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NOTES ON BAER MODULES AND THEIR DUAL

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Abstract. In this paper, we give new categorical characterizations of (dual-)Baer modules and then several applications of them are presented. Among other things, it is proved that a module M_R is Baer if and only if for every $N \leq M_R$, $\mathrm{Rej}^{-1}(N)$ is a direct summand of M_R. This shows that a module M_R is Baer and co-retractable if and only if it is semisimple. Hence, over a ring Morita equivalent to a perfect duo ring, all Baer modules are semisimple. If R is a right semi-hereditary and $\mathrm{u.dim}(R_R)$ is finite, then every finitely generated torsionless R-module M is Baer. Dually, dual-Baer modules over certain rings are also investigated. If the left R-module $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is Baer, then it is shown that $\mathrm{Hom}_R(M,N)M$ is a pure submodule M_R for any $N \leq M_R$.

1. Introduction

Throughout the paper, all rings are associative with identity and all modules are unital right modules. In [**10**], a ring R was called *Baer* if for every non-empty subset X of R, the right annihilator X in R is of the form eR for some $e = e^2 \in R$. The concept of Baer ring was extended to modules by S. T. Rizvi and C. S. Roman in [**16**] and [**18**]. A module M_R is called *Baer* if for every non-empty subset X of $\mathrm{End}_R(M)$, the right annihilator X in M is a direct summand of M_R. Baer modules and their generalizations have been studied among many other works, see [**13**] and [**3**] (therefore their references) for recent works on the subjects. The dual notion of the Baer modules was introduced and studied in [**20**], where a module M_R was called *dual-Baer* if for every $N \leq M_R$, the right ideal $\mathrm{Hom}_R(M,N)$ of $\mathrm{End}_R(M)$ is generated by an idempotent element; see [**19**] and [**4**] for some recent works on the dual-Baer modules.

In this paper, we first give new characterizations of (dual-)Baer modules in Theorems 2.2, 2.3 and 2.4. Then we show that a module M is semisimple if and only if it is Baer and co-retractable (Theorem 2.6). Baer modules and dual-Baer modules over certain rings are investigated and, among other things, it is shown that if R is a ring Morita invariant to a right duo ring (resp. semi-artinian commutative ring), then (dual-)Baer modules are precisely semisimple modules

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(see Theorems 3.3 and 3.6). Finally, we apply our characterization of Baer modules to investigate conditions on M_R under which the character left R-module M^+ is Baer and prove that if M_R is a non-zero strongly torsionfree extending and the R-module M^+ is Baer, then M_R is dual-Baer. Any unexplained terminology and all the basic results on rings and modules that are used in the sequel can be found in [**1**] and [**12**].

2. Characterizations of (dual) Baer modules and (co-)retractability

If K and L are two R-modules, then $\mathrm{Tr}(K, L)$ means $\sum\{f(K) \mid f\colon K \to L\}$ and $\mathrm{Rej}(K, L)$ means $\bigcap\{\ker f \mid f\colon K \to L\}$. If M_R is a module, then the class of R-modules that are generated, (resp., co-generated) by M is denoted by $\mathrm{Gen}(M_R)$ (resp. $\mathrm{Cog}(M_R)$). We begin with the following characterization.

Lemma 2.1. *Let M_R be a nonzero module and $\mathrm{End}_R(M) = S$. Then $N \in \mathrm{Gen}(M_R)$ if and only if $\mathrm{Hom}_R(M, N)M = N = \mathrm{Tr}(M, N)$.*

Proof. By [**21**, Theorem 13.5]. □

Theorem 2.2.

(i) *Every exact sequence $0 \to X \to M \to Y \to 0$ of R-modules with $Y \in \mathrm{Cog}(M)$ splits if and only if M is a Baer R-module.*

(ii) *Every exact sequence $0 \to X \to M \to Y \to 0$ of R-modules with $X \in \mathrm{Gen}(M)$ splits if and only if M is a dual Baer R-module.*

Proof. (i) ($\Rightarrow$). To show that a module M_R is Baer, Let $N = \mathrm{r}_M(X)$ for some $X \subseteq S$. Then $N = \bigcap_{f\in X} \ker f$. Hence, we can deduce that $M/N \in \mathrm{Cog}(M)$. Now the exact sequence $0 \to N \xrightarrow{\imath} M \xrightarrow{\pi} M/N \to 0$ splits by our assumption. This means $N \leq^{\oplus} M$.
($\Leftarrow$). Suppose that if $0 \to X \xrightarrow{f} M \to Y \to 0$ is an exact sequence of R-modules with $Y \in \mathrm{Cog}(M_R)$. Let $N = \mathrm{Im}\, f$, then there exists a one to one R-homomorphism $\theta\colon M/N \to M^{\Lambda}$ for some set Λ. For each $\lambda \in \Lambda$, let $f_\lambda = \pi_\lambda\theta$, where π_λ is the canonical projection on M^{Λ}. If $X = \{f_\lambda \mid \lambda \in \Lambda\}$, then it is easily seen that $N = \mathrm{r}_M(X)$. Thus $\mathrm{Im}\, f$ is a direct summand of M_R by the Baer condition on M. It follows that the exact sequence splits.
(ii) By Lemma 2.1, this is the dual of (i) and has a similar argument. □

Theorem 2.3. *The following conditions are equivalent for a module M_R.*

(a) *M_R is dual Baer.*

(b) *For every nonempty set Λ, every R-homomorphism $f\colon M^{(\Lambda)} \to M$ preserves direct summands.*

(c) *For every nonempty set Λ, every R-homomorphism $f\colon M^{(\Lambda)} \to M$, $\mathrm{Im}\, f$ is a direct summand of M_R.*

(d) *For every $N \leq M_R$, $\mathrm{Tr}(M, N) \leq^{\oplus} M_R$.*

Proof. This is obtained by Theorem 2.2. □

Regarding the Theorem 2.3, we introduce the dual notation for $\mathrm{Tr}(M, N)$, where $N \leq M$ and give a similar result for Baer modules. By $\mathrm{Rej}^{-1}(N)$, we mean $\pi^{-1}(\mathrm{Rej}(M/N, M))$, where $\pi \colon M \to M/N$ is the canonical epimorphism. Clearly, $N \leq \mathrm{Rej}^{-1}(N)$, and $N = \mathrm{Rej}^{-1}(N)$ if and only if $M/N \in \mathrm{Cog}(M)$. In [**18**, Proposition A3], it is proved that if M_R is a Baer module, then for every $f \in \mathrm{End}_R(M)$ the inverse image through f of any direct summand of M is again a direct summand. This shows that f is continuous with respect to a certain topology on M. In the following characterization of Baer modules, we observe in a way the converse of [**18**, Proposition A3].

Theorem 2.4. *The following conditions are equivalent for a module M_R.*

(a) *M_R is Baer.*
(b) *For every nonempty set Λ and every R-homomorphism $f \colon M \to M^\Lambda$, the inverse image $f^{-1}(D)$ is a direct summand of M, where D is a direct summand of M^Λ.*
(c) *For every nonempty set Λ and every R-homomorphism $f \colon M \to M^\Lambda$, $\ker f$ is a direct summand of M_R.*
(d) *For every $N \leq M_R$, $\mathrm{Rej}^{-1}(N) \leq^\oplus M_R$.*

Proof. (a) ⇒ (b) Let M_R be Baer and $f \colon M \to M^\Lambda$ be an R-homomorphism for some Λ. If D is a direct summand of M^Λ and $N = f^{-1}(D)$, then we have the natural monomorphism $M/N \to M/D$. It follows that $M/N \in \mathrm{Cog}(M)$. Hence, N is a direct summand of M by Theorem 2.2.
(b) ⇒ (c) is clear.
(c) ⇒ (a) Let $\{f_i\}_i \in \mathrm{End}_R(M)$. Consider the R-homomorphism $f \colon M \to M^\Lambda$ with $f(m) = \{f_i(m)\}_i$. Then by our assumption $f^{-1}(0)$ is a direct summand of M_R. Thus $\cap_i \ker f_i$ is a direct summand of M, proving that M_R is Baer.
The equivalence (d) ⇔ (a), follows by Theorem 2.2 and the above notes. □

The Theorem 2.3 shows that a module M_R is dual Baer if and only if $\mathrm{Tr}(M, X) \leq^\oplus M_R$ for every $X \leq M_R$. Clearly, $\mathrm{Tr}(M, X) \leq X$. Thus if M_R is dual Baer, then a submodule $X \leq M_R$ contains a non-zero direct summand of M_R if and only if $\mathrm{Hom}_R(M, X) \neq 0$. Modules M_R in which $\mathrm{Hom}_R(M, N) \neq 0$ for every $0 \neq N \leq M_R$ is called *retractable* [**11**]. Dually, an R-module M_R is called *co-retractable* if $\mathrm{Hom}_R(M/N, M) \neq 0$ for every $N < M_R$. Below we study the retractable (co-retractable) condition for Baer and dual Baer modules and then we shall give some applications of our results.

Lemma 2.5. *Let M_R be a nonzero module.*

(i) *If M_R is co-retractable, then $\mathrm{Rej}^{-1}(N)/N \ll M/N$ for every proper $N \leq M_R$.*
(ii) *If M_R is retractable, then $\mathrm{Tr}(M, N) \leq_{ess} N$ for every nonzero $N \leq M_R$.*

Proof. We only prove (i). Let $N < M_R$ and $K = \mathrm{Rej}^{-1}(N)$. Clearly, $N \leq K$. Suppose that $K/N + L/N = M/N$. If $L \neq M$, since M_R is co-retractable, there exists nonzero homomorphism $g\colon M/L \to M$. Consider the natural epimorphism $p\colon M/N \to M/L$, then $0 \neq gp \in \mathrm{Hom}_R(M/N, M)$ and $gp(L/N) = 0$. On the other hand, by the definition of K, we have $gp(K/N) = 0$. It follows that $gp(M/N) = 0$, a contradiction. Thus $L = M$ and we are done.

The equivalences (i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iii) below are dual of each others. The equivalence (i) $\Leftrightarrow$ (iii) appeared in [**19**, Corollary 2.19]. $\square$

Theorem 2.6. *The following statements are equivalent for a module M_R.*

(i) *M_R is semisimple.*

(ii) *M_R is co-retractable and Baer.*

(ii) *M_R is retractable and dual Baer.*

Proof. We need to show that (ii) or (iii) $\Rightarrow$ (i). Let M_R be co-retractable and Baer (the other case is similar). If $N \leq M_R$ and $K = \mathrm{Rej}^{-1}(N)$, then $K \leq^{\oplus} M_R$ by Theorem 2.4. This shows that $K/N \leq^{\oplus} M/N$. On the other hand, $K/N \ll M/N$ by Lemma 2.5. Hence, $N = K$. Therefore, every submodule of M is a direct of M, as desired. $\square$

Corollary 2.7. *Let M_R be a non-zero quasi-projective module. Then the R-module $M/J(M)$ is dual Baer if and only if it is a semisimple R-module.*

Proof. By [**5**, (3.4)], any quasi-projective module with zero Jacobson radical is retractable. Hence, the result is obtained by Theorem 2.6. $\square$

An R-module M is said to be *torsionless* if M is cogenerated by R.

Proposition 2.8.

(a) *A ring R is Baer if and only if every cyclic torsionless right (left) R-module is projective.*

(b) *Every n-generated torsionless right(left) R-module is projective if and only if $M_n(R)$ is a Baer ring.*

Proof. (a) Since a cyclic R-module R/I is projective if and only if the exact sequence $0 \to I \to R \to R/I \to 0$ splits, the result is an application of Theorem 2.2 for $M = R$.
(b) This is now obtained by (a) and the fact that the standard Morita equivalent between R and $\mathrm{M}_n(R)$ corresponds n-generated R-modules to cyclic $\mathrm{M}_n(R)$-modules. $\square$

We end this section with a result on the direct sum of dual-Baer modules similar to [**17**, Proposition 3.20]. Some applications of Theorem 2.2 will be given in the next section. The following lemma is used in Proposition 2.10 and Remark 2.11(2).

Lemma 2.9. *Let $M = \bigoplus_{i\in I} M_i$ (I an index set) such that $\mathrm{Hom}_R(M_i, M_j) = 0$ ($i \neq j$). Assume $0 \to X \xrightarrow{f} M \xrightarrow{g} Y \to 0$ is an exact sequence of R-modules. For each i, replace $\iota_i(M_i)$ with M_i and let $g_i = g|_{M_i}$, $K_i = \ker g_i$, $X_i = f^{-1}(K_i)$ and $f_i = f|_{X_i}$. Then we have:*

(a) *For each i, the sequence $0 \to X_i \xrightarrow{f_i} M_i \xrightarrow{g_i} g(M_i) \to 0$ is exact.*

(b) *If* $Y \in \mathrm{Cog}(M)$, *then for each* i, $g(M_i) \in \mathrm{Cog}(M_i)$ *and* $g(M) = \bigoplus_{i \in I} g(M_i)$.
(c) *If* $X \in \mathrm{Gen}(M)$, *then for each* i, $X_i \in \mathrm{Gen}(M_i)$ *and* $X = \bigoplus_{i \in I} X_i$.

Proof. (a) It has a routine argument.
(b) Let $Y \overset{\theta}{\hookrightarrow} \bigoplus_i (M_i)^{\Lambda}$. Then by hypothesis, $\theta g(M_i) \subseteq (M_i)^{\Lambda}$ for each $i \in I$. It follows that $g(M) = \bigoplus_{i \in I} g(M_i)$ and $g(M_i) \in \mathrm{Cog}(M_i)$.
(c) Clearly, $\{X_i\}_{i \in I}$ are R-linear independent. Let $\bigoplus_i (M_i)^{(\Lambda)} \overset{\alpha}{\to} X$ be a surjective R-homomorphism. By hypothesis, $f\alpha\psi_i((M_i)^{(\Lambda_i)} \subseteq M_i$, where $\psi_i \colon M_i^{(\Lambda_i)} \to \bigoplus_i (M_i)^{(\Lambda_i)}$ is the natural R-monomorphism. If $x \in X$, then there are $u_i \in M_i^{(\Lambda_i)}$ $(i = 1, \dots, n)$ such that $\alpha(\sum_i u_i) = x$. Since $f\alpha(u_i) \in M_i$, we have $f\alpha(u_i) \in K_i$. Thus $x \in \sum_i X_i$. It follows that $X = \sum_i X_i$ and so $X_i \in \mathrm{Gen}(M_i)$. □

Proposition 2.10. *Let* $M = \bigoplus_{i \in I} M_i$ *(I an index set) such that* $\mathrm{Hom}_R(M_i, M_j) = 0$ *$(i \neq j)$. If every M_i is a dual Baer R-module, then M_R is dual Baer.*

Proof. These are obtained by Lemma 2.9 and Theorem 2.2. □

Remarks 2.11. If we consider the equivalent conditions presented in Theorem 2.2 for (dual) Baer modules, then:
1) We can give a simultaneous proof for [**16**, Theorem 2.17] and [**20**, Corollary 2.5] that state “a direct summand of a Baer (resp. dual Baer) module is a Baer (resp. dual Baer) module”. In fact, these are obtained by Theorem 2.2 and the fact that an exact sequence $0 \to X \overset{f}{\to} N \overset{g}{\to} Y \to 0$ splits if the exact sequence $0 \to X \oplus L \overset{f \oplus 1_L}{\to} N \oplus L \overset{\beta}{\to} Y \to 0$ with $\beta(n, l) = g(n)$ splits. Note that if there exists $N \oplus L \overset{\alpha}{\to} X \oplus L$ such that $\alpha(f \oplus 1_L) = 1_{X \oplus L}$, then we have $hf = 1_X$, where $h \colon N \to X$ with $h(n) = \pi\alpha(n, 0)$ and $\pi \colon X \oplus L \to X$ is the natural projection.
2) Let $M = \bigoplus_{i \in I} M_i$ (I an index set) such that $\mathrm{Hom}_R(M_i, M_j) = 0$ $(i \neq j)$. In [**17**, Proposition 3.20] it is proved that if every M_i is a Baer R-module, then M_R is Baer. Now using Theorem 2.2 and Lemma 2.9, we can give an alternative proof for [**17**, Proposition 3.20].

3. Applications

In this section, we give some applications of our results. A ring R is called *von Neumann regular* (regular for short) if for every $a \in R$ there is $b \in R$ such that $a = aba$.

Proposition 3.1. *If R is a Baer ring, then* $\mathrm{ann}_R(P)$ *is a direct summand of R_R for any projective right R-module P. The converse is not true in general.*

Proof. Let R be a Baer ring and P_R be projective. If $I = \mathrm{ann}_R(P)$, then $R/I \in \mathrm{Cog}(R_R)$. Hence, I is a direct summand of R_R by Theorem 2.2. For the last statement, note that if R is any simple ring, then the annihilator of any nonzero R-module is zero. But there exist simple rings that are not Baer. For if R is a right self-injective simple ring, then by [**12**, Corollary 13.5], we have $A = lr(A)$

for any finitely left ideal A of R. Thus if further R is Baer, then R must be a regular ring. However, there exists self-injective simple ring which is not regular; see [**7**]. □

A ring R is said to be *right duo* if every right ideal in R is a two sided ideal. The following lemma may be appeared in the literature, we give a proof for completeness.

Lemma 3.2. *If R is a ring Morita invariant to a right duo perfect ring the every nonzero R-module is co-retractable.*

Proof. We may suppose that R is a right duo and a perfect ring. Let N be a proper submodule of a nonzero module M_R. Since R is right perfect, then the nonzero module M/N has a maximal submodule K/N. On the other hand, by [**2**, Theorem 2.14], the simple R-module M/K can be embedded in M_R. It follows that $\mathrm{Hom}_R(M/N, M)$ is nonzero, as desired. □

Theorem 3.3. *Let R be a ring Morita invariant to a right duo perfect ring, then Baer R-modules are precisely semisimple R-modules.*

Proof. The result follows from Lemma 3.2 and Theorem 2.6. □

Recall that the singular submodule $\mathrm{Z}(M_R)$ of an R-module M is defined by $\mathrm{Z}(M_R) = \{m \in M_R \mid mA = 0 \text{ for some essential right ideal } A \text{ of } R\}$. The module M_R is called *singular* (resp. *nonsingular*) if $\mathrm{Z}(M_R) = M$ (resp. $\mathrm{Z}(M_R) = 0$). It is well known that if N is an essential submodule of M_R, then M/N is a singular R-module. Also, if R is a right semi-hereditary ring, then it is known that $\mathrm{Z}(R_R) = 0$. A submodule K of M_R is said to be closed in M, whenever if K is essential in a submodule L of M_R, then $K = L$.

Proposition 3.4. *If R is a right semi-hereditary and $u.\mathrm{dim}(R_R)$ is finite, then every finitely generated torsionless R-module M is Baer.*

Proof. Let M_R be a finitely generated torsionless R-module and $0 \to X \to M \to Y \to 0$ be an exact sequence of R-modules with $Y \in \mathrm{Cog}(M)$. Since M_R is torsionless, Y is also a torsionless R-module. By our assumption $\mathrm{Z}(R_R) = 0$, hence $\mathrm{Z}(Y_R) = 0$. Now suppose that $Y = R^{(n)}/K$, then K is a closed submodule of $R^{(n)}$. Because if K is essential in L for some submodule L of $R^{(n)}$, then L/K is singular submodule of Y, and so $K = L$. It follows that the uniform dimension of Y_R is finite by [**12**, Theorem 6.35]. Therefore, Y can be embedded in a free R-module by [**14**, Proposition 3.4.3]. Now Y_R must be projective because R is right semi-hereditary. Hence, the exact sequence splits, as desired. □

Proposition 3.5. *If R is a ring such that $R_R^{(n)}$ is extending (e.g., R is right self-injective), then every n-generated nonsingular R-module M_R is Baer.*

Proof. Just note that if $Y = R^{(n)}/K$ is an n-generated nonsingular R-module, then K is a direct summand of $R^{(n)}$. Hence, Y_R is projective, see the proof of Proposition 3.4. □

In [**15**], rings over which all nonzero modules are retractable are studied. Hence, over such rings (e.g., commutative semi-Artinian rings), dual Baer modules are precisely semisimple modules.

Theorem 3.6. *Let R be a ring Morita invariant to a commutative ring.*

(i) *A finitely generated R-module is dual Baer if and only if it is semisimple.*
(ii) *If R is semi-Artinian, then dual Baer R-modules are precisely semisimple R-modules.*

Proof. By Theorem 2.2, we can suppose that R is a commutative ring. Thus by [**9**, Theorem 2.7], every finitely generated R-module is retractable. Also if R is semi-Artinian, then all nonzero R-modules are retractable [**9**, Theorem 2.8]. Thus the result is obtained by Theorem 2.6. □

An R-module M is called *divisible* if $Mc = M$ for every regular element in $c \in R$. Note that the class of divisible R-modules is closed under arbitrary direct sums and homomorphic images. Hence, if M is a divisible R-module and $X \in \mathrm{Gen}(M_R)$, then X is also a divisible R-module.

Proposition 3.7. *If R is a semiprime right Goldie ring, then every torsionfree injective module is dual Baer.*

Proof. Let M_R be a torsionfree injective R-module and $0 \to X \to M \to Y \to 0$ be an exact sequence of R-modules with $X \in \mathrm{Gen}(M)$. Since M_R is injective, it is divisible. It follows that X_R is divisible (for X is generated by M). Thus X_R is injective by [**8**, Proposition 6.12]. Hence, the exact sequence splits and the result holds by Theorem 2.2. □

Proposition 3.8. *If R is a right hereditary right Noetherian ring, then injective module is dual Baer.*

Proof. Note that since R is assumed to be right Noetherian, every direct sum of injective R-modules is injective. Hence, if M_R is injective, then so is every R-module in $\mathrm{Gen}(M)$ by the hereditary condition. The rest of the proof is similar to the proof of Proposition 3.7. □

For any right R-module M, the left R-module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is called the *character of* M and is denoted by M^+. We conclude the paper with further application of the Theorem 2.2 to show that if ${}_RM^+$ is Baer, then M_R has a condition close to the dual Baer condition. A submodule N of a module M_R is called *pure* if any system of equations $\sum_{i=1}^{t} x_i a_{ij} = n_j \in N$ $(j = 1, \dots, m,\ a_{ij} \in R)$ which is solvable in M, is also solvable in N, equivalently, if for any left R-module A, the homomorphism $i \otimes 1_A$ is one to one, where $i\colon N \to M$ is the inclusion map and $1_A\colon A \to A$ is the identity map. The exact sequence $0 \to X \xrightarrow{f} M \to Y \to 0$ is then called *pure exact* if the $\mathrm{Im}\, f$ is a pure submodule of M. Clearly, every direct summand of M_R is a pure submodule of M. Hence, in view of Theorem 2.3, we may consider the condition weaker than the dual Baer condition for a module M: $\mathrm{Tr}(M, X)$ *is a pure submodule* M_R *for any* $X \leq M_R$.

Proposition 3.9. *If M_R is a generator for Mod-R, then the left R-module M^+ is Baer if and only if it is a semisimple left R-module.*

Proof. Let M_R be a generator for Mod-R. Note that for every left R-module L, there exists a natural R-monomorphism from L into the left R-module L^{++}. It follows that M^+ is a co-generator for R-Mod. Because if X is a left R-module, then $X^+ \in \mathrm{Gen}(M)$ and so $X^{++} \in \mathrm{Cog}(M^+)$. Thus the result is obtained by Theorem 2.2. □

Proposition 3.10. *Let M be a non-zero R-module.*

(i) *The exact sequence $0 \to N \to M$ of R-modules is pure exact if and only if the sequence $M^+ \to N^+ \to 0$ splits.*
(ii) *Let M_R be flat. An exact sequence $0 \to X \to M \to Y \to 0$ of R-modules is pure if and only if Y_R is flat.*

Proof. (i) This follows from [**6**, Proposition 5.3.8]
(ii) It is true by [**12**, Corollary 4.86]. □

Theorem 3.11. *Suppose that M_R is a non-zero module such that the left R-module M^+ is Baer. Then $\mathrm{Tr}(M, X)$ is a pure submodule M_R for any $X \leq M_R$.*

Proof. Suppose that the left R-module M^+ is Baer. Let $X \leq M_R$ and $\mathrm{Tr}(M, X) = N$. Then $N \in \mathrm{Gen}(M)$. Consider the exact sequence $0 \to N \to M \to M/N \to 0$ in Mod-R. Note that since $N \in \mathrm{Gen}(M)$, we have $N^+ \in \mathrm{Cog}(M^+)$. Thus we obtain the exact sequence $0 \to (M/N)^+ \to M^+ \to N^+ \to 0$ in R-Mod with $N^+ \in \mathrm{Cog}(M^+)$ by [**12**, Proposition 4.8]. Now by the Baer condition on M^+ and Theorem 2.2, the sequence $M^+ \to N^+ \to 0$ splits. Hence, N is a pure submodule of M_R by Proposition 3.10(i). □

Proposition 3.12. *Let M be a non-zero R-module. If M_R is self-generator and the left R-module M^+ is Baer, then all submodules of M_R are pure.*

Proof. This follows from Theorem 3.11. □

We call an R-module M strongly torsionfree if $\mathrm{ann}_R(m) = 0$ for any $0 \neq m \in M$.

Theorem 3.13. *Suppose that M is a non-zero strongly torsionfree extending R-module. If the left R-module M^+ is Baer, then M_R is dual-Baer.*

Proof. We apply Theorem 2.3. Let X be a submodule of M_R and $N = \mathrm{Tr}(M, X)$. By Theorem 3.11, N is a pure submodule of M_R. We shall show that N is a direct summand of M_R. Hence, by the extending condition, it is enough to prove that N is an essentially closed submodule of M_R. Now suppose that N is essential in $K \leq M_R$ and $0 \neq k \in K$. Then there is $r \in R$ such that $0 \neq kr \in N$. Let $n = kr$ that means the equation $xr = n$ has solution in M and so by purity condition on N, the equation must have a solution in N. Therefore, there exists $n' \in N$ such that $n'r = n$. It follows that $(n' - k)r = 0$, hence $k = n' \in N$ by our assumption on M. Proving that $N = K$, as desired. □

It is well known that R is a von Neumann regular ring if and only if every right (left) ideal is pure in R_R (${}_RR$).

Corollary 3.14. *If the left R-module R^+ is Baer (semisimple), then R is a von Neumann regular ring.*

Proof. It is obtained by Proposition 3.12 and the above notes. □

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