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ROBUST SUFFICIENT OPTIMALITY CONDITIONS AND DUALITY IN SEMI-INFINITE MULTIOBJECTIVE PROGRAMMING WITH DATA UNCERTAINTY

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Abstract. A semi-infinite multiobjective programming problem in the face of data uncertainty in constraints is considered. Robust sufficient optimality conditions for weakly robust efficient, robust efficient and properly robust efficient solutions to the problem are established. Mond-Weir type dual model is formulated and appropriate duality results are obtained. The results are illustrated with a bi-objective uncertainty semi-infinite problem.

1. Introduction

In mathematical programs, the feasible set contains all those points that satisfy constraints of the problem, and the constraints further depend on the availability of the resources and environment. In the real life optimization problems, the input information may come from an uncertain or incomplete source. For example, multiobjective optimization problems arising in the financial market may involve varying costs, returns and future demands that might be unknown or hidden at the time of the decision making. So these parameters have to be replaced with some initial forecast or measurements.

Robust optimization approach is a deterministic approach that has come out to be a potent method for studying mathematical programming under uncertainty, associating a robust counterpart to an uncertain optimization problem. This method intends to find the solution by considering all the possible values of the parameters within their prescribed set of uncertainty including the worst case of all the existing scenarios. A detailed study of robust optimization can be found in [**3**]. Researchers have explored diverse aspects of such problems. Goberna et al. [**8**] obtained characterization of efficient, properly efficient as well as strongly efficient solutions to a linear semi-infinite multiobjective program. Along with these they

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discussed various applications including a linear robust vector program. Goberna et al. [**9**] studied the radius of feasibility, optimality and duality for a robust counterpart of a linear multiobjective problem affected by data uncertainty.

Goberna et al. [**10**] characterized weakly efficient solutions to a multiobjective convex uncertain problem by estimating bounds on the radius of feasibility. Huang and Chen [**11**] studied optimality conditions of a convex multiobjective problem with data uncertainty in both objectives and constraints via the weighted sum approach. Jeyakumar et al. [**12**] discussed optimality of an uncertain convex program using robust optimization approach by exploring various properties of optimal solutions to the robust problem. Lee and Lee [**17**] presented approximated (ϵ-) solutions to a robust program. They studied robust N and S optimality conditions and duality results for ϵ-solutions to a robust programming problem.

Duality theory has played a key role in convex programming and enough literature is available for programs in which data uncertainty has not been incorporated. But since the last decade, the study of duality theory for optimization problems with uncertain parameters has gained the interest of several authors around the globe. Beck and Ben-Tal [**2**] first obtained duality results for a robust problem linked to a convex data uncertainty problem. Kim [**15**] studied Mond-Weir type duality for a robust multiobjective program associated with an uncertainty problem. To get further insight into development of duality theory in robust programs, one can refer to the work of Chen et al. [**4**], Chuong [**5**], Jeyakumar et al. [**13**], and Jeyakumar and Li [**14**].

Semi-infinite programming problems involve optimizing a finite number of objectives with finite variables in the presence of infinitely many constraints. Lee and Lee [**16**] discussed optimality and duality for a robust semi-infinite programming problem. The functions involved were assumed to be convex which is a generalization of the robust linear programs presented in [**8, 9**]. Motivated by the work of Lee and Lee [**16**] and Tung [**18**], we study the robust optimality conditions and duality results for robust semi-infinite multiobjective programs in the generalized convex setting. The pseudoconvexity and quasiconvexity assumptions on the objectives and constraint functions respectively are used to discuss the mentioned results.

A brief outline of this paper is as follows. Section 2 deals with problem formulation and preliminaries. In Section 3, robust sufficient optimality conditions for the existence of robust weakly efficient, robust efficient, and properly robust properly efficient solutions are studied for an uncertain semi-infinite multiobjective program. Section 4 contains the duality results such as weak, strong and strict converse duality theorems. In Section 5, a numerical example is discussed.

2. Problem formulation and preliminaries

In this section, we present a semi-infinite multiobjective programming problem with data uncertainty in constraints along with some notations and definitions. We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space. For $\mathbb{R}^n$, $\mathbb{R}^n_+$ denotes the corresponding non-negative orthant and $\text{int}\mathbb{R}^n_+$ denotes the interior of $\mathbb{R}^n_+$. We shall first define the ordering in $\mathbb{R}^n$, as the problem involves vector functions. For $u, v \in \mathbb{R}^n$, we have the ordering

$$\begin{aligned}
&u \geqq v \Leftrightarrow u_j \geqq v_j \text{ for every } j = 1, 2, \dots, n;\\
&u \geq v \Leftrightarrow u_j \geqq v_j \text{ for every } j = 1, 2, \dots, n \text{ and } u_i > v_i \text{ for some } i = 1, 2, \dots, n;\\
&u > v \Leftrightarrow u_j > v_j \text{ for every } j = 1, 2, \dots, n;\\
&u = v \Leftrightarrow u_j = v_j \text{ for every } j = 1, 2, \dots, n.
\end{aligned}$$

A general semi-infinite multiobjective programming problem is as follows:

$$\begin{array}{lll}
\text{Problem (SIP)} & \text{Minimize} & f(x) = (f_1(x), f_2(x), \dots, f_k(x))\\
& \text{subject to} & g_t(x) \leq 0 \text{ for all } t \in T,
\end{array}$$

where $f_i, g_t \colon \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \dots, k$, $t \in T$ are continuously differentiable functions. The set T is an arbitrary index set (possibly infinite) and the feasible set of (SIP) is given by $F_S = \{x \in \mathbb{R}^n \colon\ g_t(x) \leq 0 \text{ for all } t \in T\}$.

We shall study the semi-infinite multiobjective programming problem with data uncertainty in constraints. It is assumed that the uncertainty factors lie in some compact sets and uncertainty of each constraint is independent of the uncertainty in the other constraints. Consider the following semi-infinite multiobjective program under data uncertainty (USIP):

$$\begin{array}{lll}
\text{Problem (USIP)} & \text{Minimize} & f(x) = (f_1(x), f_2(x), \dots, f_k(x))\\
& \text{subject to} & g_t(x, v_t) \leq 0 \text{ for all } t \in T,
\end{array}$$

where $f_i \colon \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \dots, k$ are same as in (SIP) and $g_t \colon \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $t \in T$, are continuously differentiable functions. In (USIP), $v_t \in \mathbb{R}^q$ is an uncertainty parameter which lies in some convex and compact set $\mathcal{V}_t \subset \mathbb{R}^q$, $t \in T$. The set $\mathcal{V}_t$ depends on $t \in T$. The set valued mapping $\mathcal{V} \colon T \rightrightarrows \mathbb{R}^q$ is defined as $\mathcal{V}(t) := \mathcal{V}_t$ for all $t \in T$. We say that v is an element of $\mathcal{V}$, when v is a selection from the elements of $\mathcal{V}$. This occurs in such a way that $v_t \in \mathcal{V}_t$ for all $t \in T$. So the points in the graph of $\mathcal{V}_t$ are of the type (t, v_t). Throughout, we assume that T is a compact metric space with $\mathcal{V}$ being compact valued and upper semi-continuous on T.

We write the following robust optimization problem (RSIP) to deal with data uncertainty:

$$\begin{array}{lll}
\text{Problem (RSIP)} & \text{Minimize} & f(x) = (f_1(x), f_2(x), \dots, f_k(x))\\
& \text{subject to} & g_t(x, v_t) \leq 0 \text{ for all } v_t \in \mathcal{V}_t,\ t \in T.
\end{array}$$

The functions f_i, g_t are the same as defined in (USIP). The feasible solutions are those elements of $\mathbb{R}^n$ which are feasible for any possible situation of uncertainty from the given uncertainty sets. The set of the robust feasible solutions, namely the robust feasible set, is defined as

$$F := \{x \in \mathbb{R}^n :\ g_t(x, v_t) \leq 0 \text{ for all } t \in T \text{ and } v_t \in \mathcal{V}_t\}.$$

Now, we give some basic definitions which will be used in the subsequent sections.

Definition 2.1. An element $x \in F$ is called a feasible solution to (RSIP) or a robust feasible solution to (USIP).

Definition 2.2. A feasible solution $u \in F$ is called weakly robust efficient solution to (USIP) or weakly efficient solution to (RSIP) if there does not exist any $x \in F$ such that $f(x) < f(u)$.

Definition 2.3. A feasible solution $u \in F$ is called robust efficient solution to (USIP) or an efficient solution to (RSIP) if there does not exist any $x \in F$ such that $f(x) \leq f(u)$.

Definition 2.4. A feasible solution $u \in F$ is said to be properly efficient solution to problem (RSIP) or properly robust efficient solution to problem (USIP) if it is efficient for (RSIP) and there exists a scalar $M > 0$, such that for each $x \in F$ and for each $j = 1, 2, \dots, k$, $f_j(x) < f_j(u)$,

$$\frac{f_j(u) - f_j(x)}{f_i(x) - f_i(u)} \leq M,$$

or equivalently

$$f_i(x) - f_i(u) \geq M_1(f_j(u) - f_j(x)) \qquad \text{for } M_1 = \frac{1}{M}$$

for some $i = 1, 2, \dots, k$ such that $f_i(x) > f_i(u)$. So we call such solutions to be properly robust efficient solutions to (USIP).

The above notion is introduced by Geoffrion [**7**], to rule out the existence of certain anomalous efficient solutions which lead to infinite loss in one objective for a small gain in another objective. Further, let us denote by $\mathbb{R}^T$ the following space of sequences

$$\mathbb{R}^T := \{\lambda = (\lambda_t)_{t\in T} : \ t \in T\}.$$

We denote by $\mathbb{R}^{(T)}$ the linear space of mappings of the type $\mu \in \mathbb{R}^T$ such that $\{t \in T : \mu_t \neq 0\}$ is a finite set. It can also be defined as

$$\mathbb{R}^{(T)} := \{\lambda = (\lambda_t)_{t\in T} : \ \lambda_t = 0, \ t \in T, \text{ except for finitely many } t \in T\}.$$

The non-negative cone of $\mathbb{R}^{(T)}$ is defined as follows:

$$\mathbb{R}^{(T)}_+ = \{\lambda = (\lambda_t)_{t\in T} \in \mathbb{R}^{(T)} : \lambda_t \geq 0, \ t \in T\}.$$

The definitions of generalized convex functions as discussed below will be used in achieving the main results of this article.

Definition 2.5. Assume a vector function $f \colon X \to \mathbb{R}^k$ and $X \subseteq \mathbb{R}^n$.

(i) f is said to be *pseudoconvex* at $u \in X$ if for all $x \in X$, we have

$$f(x) < f(u) \quad \Longrightarrow \quad \nabla f(u)(x-u) < 0.$$

(ii) f is said to be *quasiconvex* at $u \in X$ if for all $x \in X$, we have

$$f(x) \leq f(u) \quad \Longrightarrow \quad \nabla f(u)(x-u) \leq 0.$$

(iii) f is said to be *strongly pseudoconvex* at $u \in X$ if for all $x \in X$, we have

$$f(x) \leq f(u) \quad \Longrightarrow \quad \nabla f(u)(x-u) \leq 0.$$

(iv) f is said to be *strictly pseudoconvex* at $u \in X$ if for all $x \in X$, we have

$$f(x) \leq f(u) \quad \Longrightarrow \quad \nabla f(u)(x-u) < 0.$$

(v) f is said to be *weak strictly pseudoconvex* at $u \in X$ if for all $x \in X$, we have

$$f(x) \leq f(u) \quad \Longrightarrow \quad \nabla f(u)(x-u) < 0.$$

Definition 2.6 ([**12, 16**]). Let $x \in F$ and $T_1(x) = \{t \in T : \exists\, v_t \in \mathcal{V}_t$ such that $g_t(x, v_t) = 0\}$ and let $\mathcal{V}_t(x) = \{v_t \in \mathcal{V}_t : g_t(x, v_t) = 0\}$. We say that the Extended Mangasarian-Fromovitz Constraint Qualification (EMFCQ) is satisfied at $x \in F$ if and only if there exists $d \in \mathbb{R}^n$ such that

$$\nabla_x g_t(x, v_t)^T d < 0 \qquad \text{for all } t \in T_1(x),\ v_t \in \mathcal{V}_t(x).$$

3. Robust sufficient optimality conditions

In this section, we discuss robust sufficient optimality conditions for a feasible solution to be weakly robust efficient, robust efficient, or a properly robust efficient solution.

Theorem 3.1. *Let $\bar{x}$ be a robust feasible solution to (USIP). Suppose that there exist $\mu \in \mathbb{R}^k_+ \setminus \{0\}$, $(\lambda_t)_{t\in T} \in \mathbb{R}^{(T)}_+$ and $v_t \in \mathcal{V}_t$, $t \in T$ such that*

$$\sum_{i=1}^{k} \mu_i \nabla f_i(\bar{x}) + \sum_{t\in T} \lambda_t \nabla_x g_t(\bar{x}, v_t) = 0, \tag{1}$$

$$\sum_{t\in T} \lambda_t g_t(\bar{x}, v_t) = 0. \tag{2}$$

Further, suppose that $\sum_{t\in T} \lambda_t g_t(\cdot, v_t)$ is quasiconvex at $\bar{x}$ and the function f is pseudoconvex at $\bar{x}$. Then $\bar{x}$ is a weakly robust efficient solution to (USIP).

Proof. Assume to the contrary that, $\bar{x}$ is not a weakly robust efficient solution to (USIP), i.e., $\bar{x}$ is not a weakly efficient solution to (RSIP). This means that there is an $x \in F$ such that

$$f(x) < f(\bar{x}). \tag{3}$$

The pseudoconvexity of f at $\bar{x}$ gives

$$\nabla f(\bar{x})(x - \bar{x}) < 0. \tag{4}$$

Since $\mu \geq 0$, therefore

$$\nabla(\mu^T f(\bar{x}))(x-\bar{x}) < 0 \quad \Longleftrightarrow \quad \sum_{i=1}^{k} \mu_i \nabla f_i(\bar{x})(x-\bar{x}) < 0. \tag{5}$$

Now as $x \in F$ and for $(\lambda_t)_{t\in T} \in \mathbb{R}^{(T)}_+$, we get that $\lambda_t g_t(x, \bar{v}_t) \leq 0$, $t \in T$. This along with (2) yields

$$\sum_{t\in T} \lambda_t g_t(x, v_t) - \sum_{t\in T} \lambda_t g_t(\bar{x}, v_t) \leq 0, \qquad t \in T. \tag{6}$$

The quasiconvexity of $\sum\limits_{t\in T} \lambda_t g_t(\cdot, v_t)$ at $\bar{x}$ implies

$$\sum_{t\in T} \lambda_t \nabla_x g_t(\bar{x}, v_t)(x-\bar{x}) \leq 0, \qquad t \in T \tag{7}$$

which along with (1) entails

$$\sum_{i=1}^{k} \mu_i \nabla f_i(\bar{x})(x-\bar{x}) = -\sum_{t\in T} \lambda_t \nabla_x g_t(\bar{x}, v_t)(x-\bar{x}) \geq 0. \tag{8}$$

But (8) is a contradiction to (5). Hence $\bar{x}$ is a weakly robust efficient solution to (USIP). □

Theorem 3.2. *Let $\bar{x}$ be a robust feasible solution to (USIP). Suppose that, there exist $\mu \in \mathbb{R}^k_+ \setminus \{0\}$, $(\lambda_t)_{t\in T} \in \mathbb{R}^{(T)}_+$ and $v_t \in \mathcal{V}_t$, $t \in T$ such that* (1) *and* (2) *hold at $\bar{x}$. Further, suppose that the function $\sum_{t\in T} \lambda_t g_t(\cdot, v_t)$ is quasiconvex at $\bar{x}$. If any of the following assumptions hold:*

(i) *f is weak strictly pseudoconvex at $\bar{x}$ or*
(ii) *f is strongly pseudoconvex at $\bar{x}$ and $\mu \in \operatorname{int} \mathbb{R}^k_+$.*

Then $\bar{x}$ is a robust efficient solution to (USIP).

Proof. Suppose for the sake of contradiction that $\bar{x}$ is not a robust efficient solution to (USIP). This means that there exists an $x \in F$ such that

$$f(x) \leq f(\bar{x}). \tag{9}$$

(i) Since f is weak strictly pseudoconvex at $\bar{x}$, we get $\nabla f(\bar{x})^T(x-\bar{x}) < 0$. This along with $\mu \in \mathbb{R}^k_+ \setminus \{0\}$ implies

$$\sum_{i=1}^{k} \mu_i \nabla f_i(\bar{x})(x-\bar{x}) < 0. \tag{10}$$

(ii) In this part, since f is strongly pseudoconvex at $\bar{x}$, (9) implies $\nabla f(\bar{x})^T(x-\bar{x}) \leq 0$. This along with $\mu \in \operatorname{int} \mathbb{R}^k_+$ gives

$$\sum_{i=1}^{k} \mu_i \nabla f_i(\bar{x})(x-\bar{x}) < 0. \tag{11}$$

Now using the same approach as in Theorem 3.1, we arrive at

$$\sum_{i=1}^{k} \mu_i \nabla f_i(\bar{x})(x-\bar{x}) = -\sum_{t\in T} \lambda_t \nabla_x g_t(\bar{x}, v_t)(x-\bar{x}) \geq 0, \tag{12}$$

which contradicts both equations (10) and (11). This means that $\bar{x}$ is a robust efficient solution to (USIP). □

The following lemma ([**7**]) will be used to obtain properly efficient solutions to (RSIP).

Lemma 3.3 ([**7**]). *Given a set* $F \subseteq \mathbb{R}^n$ *of feasible points and a function* $h\colon \mathbb{R}^n \to \mathbb{R}^k$*. If for some fixed* $\mu(>0)$ *in* $\mathbb{R}^k$*,* $\bar{x}$ *is an optimal solution to the following scalar minimization problem*

$$(P_\lambda) \qquad \text{Min } \mu^T h(x) \quad \textit{subject to } x \in F,$$

then $\bar{x}$ *is a properly efficient solution to the following multiobjective problem* (P)*:*

$$(P) \qquad \text{Min } h(x) = (h_1(x), h_2(x), \dots, h_k(x)) \quad \textit{subject to } x \in F.$$

Theorem 3.4. *Let* $\bar{x}$ *be a feasible solution to (RSIP). Suppose that there exist* $\mu > 0$*,* $(\lambda_t)_{t\in T} \in \mathbb{R}_+^{(T)}$ *and* $v_t \in \mathcal{V}_t$*,* $t \in T$ *such that* (1) *and* (2) *hold at* $\bar{x}$*. Further suppose that the functions* $\sum_{t\in T} \lambda_t g_t(\cdot, v_t)$ *is quasiconvex at* $\bar{x}$ *and the function* $\mu^T f$ *is pseudoconvex at* $\bar{x}$*. Then* $\bar{x}$ *is a properly efficient solution to (RSIP).*

Proof. It is given that $\bar{x} \in F$ and that $\sum\limits_{t\in T} \lambda_t g_t(\cdot, v_t)$ is quasiconvex at $\bar{x}$. Then along the lines of the proof of Theorem 3.1, we get

$$\sum_{i=1}^{k} \mu_i \nabla f_i(\bar{x})(x - \bar{x}) = -\sum_{t\in T} \lambda_t \nabla_x g(\bar{x}, v_t)(x - \bar{x}) \geq 0. \tag{13}$$

Relation (13) implies

$$\nabla(\mu^T f(\bar{x}))(x - \bar{x})^T \geq 0. \tag{14}$$

This, along with pseudoconvexity of $\mu^T f$, provides us with the following inequality

$$\mu^T f(x) \geq \mu^T f(\bar{x}) \qquad \text{for all } x \in F. \tag{15}$$

This implies that $\bar{x}$ is an optimal solution to (RSIP$_\lambda$) which by Lemma 3.3 implies that $\bar{x}$ is a properly efficient solution to (RSIP). Hence the theorem. □

4. Robust dual model

We present the following Mond-Weir type dual for robust semi-infinite program (RSIP):

$$\begin{aligned}
\text{Dual (RSIMWD)} \qquad & \text{Maximize} \quad f(u) = (f_1(u), f_2(u), \dots, f_k(u)) \\
& \text{subject to} \quad \sum_{i=1}^{k} \alpha_i \nabla f_i(u) + \sum_{t\in T} \lambda_t \nabla_x g_t(u, v_t) = 0, \\
& \qquad\qquad\quad \sum_{t\in T} \lambda_t g_t(u, v_t) \geq 0, \quad t \in T, \\
& \qquad\qquad\quad \lambda_t \in \mathbb{R}_+^{(T)}, \quad \alpha \in \mathbb{R}_+^k \setminus \{0\}, \quad v_t \in \mathcal{V}_t.
\end{aligned}$$

In the absence of uncertainty parameters, the above Mond-Weir dual is obtained by Chuong and Kim [**6**] for a semi-infinite program (RSIP). Now, we discuss the weak, strong and strict converse duality theorems under pseudoconvexity and quasiconvexity.

Theorem 4.1 (Weak Duality). *Let x be a feasible solution to (RSIP) and $(u, \alpha, \lambda_t, v_t)$ be feasible solution to (RSIMWD). Suppose that $\sum_{t\in T} \lambda_t g_t(\cdot, v_t)$ is quasiconvex at u. Furthermore,*

(i) *if $f(\cdot)$ is pseudoconvex at u, then the following cannot hold*

$$f(x) < f(u), \tag{16}$$

(ii) *if $f(\cdot)$ is strongly pseudoconvex at u and $\alpha > 0$, then the following cannot hold*

$$f(x) \leq f(u), \tag{17}$$

(iii) *if $f(\cdot)$ is weak strictly pseudoconvex at u, then the following cannot hold*

$$f(x) \leq f(u). \tag{18}$$

Proof. The feasibility of x for (RSIP) implies, $\sum_{t\in T} \lambda_t g_t(x, v_t) \leq 0$ and the dual feasibility of $(u, \alpha, \lambda_t, v_t)$ gives $\sum_{t\in T} \lambda_t g_t(u, v_t) \geq 0$ for $\lambda_t \in \mathbb{R}_+^{(T)}$. Combining, we get

$$\sum_{t\in T} \lambda_t g_t(x, v_t) - \sum_{t\in T} \lambda_t g_t(u, v_t) \leq 0. \tag{19}$$

Since $\sum_{t\in T} \lambda_t g_t(\cdot, v_t)$ is quasiconvex at u, we get

$$\nabla_x\Big(\sum_{t\in T} \lambda_t g_t(u, v_t)\Big)(x-u) \leq 0 \quad \Longleftrightarrow \quad \sum_{t\in T} \lambda_t \nabla_x g_t(u, v_t)(x-u) \leq 0. \tag{20}$$

Now by first dual feasibility condition, we have

$$\sum_{i=1}^{k} \alpha_i \nabla f_i(u) = -\sum_{t\in T} \lambda_t \nabla_x g_t(u, v_t), \tag{21}$$

hence

$$\sum_{i=1}^{k} \alpha_i \nabla f_i(u)(x-u) = -\sum_{t\in T} \lambda_t \nabla_x g_t(u, v_t)(x-u) \geq 0 \qquad \text{(by (20))}. \tag{22}$$

Now, we justify (i)–(iii).

(i) Suppose to the contrary that $f(x) - f(u) < 0$. The pseudoconvexity of f at u implies

$$\nabla f(u)(x-u) < 0 \quad \Longrightarrow \quad \sum_{i=1}^{k} \alpha_i \nabla f_i(u)(x-u) < 0 \quad (\text{for } \alpha \in \mathbb{R}_+^k \setminus \{0\}). \tag{23}$$

But (23) and (22) contradict each other. So, we conclude that $f(x) - f(u) < 0$ is not true. Hence, we have proved the first part of the theorem.

(ii) In this part also, if we assume that the result does not hold, i.e., $f(x) \leq f(u)$. As f is strongly pseudoconvex at u, we have

$$\nabla f(u)(x-u) \leq 0.$$

This along with $\alpha > 0$ implies

$$\sum_{i=1}^{k} \alpha_i \nabla f_i(u)(x-u) < 0. \tag{24}$$

The rest of the proof is the same as in part (i).

(ii) The proof of this part also is the same as in part (i). □

Theorem 4.2 (Strict Converse Duality). *Let $\bar{x}$ be a feasible solution to (RSIP) and $(\bar{u}, \bar{\alpha}, \bar{\lambda}_t, \bar{v}_t)$ be a feasible solution to (RSIMWD). Let $\sum_{t\in T} \bar{\lambda}_t g_t(\cdot, \bar{v}_t)$ be quasiconvex at $\bar{u}$ and $\bar{\alpha}^T f$ be strictly pseudoconvex at $\bar{u}$. If*

$$\sum_{i=1}^{k} \bar{\alpha}_i f_i(\bar{x}) = \sum_{i=1}^{k} \bar{\alpha}_i f_i(\bar{u}), \tag{25}$$

then $\bar{x} = \bar{u}$.

Proof. Assume that $\bar{x} \neq \bar{u}$. As $(\bar{u}, \bar{\alpha}, \bar{\lambda}_t, \bar{v}_t)$ is a feasible solution to (RSIMWD) and $\bar{x}$ is a feasible solution to (RSIP) for $\bar{\lambda}_t \in \mathbb{R}_+^{(T)}$, we obtain

$$\sum_{t\in T} \bar{\lambda}_t g_t(\bar{x}, \bar{v}_t) - \sum_{t\in T} \bar{\lambda}_t g_t(\bar{u}, \bar{v}_t) \leq 0, \qquad t \in T. \tag{26}$$

The above inequality along with the quasiconvexity of $\sum_{t\in T} \bar{\lambda}_t g_t(\cdot, \bar{v}_t)$ gives

$$\sum_{t\in T} \bar{\lambda}_t \nabla_x g_t(\bar{u}, \bar{v}_t)(\bar{x} - \bar{u}) \leq 0, \qquad t \in T. \tag{27}$$

Inequality (27) and the first dual feasibility condition yield

$$\sum_{i=1}^{k} \bar{\alpha}_i \nabla f_i(\bar{u})(\bar{x} - \bar{u}) \geq 0. \tag{28}$$

Since $\bar{\alpha}^T f$ is strictly pseudoconvex at $\bar{u}$, (28) implies

$$\sum_{i=1}^{k} \bar{\alpha}_i f_i(\bar{x}) > \sum_{i=1}^{k} \bar{\alpha}_i f_i(\bar{u}). \tag{29}$$

Comparing (25) with (29), we arrive at a contradiction. Thus the proof is completed. □

We assume that $g_{t_m}(x_m, v_{t_m}) \to g_t(x, v_t)$ and $\nabla g_{t_m}(x_m, v_{t_m}) \to \nabla g_t(x, v_t)$ whenever $T \ni t_m \to t \in T$, $\mathcal{V}_{t_m} \ni v_{t_m} \to v_t \in \mathcal{V}_t$, and $\mathbb{R}^n \ni x_m \to x \in \mathbb{R}^n$ as $m \to \infty$, in view of necessary optimality conditions being used for strong duality theorem (as discussed by Lee and Lee [**16**]).

Theorem 4.3 (Strong Duality). *Let $\bar{u}$ be a weakly efficient solution to (RSIP) and for each $x \in \mathbb{R}^n$ and $t \in T$, $g_t(x, \cdot)$ be concave on $\mathcal{V}_t$. Also, suppose that (EMFCQ) hold at $\bar{u}$. Then there exist $(\bar{\alpha}, \bar{\lambda}_t, \bar{v}_t) \in \mathbb{R}_+^k \setminus \{0\} \times \mathbb{R}_+^{(T)} \times \mathcal{V}_t$ such that $(\bar{u}, \bar{\alpha}, \bar{\lambda}, \bar{v}_t)$ is a feasible solution to the Mond-Weir dual problem (RSIMWD). Further, if the assumptions of weak duality* Theorem 4.1(i) *hold at every feasible solution of dual, then $(\bar{u}, \bar{\alpha}, \bar{\lambda}_t, \bar{v}_t)$ is also a weakly efficient solution to (RSIMWD).*

Proof. If $\bar{u}$ is a weakly efficient solution to (RSIP) and the constraint qualification (EMFCQ) is satisfied at $\bar{u}$, then by necessary optimality conditions [**16**], there exist $\bar{\alpha} \geq 0$, $\bar{\lambda}_t \in \mathbb{R}_+^{(T)}$, $\bar{v}_t \in \mathcal{V}_t$, $t \in T$ and $\sum_{i=1}^k \bar{\alpha}_i = 1$ such that

$$\sum_{i=1}^{k} \bar{\alpha}_i \nabla f_i(\bar{u}) + \sum_{t \in T} \bar{\lambda}_t \nabla_x g_t(\bar{u}, \bar{v}_t) = 0, \tag{30}$$

$$\sum_{t \in T} \bar{\lambda}_t g_t(\bar{u}, \bar{v}_t) = 0. \tag{31}$$

Equations (30) and (31) give that $(\bar{u}, \bar{\alpha}, \bar{\lambda}_t, \bar{v}_t)$ is a feasible solution to (RSIMWD). Now, we will show that this point is a weakly efficient solution to (RSIMWD). The proof is as follows:

If this is not a weakly efficient solution to the dual, then there exists a feasible solution to (RSIMWD), say $(\tilde{u}, \tilde{\alpha}, \tilde{\lambda}_t, \tilde{v}_t)$, such that $f(\tilde{u}) > f(\bar{u})$. In wake of $\bar{u}$ being a feasible solution to primal, we get a contradiction to weak duality Theorem 4.1(i). Hence $(\bar{u}, \bar{\alpha}, \bar{\lambda}_t, \bar{v}_t)$ is a weakly efficient solution to (RSIMWD). This completes the proof of the theorem. □

Note. Fractional programming problems have been of better advantage in real life, as ratio optimization often describe an efficiency measure for a system. Such problems arise in management decision making, warehouse personnel allocation, information theory and in routing problems which tend to the maximization of profit to time ratio or the minimization of cost to time ratio. A recent study in fractional programs may be found in [**1**]. The results of Sections 3 and 4 may be extended for a fractional analogue of the problem (USIP). However the proofs of parallel results are left as an exercise to the readers.

5. An illustrative example

We discuss the main results obtained in the paper with the help of the following robust bi-objective program:

$$\text{(UP)} \qquad \begin{aligned} &\text{Minimize} \quad f(x) = (x^3 + x, -2x) \\ &\text{subject to} \quad 2xv_t - 3 \leq 0, \end{aligned}$$

where v_t is an uncertainty parameter such that $v_t \in \mathcal{V}_t = [-t, t+1]$ and $t \in T = [1, 2]$. The robust counterpart is

$$\text{(RP)} \qquad \begin{aligned} &\text{Minimize} \quad f(x) = (x^3 + x, -2x) \\ &\text{subject to} \quad 2xv_t - 3 \leq 0, \\ &\qquad\qquad\quad \text{for all } v_t \in \mathcal{V}_t,\ t \in T,\ x \in \mathbb{R}, \end{aligned}$$

where the set of feasible solutions is given by

$$F_{RP} = \{x \in \mathbb{R},\ 2xv_t - 3 \leq 0,\ v_t \in [-t, t+1],\ t \in T = [1, 2]\}.$$

We obtain the feasible region to be $[-\frac{3}{4}, \frac{1}{2}]$ for the set F_{RP}. We consider the point $\bar{x} = \frac{1}{2}$ such that

$$\begin{aligned}\sum_{t\in T} \lambda_t g_t(x, v_t) - \sum_{t\in T} \lambda_t g_t\Big(\frac{1}{2}, v_t\Big) &= \sum_{t\in T} \lambda_t (2xv_t - 3 - \Big(2\Big(\frac{1}{2}\Big)(v_t) - 3)\Big) \\ &= \sum_{t\in T} \lambda_t v_t (2x - 1) \leq 0.\end{aligned} \tag{32}$$

It follows

$$\begin{aligned}\nabla_x\Big(\sum_{t\in T} \lambda_t g_t(x, v_t)\Big)(x - \bar{x}) &= \nabla_x\Big(\sum_{t\in T} \lambda_t (2xv_t - 3)\Big)\Big(x - \frac{1}{2}\Big) \\ &= \sum_{t\in T} \lambda_t (2v_t)\Big(x - \frac{1}{2}\Big) \\ &= \sum_{t\in T} \lambda_t v_t (2x - 1) \leq 0 \quad \text{(by (32))}.\end{aligned} \tag{33}$$

Hence, $\sum_{t\in T} \lambda_t g_t(\cdot, v_t)$ is quasiconvex at $\bar{x} = \frac{1}{2}$. The function $f(x) = (x^3+x, -2x)$ is pseudoconvex at $\bar{x} = \frac{1}{2}$. For $\bar{x} = \frac{1}{2}$, $\mu = (\frac{1}{2}, \frac{1}{2})$, $\lambda_t = \frac{1}{48}$ for $t = 2$ and $\lambda_t = 0$ for all $t \in [1, 2)$, and $v_t = 3$. So the equations (1) and (2) are satisfied. Hence, we conclude that $\bar{x} = \frac{1}{2}$ is a weakly efficient solution to (RP) and hence a weakly robust efficient solution of (UP).

The robust Mond-Weir type dual for the above primal (RP) is written as follows:

$$\begin{aligned}\text{(RD)} \qquad \text{Maximize} \quad & f(x) = (u^3 + u, -2u) \\ \text{subject to} \quad & 3\alpha_1 u^2 + \alpha_1 - 2\alpha_2 + 2\sum_{t\in T} \lambda_t v_t = 0, \\ & \sum_{t\in T} \lambda_t (2xv_t - 3) \geq 0, \\ & v_t \in [-t, t+1], \quad t \in [1, 2], \\ & x \in \mathbb{R}, \quad \alpha_1, \alpha_2 \geq 0, \quad \alpha \neq 0.\end{aligned}$$

$u = \frac{1}{2}$ is a feasible solution to (RP) and $(\frac{1}{2}, (\frac{1}{2}, \frac{1}{2}), \frac{1}{48}, 3)$ satisfies the feasibility conditions of the dual model (RD). The assumptions of weak duality theorem (i) hold at these points. Hence the weak duality relation holds between (RP) and (RD).

Now at $u = \frac{1}{2}$, we have $T_1(x) = \{t \in T : \exists\, v_t \in \mathcal{V}_t \text{ s.t. } g_t(x, v_t) = 0\} = \{2\}$ and $\mathcal{V}_t(x) = \{v_t \in \mathcal{V}_t : g_t(x, v_t) = 0\} = \{3\}$. Then for $d = -1$, $\nabla_x g_t(x, v_t)^T d < 0$, for all $t \in T_1(x)$, $v_t \in \mathcal{V}_t(x)$. Rest of the assumptions followed from weak duality imply that $(\frac{1}{2}, (\frac{1}{2}, \frac{1}{2}), \frac{1}{48}, 3)$ is a weakly efficient solution of the dual (RD).

6. Conclusions

In this paper, a robust counterpart of an uncertain semi-infinite multiobjective programming problem has been studied. By using pseudoconvexity and quasiconvexity assumptions on the functions involved, we devised optimality conditions that identify the efficient solutions to the given program. We have studied sufficient optimality conditions for three types of solutions to (RSIP) namely, weakly robust efficient solution, robust efficient solution, and properly robust efficient solution. Then, we have constructed the Mond-Weir type dual program for a robust semi-infinite programming problem and discussed weak, strong, and strict converse duality theorems to establish the relationship between the primal and its dual problem. A numerical example of weak duality theorem has been also provided. We have concluded this paper with an extension of the above-discussed program to fractional programming problems. This paper opens different avenues for the future work of the researchers. For example, this work can be extended to the study of a problem with uncertainty into the objective function also. Also, a new program can be introduced which has non-differentiable objective functions and constraints or adding some non-differentiable functions to the respective functions.

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