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DOUBLE WIJSMAN ASYMPTOTIC $\mathcal{I}_2$-INVARIANT EQUIVALENCE

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Abstract. In this study, for double set sequences, we present the notions of Wijsman asymptotic invariant equivalence, Wijsman asymptotic $\mathcal{I}_2$-invariant equivalence, and Wijsman asymptotic $\mathcal{I}_2^*$-invariant equivalence. Also, we examine the relations between these notions and Wijsman asymptotic invariant statistical equivalence studied in this field before.

1. Preliminaries

For double sequences, the notion of convergence was introduced by Pringsheim [**28**]. Then, this notion was extended to the notions of statistical convergence and $\mathcal{I}$-convergence for double sequences by Mursaleen and Edely [**13**] and Das et al. [**5**], respectively. Also, many authors [**15, 16, 17, 18, 19**] studied ideal convergence in some different spaces.

The notion of asymptotic equivalence for double sequences was introduced by Patterson [**26**]. Then for double sequences, this notion was extended to the notions of asymptotic statistical equivalence and asymptotic $\mathcal{I}$-equivalence by Esi and Açıkgöz [**9**] and Hazarika and Kumar [**10**], respectively.

Over the years, many authors have studied on the concepts of convergence for set sequences. One of them, discussed in this study, is the notion of Wijsman convergence [**3**]. Using the concepts of statistical convergence, invariant mean, and $\mathcal{I}$-convergence, the notion of Wijsman convergence has been extended to new convergence notions for double set sequences by many authors.

For double set sequences, the notions of asymptotic equivalence in Wijsman sense were firstly introduced by Nuray et al. [**22**] and then have been studied by many authors. In this paper, using the concept of invariant mean, we study on new asymptotic equivalence notions for double set sequences.

Now, before giving the main part of the study, we recall some basic concepts such as ideal convergence, invariant, and asymptotic equivalence for real sequences

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or set sequences that can be found in [**1, 2, 4, 6, 7, 8, 11, 12, 14, 20, 21, 23, 24, 25, 27, 29, 30, 31, 32, 33, 34**].

Let σ be a mapping such that $\sigma\colon \mathbb{N}^+ \to \mathbb{N}^+$, the set of positive integers. A continuous linear functional ψ on ℓ_∞ is called an invariant mean or a σ-mean if it satisfies following conditions:

1) $\psi(s_n) \geq 0$ when the sequence (s_n) has $s_n \geq 0$ for all n,
2) $\psi(e) = 1$, where $e = (1, 1, 1, \ldots)$, and
3) $\psi(s_{\sigma(n)}) = \psi(s_n)$ for all $(s_n) \in \ell_\infty$.

The mappings σ are assumed to be one-to-one and such that $\sigma^m(j) \neq j$ for all $m, j \in \mathbb{N}^+$, where $\sigma^m(j)$ denotes the m th iterate of the mapping σ at j. Thus, ψ extends the limit functional on c, in the sense that $\psi(s_n) = \lim s_n$ for all $(s_n) \in c$.

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if it satisfies the following conditions:

i) $\emptyset \in \mathcal{I}$,
ii) $U, V \in \mathcal{I} \Rightarrow U \cup V \in \mathcal{I}$, and
iii) $(U \in \mathcal{I}) \wedge (V \subseteq U) \Rightarrow V \in \mathcal{I}$.

An ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and a non-trivial ideal is called admissible if $\{i\} \in \mathcal{I}$ for each $i \in \mathbb{N}$.

A non-trivial ideal $\mathcal{I}_2 \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ is called strong admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$. Obviously a strong admissible ideal is admissible.

Let $\mathcal{I}_2^0 = \{E \subset \mathbb{N} \times \mathbb{N} : (\exists m(E) \in \mathbb{N})(i, j \geq m(E) \Rightarrow (i, j) \notin E)\}$. Then, $\mathcal{I}_2^0$ is a strong admissible ideal and clearly an ideal $\mathcal{I}_2 \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ is strong admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

An admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2) if for every countable family of mutually disjoint sets $\{U_1, U_2, \ldots\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{V_1, V_2, \ldots\}$ such that $U_i \Delta V_i \in \mathcal{I}_2^0$, that is, $U_i \Delta V_i$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $i \in \mathbb{N}$ and $V = \bigcup_{i=1}^{\infty} V_i \in \mathcal{I}_2$ (hence, $V_i \in \mathcal{I}_2$ for each $i \in \mathbb{N}$).

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is called a filter if it satisfies the following conditions:

i) $\emptyset \notin \mathcal{F}$,
ii) $U, V \in \mathcal{F} \Rightarrow U \cap V \in \mathcal{F}$, and
iii) $(U \in \mathcal{F}) \wedge (V \supseteq U) \Rightarrow V \in \mathcal{F}$.

For any ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$, there is a filter $\mathcal{F}(\mathcal{I})$ corresponding with $\mathcal{I}$ such that

$$\mathcal{F}(\mathcal{I}) = \{E \subset \mathbb{N} : (\exists U \in \mathcal{I})(E = \mathbb{N} \backslash U)\}.$$

Throughout the study, $|A|$ denotes the number of elements of the set A.

Let $E \subseteq \mathbb{N} \times \mathbb{N}$, and

$$s_{mn} = \min_{j,k} \Big\{ \big| E \cap \big\{ \big(\sigma(j), \sigma(k)\big), \big(\sigma^2(j), \sigma^2(k)\big), \ldots, \big(\sigma^m(j), \sigma^n(k)\big) \big\} \big| \Big\},$$

and

$$S_{mn} = \max_{j,k} \Big\{ \big| E \cap \big\{ \big(\sigma(j), \sigma(k)\big), \big(\sigma^2(j), \sigma^2(k)\big), \ldots, \big(\sigma^m(j), \sigma^n(k)\big) \big\} \big| \Big\}.$$

If the following limits

$$\underline{V_2}(E) = \lim_{m,n\to\infty} \frac{s_{mn}}{mn} \qquad \text{and} \qquad \overline{V_2}(E) = \lim_{m,n\to\infty} \frac{S_{mn}}{mn}$$

exist, then they are called a lower and an upper σ-uniform density of the set E, respectively. If $\underline{V_2}(E) = \overline{V_2}(E)$, then $V_2(E) = \underline{V_2}(E) = \overline{V_2}(E)$ is called σ-uniform density of the set E.

Denote by $\mathcal{I}_2^\sigma$ the class of all $E \subseteq \mathbb{N} \times \mathbb{N}$ with $V_2(A) = 0$. Obviously $\mathcal{I}_2^\sigma$ is a strong admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Two non-negative double sequences (a_{mn}) and (b_{mn}) are called asymptotic equivalent if

$$\lim_{m,n\to\infty} \frac{a_{mn}}{b_{mn}} = 1.$$

It is denoted by $a_{mn} \sim b_{mn}$.

Let Y be any non-empty set. The function $g \colon \mathbb{N} \to 2^Y$ is defined by $g(i) = B_i \in 2^Y$ for each $i \in \mathbb{N}$. The sequence $\{B_i\} = (B_1, B_2, \ldots)$, which is the range's elements of g, is called set sequences.

Let (Y, d) be a metric space. For any $y \in Y$ and any non-empty $B \subseteq Y$, the distance from y to B is defined by

$$\rho(y, B) = \inf_{b\in B} d(y, b).$$

Throughout the paper, (Y, d) is taken as a metric space and B, B_{mn}, D_{mn} as any nonempty closed subsets of Y.

A double set sequence $\{B_{mn}\}$ is called Wijsman convergent to B if for each $y \in Y$,

$$\lim_{m,n\to\infty} \rho(y, B_{mn}) = \rho(y, B).$$

A double set sequence $\{B_{mn}\}$ is called Wijsman invariant convergent to B if for each $y \in Y$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{j,k=1,1}^{m,n} \rho(y, B_{\sigma^j(s)\sigma^k(t)}) = \rho(y, B)$$

uniformly in $s, t = 1, 2, \ldots$.

Let $0 < r < \infty$. A double set sequence $\{B_{mn}\}$ is called Wijsman strong r-invariant convergent to B if for each $y \in Y$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{j,k=1,1}^{m,n} |\rho(y, B_{\sigma^j(s)\sigma^k(t)}) - \rho(y, B)|^r = 0$$

uniformly in $s, t = 1, 2, \ldots$.

If $r = 1$, then the double set sequence is simply called Wijsman strong invariant convergent to B.

A double set sequences $\{B_{mn}\}$ is called Wijsman $\mathcal{I}$-invariant convergent to B if for every $\varepsilon > 0$ and each $y \in Y$,

$$E_y(\varepsilon) = \big\{(m, n) : |\rho(y, B_{mn}) - \rho(y, B)| \geq \varepsilon\big\} \in \mathcal{I}_2^\sigma,$$

that is, $V_2\big(E_y(\varepsilon)\big) = 0$.

The term $\rho_y\Big(\dfrac{B_{mn}}{D_{mn}}\Big)$ is defined as follows:

$$\rho_y\Big(\frac{B_{mn}}{D_{mn}}\Big) = \begin{cases} \dfrac{\rho(y, B_{mn})}{\rho(y, D_{mn})}, & y \notin B_{mn} \cup D_{mn}, \\ \lambda, & y \in B_{mn} \cup D_{mn}. \end{cases}$$

Two double set sequences $\{B_{mn}\}$ and $\{D_{mn}\}$ are called Wijsman asymptotic equivalent of multiple λ if for each $y \in Y$,

$$\lim_{m,n\to\infty} \rho_y\Big(\frac{B_{mn}}{D_{mn}}\Big) = \lambda.$$

It is denoted by $B_{mn} \overset{W^\lambda}{\sim} D_{mn}$ and simply Wijsman asymptotic equivalent if $\lambda = 1$.

As an example, consider the following double sequences of circles in the (x, y)-plane:

$$B_{mn} = \big\{(a, b) : a^2 + b^2 + 2ma + 2nb = 0\big\}$$

and

$$D_{mn} = \big\{(a, b) : a^2 + b^2 - 2ma - 2nb = 0\big\}.$$

The double set sequences are Wijsman asymptotic equivalent, that is, $B_{mn} \sim D_{mn}$.

Two double set sequences $\{B_{mn}\}$ and $\{D_{mn}\}$ are called Wijsman asimptotic invariant statistical equivalent to multiple λ if for every $\varepsilon > 0$ and each $y \in Y$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \Bigg| \Bigg\{ (j, k) : j \le m, k \le n, \Big| \rho_y\Big(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\Big) - \lambda \Big| \ge \varepsilon \Bigg\} \Bigg| = 0$$

uniformly in s, t. It is denoted by $B_{mn} \overset{W^\lambda(S_2^\sigma)}{\sim} D_{mn}$ and simply Wijsman asymptotic invariant statistical equivalent if $\lambda = 1$.

2. Main results

In this section, for double set sequences, we present the notions of Wijsman asymptotic invariant equivalence $\big(W^\lambda(V_2^\sigma), W^\lambda[V_2^\sigma], W^\lambda[V_2^\sigma]^r\big)$, Wijsman asymptotic $\mathcal{I}_2$-invariant equivalence, and Wijsman asymptotic $\mathcal{I}_2^*$-invariant equivalence. Also, we examine the relations between these notions and Wijsman asymptotic invariant statistical equivalence studied in this field before.

Definition 2.1. Two double set sequences $\{B_{mn}\}$ and $\{D_{mn}\}$ are Wijsman asymptotic invariant equivalent of multiple λ if for each $y \in Y$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{j,k=1,1}^{m,n} \rho_y\Big(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\Big) = \lambda$$

uniformly in s, t. We denote this in $B_{mn} \overset{W^\lambda(V_2^\sigma)}{\sim} D_{mn}$ format and simply called Wijsman asymptotic invariant equivalent if $\lambda = 1$.

Definition 2.2. Two double sequences $\{B_{mn}\}$ and $\{D_{mn}\}$ are Wijsman asymptotic $\mathcal{I}_2$-invariant equivalent of multiple λ if for every $\varepsilon > 0$ and each $y \in Y$,

$$E_y^{\sim}(\varepsilon) := \left\{(m,n) : \left|\rho_y\Big(\frac{B_{mn}}{D_{mn}}\Big) - \lambda\right| \geq \varepsilon\right\} \in \mathcal{I}_2^{\sigma},$$

that is, $V_2\big(E_y^{\sim}(\varepsilon)\big) = 0$. We denote this in $B_{mn} \overset{W^{\lambda}(\mathcal{I}_2^{\sigma})}{\sim} D_{mn}$ format and simply called Wijsman asymptotic $\mathcal{I}_2$-invariant equivalent if $\lambda = 1$.

Theorem 2.1. *If $\rho_y(B_{mn}) = \mathcal{O}\big(\rho_y(D_{mn})\big)$, then*

$$B_{mn} \overset{W^{\lambda}(\mathcal{I}_2^{\sigma})}{\sim} D_{mn} \qquad \Longrightarrow \qquad B_{mn} \overset{W^{\lambda}(V_2^{\sigma})}{\sim} D_{mn}.$$

Proof. Let $m, n, s, t \in \mathbb{N}$ be arbitrary and $\varepsilon > 0$ be given. Also, we assume that $B_{mn} \overset{W^{\lambda}(\mathcal{I}_2^{\sigma})}{\sim} D_{mn}$. Now, we calculate

$$\mathcal{R}(m,n,s,t) := \left|\frac{1}{mn}\sum_{j,k=1,1}^{m,n} \rho_y\Big(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\Big) - \lambda\right|.$$

For every $m, n = 1, 2, \ldots,$ $s, t = 1, 2, \ldots,$ and each $y \in Y$, we have

$$\mathcal{R}(m,n,s,t) \leq \mathcal{R}_1(m,n,s,t) + \mathcal{R}_2(m,n,s,t),$$

where

$$\mathcal{R}_1(m,n,s,t) = \frac{1}{mn} \sum_{\substack{j,k=1,1 \\ \left|\rho_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right)-\lambda\right| \geq \varepsilon}}^{m,n} \left|\rho_y\Big(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\Big) - \lambda\right|$$

and

$$\mathcal{R}_2(m,n,s,t) = \frac{1}{mn} \sum_{\substack{j,k=1,1 \\ \left|\rho_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right)-\lambda\right| < \varepsilon}}^{m,n} \left|\rho_y\Big(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\Big) - \lambda\right|.$$

For every $m, n = 1, 2, \ldots,$ $s, t = 1, 2, \ldots,$ and each $y \in Y$, it is obvious that $\mathcal{R}_2(m,n,s,t) < \varepsilon$. Since $\rho_y(B_{mn}) = \mathcal{O}\big(\rho_y(D_{mn})\big)$, there exists an $M > 0$ such that

$$\left|\rho_y\Big(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\Big) - \lambda\right| \leq M, \qquad (j,k = 1,2,\ldots;\ s,t = 1,2,\ldots)$$

for each $y \in Y$, and so we have

$$\begin{aligned}
\mathcal{R}_1(m,n,s,t) &\leq \frac{M}{mn}\left|\left\{(j,k) : 1 \leq j \leq m, 1 \leq k \leq n, \left|\rho_y\Big(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\Big) - \lambda\right| \geq \varepsilon\right\}\right| \\
&\leq M\frac{\max_{s,t}\left\{\left|\left\{(j,k) : 1 \leq j \leq m, 1 \leq k \leq n, \left|\rho_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right) - \lambda\right| \geq \varepsilon\right\}\right|\right\}}{mn} \\
&= M\frac{S_{mn}}{mn}.
\end{aligned}$$

Hence, due to our assumption, $B_{mn} \overset{W^{\lambda}(V_2^{\sigma})}{\sim} D_{mn}$. $\square$

Definition 2.3. Let $0 < r < \infty$. Two double set sequences $\{B_{mn}\}$ and $\{D_{mn}\}$ are Wijsman asymptotic strong r-invariant equivalent of multiple λ if for each $y \in Y$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{j,k=1,1}^{m,n} \left| \rho_y\Big(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\Big) - \lambda \right|^r = 0$$

uniformly in s,t. We denote this in $B_{mn} \overset{W^{\lambda}[V_2^{\sigma}]^r}{\sim} D_{mn}$ format and simply called Wijsman asymptotic strong r-invariant equivalent if $\lambda = 1$.

If $r = 1$, then the double set sequences are simply called Wijsman asymptotic strong invariant equivalent of multiple λ.

Theorem 2.2. *If* $B_{mn} \overset{W^{\lambda}[V_2^{\sigma}]^r}{\sim} D_{mn}$, *then* $B_{mn} \overset{W^{\lambda}(\mathcal{I}_2^{\sigma})}{\sim} D_{mn}$.

Proof. Let $0 < r < \infty$ and $\varepsilon > 0$ be given. Also, we assume that $B_{mn} \overset{W^{\lambda}[V_2^{\sigma}]^r}{\sim} D_{mn}$. For every $m,n = 1,2,\ldots,$ $s,t = 1,2,\ldots$ and each $y \in Y$, we have

$$\begin{aligned}
&\sum_{j,k=1,1}^{m,n} \left| \rho_y\Big(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\Big) - \lambda \right|^r \\
&\geq \sum_{\substack{j,k=1,1 \\ \left|\rho_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right)-\lambda\right|\geq\varepsilon}}^{m,n} \left| \rho_y\Big(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\Big) - \lambda \right|^r \\
&\geq \varepsilon^r \left| \left\{ (j,k) : 1 \leq j \leq m, 1 \leq k \leq n, \left| \rho_y\Big(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\Big) - \lambda \right| \geq \varepsilon \right\} \right| \\
&\geq \varepsilon^r \max_{s,t} \left\{ \left| \left\{ (j,k) : 1 \leq j \leq m, 1 \leq k \leq n, \left| \rho_y\Big(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\Big) - \lambda \right| \geq \varepsilon \right\} \right| \right\},
\end{aligned}$$

and so

$$\begin{aligned}
&\frac{1}{mn} \sum_{j,k=1,1}^{m,n} \left| \rho_y\Big(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\Big) - \lambda \right|^r \\
&\geq \varepsilon^r \frac{\max_{s,t} \left\{ \left| \left\{ (j,k) : 1 \leq j \leq m, 1 \leq k \leq n, \left| \rho_y\Big(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\Big) - \lambda \right| \geq \varepsilon \right\} \right| \right\}}{mn} \\
&= \varepsilon^r \frac{S_{mn}}{mn}.
\end{aligned}$$

Hence, due to our assumption, $B_{mn} \overset{W^{\lambda}(\mathcal{I}_2^{\sigma})}{\sim} D_{mn}$. $\square$

Theorem 2.3. *If* $\rho_y(B_{mn}) = \mathcal{O}\big(\rho_y(D_{mn})\big)$, *then*

$$B_{mn} \overset{W^{\lambda}(\mathcal{I}_2^{\sigma})}{\sim} D_{mn} \quad \Longrightarrow \quad B_{mn} \overset{W^{\lambda}[V_2^{\sigma}]^r}{\sim} D_{mn}.$$

Proof. Let $0 < r < \infty$ and $\varepsilon > 0$ is given. Also, we assume that $\rho_y(B_{mn}) = \mathcal{O}\big(\rho_y(D_{mn})\big)$ and $B_{mn} \overset{W^{\lambda}(\mathcal{I}_2^{\sigma})}{\sim} D_{mn}$. Since $\rho_y(B_{mn}) = \mathcal{O}\big(\rho_y(D_{mn})\big)$, there exists $M > 0$ such that

$$\left|\rho_y\Big(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\Big) - \lambda\right| \le M, \qquad (j,k = 1,2,\ldots;\ s,t = 1,2,\ldots)$$

for each $y \in Y$, and so for every $m,n = 1,2,\ldots$, we have

$$\begin{aligned}
&\frac{1}{mn}\sum_{j,k=1,1}^{m,n}\left|\rho_y\Big(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\Big) - \lambda\right|^r \\
&= \frac{1}{mn}\sum_{\substack{j,k=1,1 \\ \left|\rho_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right)-\lambda\right|\ge\varepsilon}}^{m,n}\left|\mu_x\Big(\frac{B_{\sigma^k(m)\sigma^j(n)}}{D_{\sigma^k(m)\sigma^j(n)}}\Big) - \lambda\right|^r \\
&\quad + \frac{1}{mn}\sum_{\substack{j,k=1,1 \\ \left|\rho_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right)-\lambda\right|<\varepsilon}}^{m,n}\left|\mu_x\Big(\frac{B_{\sigma^k(m)\sigma^j(n)}}{D_{\sigma^k(m)\sigma^j(n)}}\Big) - \lambda\right|^r \\
&\le M\frac{\max\limits_{s,t}\left\{\left|\left\{(j,k) : 1 \le j \le m, 1 \le k \le n, \left|\rho_y\left(\frac{B_{\sigma^j(s)\sigma^k(t)}}{D_{\sigma^j(s)\sigma^k(t)}}\right) - \lambda\right| \ge \varepsilon\right\}\right|\right\}}{mn} + \varepsilon^r \\
&\le M\frac{S_{mn}}{mn} + \varepsilon^r.
\end{aligned}$$

Hence, due to our assumption, $B_{mn} \overset{W^{\lambda}[V_2^{\sigma}]^r}{\sim} D_{mn}$. □

Theorem 2.4. *If* $\rho_y(B_{mn}) = \mathcal{O}\big(\rho_y(D_{mn})\big)$, *then*

$$B_{mn} \overset{W^{\lambda}(\mathcal{I}_2^{\sigma})}{\sim} D_{mn} \quad \Longleftrightarrow \quad B_{mn} \overset{W^{\lambda}[V_2^{\sigma}]^r}{\sim} D_{mn}.$$

Proof. This is immediate consequence of Theorem 2.2 and Theorem 2.3. □

Now, without proof, we present a theorem that gives a relationship between the notions of Wijsman asymptotic $\mathcal{I}_2$-invariant equivalence of multiple λ and Wijsman asymptotic invariant statistical equivalence of multiple λ.

Theorem 2.5. *For any double set sequences* $\{B_{mn}\}$ *and* $\{D_{mn}\}$,

$$B_{mn} \overset{W^{\lambda}(\mathcal{I}_2^{\sigma})}{\sim} D_{mn} \quad \Longleftrightarrow \quad B_{mn} \overset{W^{\lambda}(S_2^{\sigma})}{\sim} D_{mn}.$$

Definition 2.4. Two double set sequences $\{B_{mn}\}$ and $\{D_{mn}\}$ are Wijsman asymptotic $\mathcal{I}_2^*$-invariant equivalent of multiple λ if and only if there exists a set $G \in \mathcal{F}(\mathcal{I}_2^\sigma)$ $(\mathbb{N}\times\mathbb{N}\setminus G = H \in \mathcal{I}_2^\sigma)$ such that for each $y \in Y$,

$$\lim_{\substack{m,n\to\infty \\ (m,n)\in G}} \rho_y\Big(\frac{B_{mn}}{D_{mn}}\Big) = \lambda.$$

We denote this in $B_{mn} \overset{W^\lambda(\mathcal{I}_2^{*\sigma})}{\sim} D_{mn}$ format and simply called Wijsman asymptotic $\mathcal{I}_2^*$-invariant equivalent if $\lambda = 1$.

Theorem 2.6. *If* $B_{mn} \overset{W^\lambda(\mathcal{I}_2^{*\sigma})}{\sim} D_{mn}$*, then* $B_{mn} \overset{W^\lambda(\mathcal{I}_2^{\sigma})}{\sim} D_{mn}$.

Proof. Let $B_{mn} \overset{W^\lambda(\mathcal{I}_2^{*\sigma})}{\sim} D_{mn}$ and $\varepsilon > 0$ be given. Then, there exists a set $G \in \mathcal{F}(\mathcal{I}_2^\sigma)$ $(\mathbb{N}\times\mathbb{N}\setminus G = H \in \mathcal{I}_2^\sigma)$ such that for each $y \in Y$,

$$\lim_{\substack{m,n\to\infty \\ (m,n)\in G}} \rho_y\Big(\frac{B_{mn}}{D_{mn}}\Big) = \lambda,$$

and so there exists $u_0 \in \mathbb{N}$ such that

$$\Big|\rho_y\Big(\frac{B_{mn}}{D_{mn}}\Big) - \lambda\Big| < \varepsilon$$

for all $(m,n) \in G$, where $m,n \geq u_0$. Hence, for every $\varepsilon > 0$ and each $y \in Y$, it is obvious that

$$\begin{aligned}\mathcal{A}(\varepsilon) &:= \Big\{(m,n) : \Big|\rho_y\Big(\frac{B_{mn}}{D_{mn}}\Big) - \lambda\Big| \geq \varepsilon\Big\} \\ &\subset H \cup \Big(G \cap \Big(\big(\{1,2,\ldots,(u_0-1)\}\times\mathbb{N}\big) \cup \big(\mathbb{N}\times\{1,2,\ldots,(u_0-1)\}\big)\Big)\Big).\end{aligned}$$

Since $\mathcal{I}_2^\sigma \subset 2^{\mathbb{N}\times\mathbb{N}}$ is a strong admissible ideal,

$$H \cup \Big(G \cap \Big(\big(\{1,2,\ldots,(u_0-1)\}\times\mathbb{N}\big) \cup \big(\mathbb{N}\times\{1,2,\ldots,(u_0-1)\}\big)\Big)\Big) \in \mathcal{I}_2^\sigma,$$

and so we have $\mathcal{A}(\varepsilon) \in \mathcal{I}_2^\sigma$. Consequently, $B_{mn} \overset{W^\lambda(\mathcal{I}_2^{\sigma})}{\sim} D_{mn}$. □

The converse of Theorem 2.6 holds if $\mathcal{I}_2^\sigma$ has property $(AP2)$.

Theorem 2.7. *If* $\mathcal{I}_2^\sigma$ *has property* $(AP2)$*, then*

$$B_{mn} \overset{W^\lambda(\mathcal{I}_2^{\sigma})}{\sim} D_{mn} \quad\Longrightarrow\quad B_{mn} \overset{W^\lambda(\mathcal{I}_2^{*\sigma})}{\sim} D_{mn}.$$

Proof. Let $\mathcal{I}_2^\sigma$ satisfies condition $(AP2)$ and $\varepsilon > 0$ be given. Also, we assume that $B_{mn} \overset{W^\lambda(\mathcal{I}_2^{\sigma})}{\sim} D_{mn}$. Then for every $\varepsilon > 0$ and each $y \in Y$, we have

$$\Big\{(m,n) : \Big|\rho_y\Big(\frac{B_{mn}}{D_{mn}}\Big) - \lambda\Big| \geq \varepsilon\Big\} \in \mathcal{I}_2^\sigma.$$

For every $y \in Y$, denote $E_1, \ldots, E_v$ as follows:

$$E_1 := \left\{(m,n) : \left|\rho_y\left(\frac{B_{mn}}{D_{mn}}\right) - \lambda\right| \geq 1\right\}$$

and

$$E_v := \left\{(m,n) : \frac{1}{v} \leq \left|\rho_y\left(\frac{B_{mn}}{D_{mn}}\right) - \lambda\right| < \frac{1}{v-1}\right\},$$

where $v \geq 2$ ($v \in \mathbb{N}$). For each $y \in Y$, note that $E_i \cap E_j = \emptyset$ ($i \neq j$) and $E_i \in \mathcal{I}_2^\sigma$ (for each $i \in \mathbb{N}$). Since $\mathcal{I}_2^\sigma$ satisfies the condition $(AP2)$, there exists a sequence of sets $\{F_i\}_{i\in\mathbb{N}}$ such that $E_i \Delta F_i$ is included in finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ (for each $i \in \mathbb{N}$) and $F = \left(\bigcup_{i=1}^{\infty} F_i\right) \in \mathcal{I}_2^\sigma$.

Now, to complete the proof, it is enough to prove that for each $y \in Y$,

$$\lim_{\substack{m,n\to\infty \\ (m,n)\in G}} \rho_y\left(\frac{B_{mn}}{D_{mn}}\right) = \lambda, \tag{1}$$

where $G = \mathbb{N}\times\mathbb{N} \setminus F$.

Let $\gamma > 0$ be given. Choose $v \in \mathbb{N}$ such that $\frac{1}{v} < \gamma$. Then for each $y \in Y$, we have

$$\left\{(m,n) : \left|\rho_y\left(\frac{B_{mn}}{D_{mn}}\right) - \lambda\right| \geq \gamma\right\} \subset \bigcup_{i=1}^{v} E_i.$$

Since $E_i \Delta F_i$ $(i = 1, 2, \ldots)$ are included in finite union of rows and columns, there exists $u_0 \in \mathbb{N}$ such that for each $y \in Y$,

$$\begin{aligned} &\left(\bigcup_{i=1}^{v} E_i\right) \cap \{(m,n) : m \geq u_0 \wedge n \geq u_0\} \\ &= \left(\bigcup_{i=1}^{v} F_i\right) \cap \{(m,n) : m \geq u_0 \wedge n \geq u_0\}. \end{aligned} \tag{2}$$

If $m, n > u_0$ and $(m,n) \notin F$, then

$$(m,n) \notin \bigcup_{i=1}^{v} F_i \quad \text{and by (2) for each } y \in Y \quad (m,n) \notin \bigcup_{i=1}^{v} E_i.$$

This implies that for each $y \in Y$,

$$\left|\rho_y\left(\frac{B_{mn}}{D_{mn}}\right) - \lambda\right| < \frac{1}{v} < \gamma$$

and so (1) holds. Consequently, $B_{mn} \overset{W^\lambda(\mathcal{I}_2^{*\sigma})}{\sim} D_{mn}$. □

Theorem 2.8. *If $\mathcal{I}_2^\sigma$ has property $(AP2)$, then*

$$B_{mn} \overset{W^\lambda(\mathcal{I}_2^{\sigma})}{\sim} D_{mn} \quad \Longleftrightarrow \quad B_{mn} \overset{W^\lambda(\mathcal{I}_2^{*\sigma})}{\sim} D_{mn}.$$

Proof. This is immediate consequence of Theorem 2.6 and Theorem 2.7. □

3. Conclusions

We have investigated the notions of Wijsman asymptotic invariant equivalence, Wijsman asymptotic $\mathcal{I}_2$-invariant equivalence, and Wijsman asymptotic $\mathcal{I}_2^*$-invariant equivalence of double set sequences. These concepts can also be studied for lacunary sequences in the future.

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