the nth place. Then $X_n \in \mathrm{dg}(\mathcal{GI})$ (by Lemma 3.1), $X_n \subset X_{n+1}$, and $E = \varinjlim X_n$.

Let $D \xrightarrow{\phi} E$ be a dg-Gorenstein injective cover, and let $K = \operatorname{Ker}\phi$. Then $0 \to \mathrm{Hom}(A, K) \to \mathrm{Hom}(A, D) \to \mathrm{Hom}(A, E) \to 0$ is exact for any dg-Gorenstein injective complex A. In particular, for all $n \geq 0$, the sequence $0 \to \mathrm{Hom}(X_n, K) \to \mathrm{Hom}(X_n, D) \to \mathrm{Hom}(X_n, E) \to 0$ is exact.

For each $n \geq 0$, we have an exact sequence of dg-Gorenstein injective complexes: $0 \to X_n \to X_{n+1} \to \frac{X_{n+1}}{X_n} \to 0$ (since $(\widetilde{{}^{\perp}\mathcal{GI}}, \mathrm{dg}(\mathcal{GI}))$ is a hereditary cotorsion pair, the class $\mathrm{dg}(\mathcal{GI})$ is closed under cokernels of monomorphisms).

Therefore, we have an exact sequence:
$0 \to \mathrm{Hom}(\frac{X_{n+1}}{X_n}, K) \to \mathrm{Hom}(X_{n+1}, K) \to \mathrm{Hom}(X_n, K) \to \mathrm{Ext}^1(\frac{X_{n+1}}{X_n}, K) = 0$ (by Wakamatsu's Lemma). So $\mathrm{Hom}(X_{n+1}, K) \to \mathrm{Hom}(X_n, K) \to 0$ is exact for any $n \geq 0$.

It follows ([**10**, Theorem 1.5.13]) that the sequence

$$0 \to \varprojlim \mathrm{Hom}(X_n, K) \to \varprojlim \mathrm{Hom}(X_n, D) \to \varprojlim \mathrm{Hom}(X_n, E) \to 0$$

is exact. But

$$\varprojlim \mathrm{Hom}(X_n, K) \simeq \mathrm{Hom}(E, K), \qquad \varprojlim \mathrm{Hom}(X_n, D) \simeq \mathrm{Hom}(E, D),$$
$$\varprojlim \mathrm{Hom}(X_n, E) \simeq \mathrm{Hom}(E, E).$$

Since the sequence $0 \to \mathrm{Hom}(E, K) \to \mathrm{Hom}(E, D) \to \mathrm{Hom}(E, E) \to 0$ is exact, it follows that there is $r \in \mathrm{Hom}(E, D)$ such that $\phi r = \mathrm{Id}_E$. So ϕ is surjective and the sequence $0 \to K \to D \xrightarrow{\phi} E \to 0$ is split exact. Since E is isomorphic to a direct summand of a dg-Gorenstein injective complex, E is dg-Gorenstein injective. □

Theorem 3.5. *Assume that the class of Gorenstein injective modules is closed under direct limits. Then the following are equivalent:*

1. $\mathrm{dg}(\mathcal{GI})$ *is covering.*
2. $\mathrm{dw}(\mathcal{GI}) = \mathrm{dg}(\mathcal{GI})$.

Proof. (1) ⇒ (2) is Lemma 3.4 above.
(2) ⇒ (1) follows from Proposition 3.3, since the class of Gorenstein injective complexes, $\mathrm{dw}(\mathcal{GI})$, is covering. □

It is known that over a commutative noetherian ring with a dualizing complex, the class of Gorenstein injective modules is closed under direct limits.

Theorem 3.6. *Let R be a commutative ring with a dualizing complex. The following are equivalent:*

1. $\mathrm{dg}(\mathcal{GI})$ *is covering.*
2. *R is a Gorenstein ring.*

Proof. (1) ⇒ (2) By Corollary 3.2, the ring R is noetherian. By Theorem 3.5, we have that every complex of Gorenstein injective modules is dg-Gorenstein injective. By [**12**, Theorem 2], this is equivalent to every exact complex of injectives being a totally acyclic complex. By [**18**, Corollary 5.5], R is a Gorenstein ring.
(2) ⇒ (1) It is known that over a Gorenstein ring, every exact complex of injective modules is totally acyclic. By [**12**, Theorem 2], it follows that every complex of Gorenstein injective modules is dg-Gorenstein injective. Then, by Theorem 3.5, $\mathrm{dg}(\mathcal{GI})$ is covering (since $\mathcal{GI}$ is closed under direct limits over Gorenstein rings). □

Example. Consider the ring $R = k[[X,Y,Z]]/(X^2, XY, Y^2)$, where k is a field. As a quotient of a Gorensein ring, R has a dualizing complex. But for the prime ideal $\mathfrak{p} = (X,Y)R$, $G.i.d_{R_\mathfrak{p}} R_\mathfrak{p} = \infty$ ([**6**, Example 3.3]). So R is not a Gorenstein ring. By Theorem 3.6, the class $\mathrm{dg}(\mathcal{GI})$ is not covering in $\mathrm{Ch}(R)$.

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