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SYMPLECTIC LIE GROUPS AND DOUBLED GEOMETRY

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Abstract. A left invariant flat para-Kähler structure is constructed on the tangent Lie group of a symplectic Lie group (G, ω). Remarkably, it is shown that the left invariant para-Kähler form on TG coincides with a certain pullback of the standard symplectic form on T^*G. The immediate upshot of this is that T^*G can be equipped with a Lie group structure for which the standard symplectic form is left invariant. Lastly, the double field theory geometry of the double manifold TG is studied using the geometric framework of Vaisman [**22, 23**].

1. Introduction

Paracomplex geometry (with particular emphasis on para-Hermitian and para-Kähler geometry) (cf. [**6**]) has seen something of a resurgence in the last decade due to an idea from string theory called double field theory (see [**12, 10, 11**] and the references therein). Formally, double field theory is a field theory where the symmetry of T-duality is made manifest. The mathematical framework of double field theory centers around a 'double manifold' where one takes the local coordinates $(x^1, \dots, x^n)$ of ordinary spacetime and adds 'dual' coordinates $(\tilde{x}_1, \dots, \tilde{x}_n)$. The physics literature on double field theory implicitly associates the following characteristics to a double manifold:

1. The double manifold is covered by a set of 'distinguished' local coordinate systems $(x^i, \tilde{x}_j)$. If $(x^i, \tilde{x}_j)$ and $(y^i, \tilde{y}_j)$ are overlapping distinguished coordinate systems, then
$$\frac{\partial y^i}{\partial \tilde{x}_j} = 0, \qquad \frac{\partial \tilde{y}_j}{\partial x^i} = 0, \qquad \frac{\partial \tilde{x}_i}{\partial \tilde{y}_j} = \frac{\partial y^j}{\partial x^i}.$$
As Vaisman noted in [**22**], the third condition necessarily implies that $\frac{\partial \tilde{x}_i}{\partial \tilde{y}_j}$ and $\frac{\partial y^j}{\partial x^i}$ are locally constant. In particular, a double manifold must be a type of affine manifold.

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2. The double manifold admits a flat metric η of signature (n,n), where n is the dimension of the original space. With respect to distinguished local coordinates $(x^i, \tilde{x}_j)$, η is given locally by

$$\eta = \mathrm{d}x^i \otimes \mathrm{d}\tilde{x}_i + \mathrm{d}\tilde{x}_i \otimes \mathrm{d}x^i.$$

Ultimately, in [**22**] it was Vaismann who was the first to realize that double manifolds were actually a type of para-Hermitian manifold. More precisely, Vaisman identified double manifolds with flat para-Kähler manifolds. Within this framework, Vaisman expressed the basic ideas of double field theory in a differential geometric invariant manner. Shortly after [**22**], Vaisman showed that the double field theory geometry he had defined for flat para-Kähler manifolds could be generalized to (non-flat) para-Hermitian manifolds [**23**]. Whether this more general setting had any relevance to string theory was a question that Vaisman left to the physicists. Regardless, the geometry of double field theory that was formulated within this para-Hermitian framework is quite rich from a purely mathematical perspective. Notably, it has a clear relationship to Hitchin's generalized geometry [**9, 15**] and to objects such as Courant algebroids [**22, 23, 5, 8, 16**].

The work of Vaisman [**22**] serves as motivation for the study of flat para-Kähler manifolds. Following Vaisman, we use the terms 'double manifold' and 'flat para-Kähler manifold' interchangeably especially when discussing double field theory. In this paper, we show that the tangent bundle of a symplectic Lie group admits a left invariant flat para-Kähler metric, where the tangent bundle is equipped with its natural Lie group structure. Hence, the tangent bundle of a symplectic Lie group represents a genuine double manifold for double field theory. It follows immediately from this that by taking higher iterations of tangent bundles, $T^{k+1}G := T(T^kG)$, where $T^0G := G$ is a (nonabelian) symplectic Lie group, one can generate (nonabelian) Lie groups of arbitrarily high dimension which admit left invariant flat para-Kahler metrics.

The rest of this paper is organized as follows. In section 2, we give a lightening review of the necessary background for this paper. In Section 3, we construct a left invariant flat para-Kähler structure on the tangent bundle of a symplectic Lie group. In Section 4, we study the relationship between the left invariant flat para-Kähler structure on the tangent bundle TG of a symplectic Lie group, and the cotangent bundle and its standard symplectic form. In particular, we show that the para-Kähler form on TG arises as a certain pullback of the standard symplectic form on T^*G. A nice consequence of this result is that T^*G can be endowed with a Lie group structure for which the standard symplectic form is left invariant. Lastly, in Section 5, we study some of the double field theory geometry of the double manifold TG for G a symplectic Lie group.

2. Preliminaries

In this section, we recall the relevant background and fix our notation and conventions for the rest of the paper. Throughout this paper, we employ the Einstein summation convention of summing over repeated indices unless stated explicitly otherwise. For a smooth map $\varphi\colon M \to N$, we denote the corresponding pushforward between tangent bundles by $\varphi_*\colon TM \to TN$. For a vector field A on M, we denote its value at $p \in M$ by $A_p \in T_pM$. For a Lie group G, the identity element is denoted as e. The left and right translation maps by an element $g \in G$ is denoted by $l_g\colon G \to G$ and $r_g\colon G \to G$, respectively. We set $\mathfrak{g} := T_eG$ and equip $\mathfrak{g}$ with the Lie algebra structure of the space of left invariant vector fields on G. With some abuse of notation, we often regard $X \in \mathfrak{g}$ as the left invariant vector field whose value at e is X. Lastly, for a vector bundle $E \to M$, we denote the sections of E by $\Gamma(E)$.

2.1. Symplectic Lie groups

In this section, we give a quick overview of symplectic Lie groups. A more comprehensive review can be found in [**3**].

Definition 2.1. A symplectic Lie group is a Lie group G together with a left invariant symplectic form ω. The pair $(\mathfrak{g}, \omega_e)$ is called a symplectic Lie algebra.

Left invariance implies that the structure of a symplectic Lie group is essentially determined at the Lie algebra level. With abuse of notation, we drop the subscript 'e' from ω_e and simply write $(\mathfrak{g}, \omega)$ for the associated symplectic Lie algebra. The cocycle condition for ω is

$$-\mathrm{d}\omega(X,Y,Z) = \omega([X,Y],Z) + \omega([Y,Z],X) + \omega([Z,X],Y) = 0 \quad \text{for all } X,Y,Z \in \mathfrak{g}.$$

Fixing a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ and writing $[X_i, X_j] = c_{ij}^k X_k$ and $\omega_{ij} = \omega(X_i, X_j)$, the cocycle condition for ω in component form is

$$c_{ij}^l \omega_{lk} + c_{jk}^l \omega_{li} + c_{ki}^l \omega_{lj} = 0. \tag{2.1.1}$$

Denoting the inverse $(\omega^{ij}) := (\omega_{ij})^{-1}$, the cocycle condition can be rewritten as

$$c_{ia}^l \omega_{lj} \omega^{am} = c_{ij}^m + c_{ja}^l \omega_{li} \omega^{am}. \tag{2.1.2}$$

We conclude this section with some examples.

Example 2.2. The Lie group of affine transformations on $\mathbb{R}^n$ is the Lie group of $(n+1) \times (n+1)$ real matrices given by

$$\mathrm{Aff}(n) = \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \mid A \in GL(n, \mathbb{R}),\ v \in \mathbb{R}^n \right\}.$$

Its corresponding Lie algebra is then

$$\mathfrak{aff}(n) = \left\{ \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix} \mid A \in \mathfrak{gl}(n, \mathbb{R}),\ v \in \mathbb{R}^n \right\}.$$

A natural basis for $\mathfrak{aff}(n)$ is $\{E_{ij} \mid 1 \leq i \leq n,\ 1 \leq j \leq n+1\}$, where E_{ij} is the $(n+1) \times (n+1)$ matrix whose elements are all zero except for the (i,j) element which is 1. Let

$$\alpha = E_{12}^* + E_{23}^* + \cdots + E_{n,n+1}^*.$$

Let ω be the left invariant 2-form on $\mathrm{Aff}(n)$ given by

$$\omega(X,Y) := -\mathrm{d}\alpha(X,Y) = \alpha([X,Y]) \qquad \text{for all } X,Y \in \mathfrak{aff}(n).$$

As an exact 2-form, ω is automatically closed. With some additional work, one can also show that ω is nondegenerate (cf. [**1**]). Hence, $(\mathrm{Aff}(n),\omega)$ is a symplectic Lie group.

Example 2.3. Let $\mathfrak{h}_4$ be the 4-dimensional Lie algebra with basis e_1, e_2, e_3, e_4 whose nonzero bracket relations are given by

$$[e_1,e_2] = e_3, \quad [e_4,e_3] = e_3, \quad [e_4,e_1] = \frac{1}{2}e_1, \quad [e_4,e_2] = e_1 + \frac{1}{2}e_2.$$

Define $\omega \in \wedge^2\mathfrak{h}_4^*$ by

$$\omega = a(e_1^* \wedge e_2^* - e_3^* \wedge e_4^*) + be_1^* \wedge e_4^* + ce_2^* \wedge e_4^* \qquad a \neq 0.$$

One can show that the pair $(\mathfrak{h}_4,\omega)$ is a symplectic Lie algebra (cf. [**18**]). Hence, any Lie group whose Lie algebra is (isomorphic to) $\mathfrak{h}_4$ is a symplectic Lie group with left invariant symplectic form given by ω.

2.2. Para-Kähler Geometry

Paracomplex geometry (cf. [**6**]) is essentially the 'real' analog of complex geometry. Recall that in complex geometry, one begins with the notion of an almost complex manifold which is a smooth manifold M together with an endomorphism $J\colon TM \to TM$ satisfying $J^2 = -\,\mathrm{id}$. The morphism implies a decomposition of the complexified tangent bundle into its $\pm i$ eigenbundles:

$$TM_{\mathbb{C}} = TM^{(1,0)} \oplus TM^{(0,1)}.$$

The almost complex manifold (M,J) is a complex manifold if and only if $TM^{(1,0)}$ (and hence $TM^{(0,1)}$) are involutive. In this case, J is then called an integrable almost complex structure or simply a complex structure. The integrability of J is equivalent to the vanishing of the Nijenhuis tensor

$$N_J(X,Y) := J[X,JY] + J[JX,Y] + [X,Y] - [JX,JY] \qquad \text{for all } X,Y \in \Gamma(TM).$$

In paracomplex geometry, one has the following analog of this idea.

Definition 2.4. An almost paracomplex manifold is a pair (M,K), where M is a smooth manifold of dimension $2n$ and $K\colon TM \to TM$ is an endomorphism satisfying $K^2 = \mathrm{id}$ such that the kernels of $P_+ := \frac{1}{2}(\mathrm{id} + K)$ and $P_- := \frac{1}{2}(\mathrm{id} - K)$ are both subbundles of TM of rank n. The endomorphism K is called an almost paracomplex structure. (M,K) is a paracomplex manifold if the tensor

$$N_K(X,Y) := K[X,KY] + K[KX,Y] - [X,Y] - [KX,KY] \quad \text{for all } X,Y \in \Gamma(TM)$$

vanishes. In this case, K is called an integrable almost paracomplex structure or simply a paracomplex structure.

The kernels of P_- and P_+, respectively, are the $+1$ and -1 eigenbundles of K:

$$L := \ker P_-, \qquad \widetilde{L} := \ker P_+.$$

The ± 1 eigenbundles then yield a direct sum decomposition of the (real) tangent bundle.

$$TM = L \oplus \widetilde{L}.$$

It is easy to see that $N_K \equiv 0$ if and only if L and $\widetilde{L}$ are both involutive distributions. Unlike the complex case, however, the involutivity of L and $\widetilde{L}$ are independent of one another.

Example 2.5. Let S_1 and S_2 be smooth manifolds of dimension n. Then $S_1 \times S_2$ has a natural paracomplex structure K. A product coordinate system $(W_1 \times W_2, x^i, y^j)$ consists of natural paracomplex coordinates. On this coordinate system, K is locally given by

$$K|_{W_1 \times W_2} = \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^i - \frac{\partial}{\partial y^i} \otimes \mathrm{d}y^i.$$

On $W_1 \times W_2$, the $+1$ eigenbundle L has local frame $\frac{\partial}{\partial x^i}$, $i = 1, \ldots, n$, and the -1 eigenbundle has local frame $\frac{\partial}{\partial y^i}$, $i = 1, \ldots, n$.

For a general paracomplex manifold (M, K), it follows from the Frobenius theorem that the involutivity of the $+1$ and -1 eigebundles L and $\widetilde{L}$ implies that for every $p \in M$, there is a coordinate neighborhood (U, x^i, y^j) such that $\frac{\partial}{\partial x^i}$, $i = 1, \ldots, n$ is a local frame for L and $\frac{\partial}{\partial y^i}$, $i = 1, \ldots, n$ is a local frame for $\widetilde{L}$. This natural division of coordinates is one of the requirements for the double manifold of double field theory.

Example 2.5 is a very trivial way to construct a paracomplex manifold. A more interesting way (which is relevant to the current paper) is as follows. Starting with a manifold M and a connection ∇ on M, we can always turn the tangent bundle TM (regarded as a manifold in its own right) into both an almost complex and an almost paracomplex manifold. Recall that a connection ∇ determines a direct sum decomposition of $TTM := T(TM)$ into a horizontal subbundle H and a vertical subbundle V (cf. [**7**]):

$$TTM = H \oplus V.$$

The vertical subbundle is simply $\ker \pi_*$ where $\pi\colon TM \to M$ is the natural projection. The horizontal subbundle H is determined by the connection ∇. For the convenience of the reader, we briefly recall the details of this fact. From ∇, we construct a map

$$F^\nabla\colon TTM \to TM$$

called the *connection map* which is defined as follows. Let $X \in TM$ and let $\xi \in T_X(TM)$. Let $\sigma(t)\colon (-\varepsilon, \varepsilon) \to TM$ be any smooth curve such that $\sigma(0) = X$ and $\dot\sigma(0) = \xi$. Then

$$F^\nabla(\xi) := \frac{D\sigma}{\mathrm{d}t}\Big|_{t=0},$$

where $D/\mathrm{d}t$ is the covariant derivative associated to ∇ of the vector field $\sigma(t)$ along the curve $\pi(\sigma(t))$. The horizontal subbundle is then defined by $H := \ker F^\nabla$. From the direct sum decomposition $TTM = H \oplus V$, one constructs an almost complex structure J^∇ and an almost paracomplex structure K^∇ via (cf. [**21**]):

$$\pi_*(J^\nabla \xi) = -F^\nabla \xi, \qquad F^\nabla(J^\nabla \xi) = \pi_* \xi$$

and

$$K^\nabla\Big|_H = \mathrm{id}_H, \qquad K^\nabla\Big|_V = -\,\mathrm{id}_V\,.$$

The above definitions imply the following compatibility relation:

$$J^\nabla K^\nabla = -K^\nabla J^\nabla. \tag{2.2.1}$$

By direct calculation, one obtains necessary and sufficient conditions for the integrability of J^∇ and K^∇ (cf. [**7, 21**]).

Proposition 2.6.

(i) J^∇ *is an integrable almost complex structure iff* ∇ *is flat and torsion free.*

(ii) K^∇ *is an integrable almost paracomplex structure iff* ∇ *is flat.*

We conclude this section by recalling the definitions of para-Hermitian and para-Kähler manifolds (which are analogous to the notions of Hermitian and Kähler manifolds in complex geometry).

Definition 2.7. A para-Hermitian manifold is a triple (M, K, g), where (M, K) is a paracomplex manifold and g is a pseudo-Riemannian metric satisfying $g(K\cdot, K\cdot) = -g(\cdot,\cdot)$. The associated fundamental 2-form is defined by

$$\omega(\cdot,\cdot) := g(K\cdot,\cdot).$$

(M, K, g) is called para-Kähler if $\mathrm{d}\omega = 0$. In this case, ω is called the para-Kähler form.

Remark 2.8. Note that the metric g of a $2n$-dimensional para-Hermitian manifold (M, K, g) has signature (n, n). Moreover, if L is the $+1$ eigenbundle and $\widetilde{L}$ is the -1 eigenbundle, then

$$g(L, L) = \omega(L, L) = g(\widetilde{L}, \widetilde{L}) = \omega(\widetilde{L}, \widetilde{L}) = 0,$$

where ω is the fundamental 2-form. Note also that $g(\cdot,\cdot) = \omega(K\cdot,\cdot)$.

2.3. The tangent Lie group

Let G be a Lie group with multiplication $m\colon G \times G \to G$ and inverse $\iota\colon G \to G$. For $(g, h) \in G \times G$, we make the following natural identification:

$$T_{(g,h)}(G \times G) \simeq T_g G \times T_h G.$$

The tangent bundle TG inherits a natural Lie group structure with group product

$$m_*\colon TG \times TG \to TG, \qquad m_*\colon T_g G \times T_h G \to T_{gh} G,$$

and inverse

$$\iota_*\colon TG \to TG, \qquad \iota_*\colon T_g G \to T_{g^{-1}} G.$$

TG with this natural Lie group structure is called the *tangent Lie group*. For $X \in \mathfrak{g}$ and $g \in G$, we define $X_g := (l_g)_* X \in T_g G$. Every element of $T_g G$ is then (uniquely) given by X_g for some $X \in \mathfrak{g}$. For $X_g \in T_gG$ and $Y_h \in T_hG$, one can show that the group product and inverse on TG are given respectively by

$$X_g \cdot Y_h = (r_h)_* X_g + Y_{gh} \in T_{gh}G$$

and

$$X_g^{-1} = -(r_{g^{-1}})_* X \in T_{g^{-1}}G.$$

The identity element on TG is then $0 \in \mathfrak{g}$. Recall that one has a smooth vector bundle isomorphism

$$\varphi\colon TG \to G \times \mathfrak{g}, \qquad X_g \mapsto (g, X).$$

Transferring the natural Lie group structure of TG to $G \times \mathfrak{g}$ via φ, we see that

$$\begin{aligned}(g,X)\cdot(h,Y) &= (gh, \mathrm{Ad}_{h^{-1}}X + Y),\\ (g,X)^{-1} &= (g^{-1}, -\mathrm{Ad}_g X).\end{aligned} \tag{2.3.1}$$

Hence, the natural Lie group structure on TG is isomorphic to the semidirect product $G \ltimes \mathfrak{g}$, where G has a right action on $\mathfrak{g}$ given by $X \cdot g := \mathrm{Ad}_{g^{-1}}X$ and $\mathfrak{g}$ is regarded as an abelian group. Using this canonical isomorphism, we set $TG = G \ltimes \mathfrak{g}$ for the remainder of the paper. The tangent space of TG at (g, X) then has the following natural identification:

$$T_{(g,X)}(TG) = T_{(g,X)}(G \ltimes \mathfrak{g}) = T_gG \times T_X\mathfrak{g} \simeq T_gG \times \mathfrak{g}.$$

The underlying vector space of the Lie algebra of TG is then identified with

$$T_{0_e}(TG) = T_{(e,0)}(G \ltimes \mathfrak{g}) \simeq \mathfrak{g} \times \mathfrak{g}.$$

For $X \in \mathfrak{g}$, one finds that the left invariant vector fields on TG determined by $(X, 0)$ and $(0, X)$ take the following values at the point $(g, Y) \in TG$:

$$(X,0)_{(g,Y)} = (X_g, -[X,Y]) \in T_gG \times \mathfrak{g} \tag{2.3.2}$$

and

$$(0,X)_{(g,Y)} = (0_g, X) \in T_gG \times \mathfrak{g}. \tag{2.3.3}$$

By direct calculation, one finds that the Lie algebra structure of TG is given by

$$[(X,0),(Y,0)] = ([X,Y],0), \quad [(X,0),(0,Y)] = (0,[X,Y]), \quad [(0,X),(0,Y)] = (0,0).$$

The left invariant vector fields $(X, 0)$ and $(0, X)$ can also be understood in terms of the notion of *complete* and *vertical* lifts (cf. [**24**]). We briefly recal the general construction. Let M be a manifold and A a vector field on M. Let ϕ_t be the flow of A. The vector field A induces two flows on TM (viewed as a manifold in its own right); one flow is given by

$$\rho_t^A\colon TM \to TM, \quad \rho_t^A(B) := B + tA_p \qquad \text{for all } B \in T_pM,\ p \in M$$

and the other flow is $\widehat{\phi}_t := (\phi_t)_*$. The vector field on TM whose flow is ρ^A is called the *vertical lift* of A and is denoted by A^v and the vector field whose flow is $\widehat{\phi}_t$ is called the *complete lift* of A and is denoted as A^c. One can show that for

$X \in \mathfrak{g}$, the left invariant vector fields $(X, 0)$ and $(0, X)$ on the tangent Lie group TG are none other than the complete and vertical lifts of X, respectively, that is,

$$X^c = (X, 0), \quad X^v = (0, X).$$

Expressing the above bracket relations in this new light, we have

$$[X^c, Y^c] = [X, Y]^c, \qquad [X^c, Y^v] = [X, Y]^v, \qquad [X^v, Y^v] = 0. \tag{2.3.4}$$

3. The tangent Lie group as a double manifold

In this section, we show that the tangent Lie group of every symplectic Lie group admits a flat para-Kähler metric and thus represents a double manifold for double field theory. As we are working with Lie groups, we limit ourselves only to left invariant structures.

3.1. Paracomplex structure

Let G be a Lie group of dimension n. As noted in Section 2.3, we make the identification $TG = G \ltimes \mathfrak{g}$ for the tangent Lie group. We also make the following identification with regard to the tangent space of TG at the point (g, Y):

$$T_{(g,Y)}(TG) \simeq T_gG \times T_Y\mathfrak{g} \simeq T_gG \times \mathfrak{g}. \tag{3.1.1}$$

Let $X_1, \ldots, X_n$ be a basis for $\mathfrak{g}$ and let ∇ be an arbitrary left invariant connection[1] on G. Write

$$[X_i, X_j] = c_{ij}^k X_k, \qquad \nabla_{X_i} X_j = \Gamma_{ij}^k X_k.$$

As noted in Section 2.3, the complete and vertical lifts of $X \in \mathfrak{g}$ are the left invariant vector fields of TG given by

$$X^c = (X, 0), \qquad X^v = (0, X).$$

Let K^∇ be the associated almost paracomplex structure defined in Section 2.2. We now determine the conditions on ∇ so that K^∇ is left invariant. We will see that there is only one possible choice for ∇ that yields left invariance for K^∇ and this choice is also integrable. Let

$$TTG = H \oplus V$$

be the direct sum decomposition of TTG into the horizontal and vertical subbundles determined by ∇. We begin with the following lemma for the connection map $F^\nabla : TTG \to TG$.

Lemma 3.1. *Let $(g, Y) \in TG$ and $(A_g, B) \in T_{(g,Y)}(TG)$, where $A, B, Y \in \mathfrak{g}$ have components*

$$Y = y^i X_i, \qquad A = a^i X_i, \qquad B = b^i X_i.$$

Then $F^\nabla(A_g, B) = \left[b^k + a^i y^j \Gamma_{ij}^k\right] (X_k)_g$.

[1] Recall that a connection ∇ on a Lie group G is left invariant if $\nabla_X Y$ is left invariant when X and Y are left invariant vector fields. Hence, the space of left invariant connections is in one to one correspondence with the space of bilinear maps $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, where $\mathfrak{g} = \mathrm{Lie}(G)$.

Proof. Let $C(t) = (c_1(t), c_2(t))$, where

$$c_1(t) := g \cdot \exp(tA), \qquad c_2(t) := Y + tB.$$

Then $C(0) = (g, Y)$ and $\dot{C}(0) = (A_g, B)$. Let $\hat{y}^j(t) := y^j$ and $\hat{b}^j(t) := tb^j$. By definition,

$$\begin{aligned} F^\nabla(A_g, B) &:= \frac{DC}{\mathrm{d}t}\Big|_{t=0} \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\hat{y}^j(t) + \hat{b}^j(t))\right](X_j)_g + (\hat{y}^j(0) + \hat{b}^j(0))a^i\nabla_{(X_i)_g}X_j \\ &= b^j(X_j)_g + a^i y^j \Gamma^k_{ij}(X_k)_g \\ &= \left[b^k + a^i y^j \Gamma^k_{ij}\right](X_k)_g \in T_gG. \end{aligned}$$

□

Proposition 3.2. *The almost paracomplex structure K^∇ is left invariant and integrable if and only if $\nabla_X Y = [X, Y]$ for $X, Y \in \mathfrak{g}$. In this case, the horizontal subbundle H has global frame $X_1^c, \dots, X_n^c$. The vertical subbundle V (which is always independent of ∇) has global frame $X_1^v, \dots, X_n^v$. In particular, $K^\nabla X_i^c = X_i^c$ and $K^\nabla X_i^v = -X_i^v$. When G is nonabelian and $\nabla_X Y = [X, Y]$ for $X, Y \in \mathfrak{g}$, the almost complex structure J^∇ is not integrable.*

Proof. Write

$$K^\nabla X_i^c = p_i^j X_j^c + q_i^j X_j^v$$

and

$$K^\nabla X_i^v = r_i^j X_j^c + s_i^j X_j^v.$$

K^∇ is then left invariant precisely when p_i^j, q_i^j, r_i^j, and s_i^j are constants. Let $(g, Y) \in TG$. Since $H = \ker F^\nabla$, it follows from Lemma 3.1 that $H_{(g,Y)}$ is spanned by

$$\mathbf{A}_i := ((X_i)_g, -y^j\Gamma^k_{ij}X_k), \qquad i = 1, \dots, n. \tag{3.1.2}$$

Since the vertical space is $\ker \pi_*$, where $\pi \colon TG \to G$ is the projection, we see that $V_{(g,Y)}$ is spanned by

$$(X_i^v)_{(g,Y)} = (0_g, X_i), \qquad i = 1, \dots, n. \tag{3.1.3}$$

In other words, $X_1^v, \dots, X_n^v$ is a global frame for V. Hence, $K^\nabla X_i^v = -X_i^v$ (and thus $r_i^j = 0$ and $s_i^j = -\delta_i^j$). Using (2.3.2) and (2.3.3), we have

$$(X_i^c)_{(g,Y)} = \mathbf{A}_i + y^j(\Gamma^k_{ij} - c^k_{ij})(X_k^v)_{(g,Y)},$$

Hence,

$$K^\nabla(X_i^c)_{(g,Y)} = \mathbf{A}_i - y^j(\Gamma^k_{ij} - c^k_{ij})(X_k^v)_{(g,Y)}. \tag{3.1.4}$$

(3.1.4) implies

$$p_i^j = \delta_i^j, \qquad q_i^k = 2y^j(c^k_{ij} - \Gamma^k_{ij}).$$

Since $Y = y^i X_i$ is arbitrary, q_i^k is a constant if and only if $\Gamma^k_{ij} = c^k_{ij}$, which is equivalent to the statement that $\nabla_X Y = [X, Y]$ for $X, Y \in \mathfrak{g}$. The Jacobi identity now implies that ∇ is flat when $\Gamma^k_{ij} = c^k_{ij}$; Proposition 2.6 implies that K^∇ is paracomplex. Note however that if $\Gamma_{ij}^k = c_{ij}^k$, then ∇ is torsion free precisely when $\mathfrak{g}$ (and hence G) is abelian. Proposition 2.6 implies that the associated almost complex structure J^∇ is not integrable when G is nonabelian.

Taking $\Gamma_{ij}^k = c_{ij}^k$, (3.1.2) and (2.3.2) imply that

$$\mathbf{A}_i = (X_i^c)_{(g,Y)}, \qquad i = 1, \ldots, n.$$

Hence, $X_1^c, \ldots, X_n^c$ is a global frame for H, which implies that

$$K^\nabla X_i^c = X_i^c. \qquad \square$$

3.2. The metric

Let (G, ω) be a symplectic Lie group and fix a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$. Write

$$[X_i, X_j] = c_{ij}^k X_k, \qquad \omega_{ij} := \omega(X_i, X_j), \qquad (\omega^{ij}) := (\omega_{ij})^{-1}.$$

A natural basis on the Lie algebra of TG (which we denote as $\mathrm{Lie}(TG)$) is then

$$X_1^c, \ldots, X_n^c,\ X_1^v, \ldots, X_n^v.$$

The Lie algebra structure on $\mathrm{Lie}(TG)$ is then given by (2.3.4). Let K denote the left invariant paracomplex structure on TG given by Proposition 3.2.

It was proved in [**19**] that the tangent Lie group of a symplectic Lie group is itself a symplectic Lie group. However, the left invariant symplectic form constructed in [**19**] is incompatible with the left invariant paracomplex structure K. Fortunately, the construction given in [**19**] can be easily modified to yield a left invariant symplectic form which is compatible with K. Let $\widehat{\omega}$ be the left invariant 2-form on TG given by

$$\text{(3.2.1)} \quad \widehat{\omega}(X^c, Y^c) = \widehat{\omega}(X^v, Y^v) = 0, \quad \widehat{\omega}(X^c, Y^v) = \omega(X, Y) \quad \text{for all } X, Y \in \mathfrak{g}.$$

Note that the definition of $\widehat{\omega}$ also implies $\widehat{\omega}(X^v, Y^c) = \omega(X, Y)$.

Proposition 3.3. *$\widehat{\omega}$ is a left invariant symplectic form on TG which satisfies $\widehat{\omega}(K\cdot, K\cdot) = -\widehat{\omega}(\cdot, \cdot)$.*

Proof. The left invariance of $\widehat{\omega}$ follows immediately from the definition of $\widehat{\omega}$ in (3.2.1) and the fact that ω is a left invariant 2-form on G. To show that $\widehat{\omega}$ is closed, it suffices to show that

$$-\mathrm{d}\widehat{\omega}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \widehat{\omega}([\mathbf{A}, \mathbf{B}], \mathbf{C}) + \widehat{\omega}([\mathbf{B}, \mathbf{C}], \mathbf{A}) + \widehat{\omega}([\mathbf{C}, \mathbf{A}], \mathbf{B}) = 0$$

for all $\mathbf{A},\ \mathbf{B},\ \mathbf{C} \in \mathrm{Lie}(TG)$. Since $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are ultimately a sum of complete and vertical lifts of left invariant vector fields on G, we just need to check that:

$$\text{(3.2.2)} \qquad \mathrm{d}\widehat{\omega}(X^v, Y^v, Z^v) = 0,$$

$$\text{(3.2.3)} \qquad \mathrm{d}\widehat{\omega}(X^v, Y^v, Z^c) = 0,$$

$$\text{(3.2.4)} \qquad \mathrm{d}\widehat{\omega}(X^v, Y^c, Z^c) = 0,$$

$$\text{(3.2.5)} \qquad \mathrm{d}\widehat{\omega}(X^c, Y^c, Z^c) = 0$$

for all $X, Y, Z \in \mathfrak{g}$. Equations (3.2.2) and (3.2.5) follow from the fact that

$$[X^c, Y^c] = [X, Y]^c, \qquad [X^v, Y^v] = 0$$

for all $X, Y \in \mathfrak{g}$ and the fact that H and V are both isotropic with respect to $\widehat{\omega}$ by definition. For (3.2.3), we have

$$\begin{aligned}-\mathrm{d}\widehat{\omega}(X^v, Y^v, Z^c) &= \widehat{\omega}([X^v, Y^v], Z^c) + \widehat{\omega}([Y^v, Z^c], X^v) + \widehat{\omega}([Z^c, X^v], Y^v) \\ &= \widehat{\omega}([Y, Z]^v, X^v) + \widehat{\omega}([Z, X]^v, Y^v) \\ &= 0,\end{aligned}$$

where we have used the fact that $[X^v, Y^c] = [X, Y]^v$. For (3.2.4), we have

$$\begin{aligned}-\mathrm{d}\widehat{\omega}(X^v, Y^c, Z^c) &= \widehat{\omega}([X^v, Y^c], Z^c) + \widehat{\omega}([Y^c, Z^c], X^v) + \widehat{\omega}([Z^c, X^v], Y^c) \\ &= \widehat{\omega}([X, Y]^v, Z^c) + \widehat{\omega}([Y, Z]^c, X^v) + \widehat{\omega}([Z, X]^v, Y^c) \\ &= \omega([X, Y], Z) + \omega([Y, Z], X) + \omega([Z, X], Y) \\ &= -\mathrm{d}\omega(X, Y, Z) \\ &= 0.\end{aligned}$$

For the non-degeneracy of $\widehat{\omega}$, let $\mathbf{A} \in \mathrm{Lie}(TG)$ be nonzero. Then $\mathbf{A} = X^c + Y^v$ for some $X, Y \in \mathfrak{g}$. Suppose first that $X \neq 0$. Then there exists $Z \in \mathfrak{g}$ such that $\omega(X, Z) \neq 0$. So

$$\widehat{\omega}(\mathbf{A}, Z^v) = \widehat{\omega}(X^c, Z^v) = \omega(X, Z) \neq 0.$$

On the other hand, if $X = 0$, then $Y \neq 0$ and there exists a $Z' \in \mathfrak{g}$ such that $\omega(Y, Z') \neq 0$. Then

$$\widehat{\omega}(\mathbf{A}, Z'^c) = \widehat{\omega}(Y^v, Z'^c) = \omega(Y, Z') \neq 0.$$

Lastly, to verify the compatibility of $\widehat{\omega}$ and K, let $\mathbf{A}, \mathbf{B} \in \mathrm{Lie}(TG)$. Then

$$\mathbf{A} = X_1^c + Y_1^v, \qquad \mathbf{B} = X_2^c + Y_2^v$$

for some $X_1, Y_1, X_2, Y_2 \in \mathfrak{g}$. Then

$$\begin{aligned}\widehat{\omega}(K\mathbf{A}, K\mathbf{B}) &= \widehat{\omega}(X_1^c - Y_1^v, X_2^c - Y_2^v) = -\widehat{\omega}(X_1^c, Y_2^v) - \widehat{\omega}(Y_1^v, X_2^c) \\ &= -\widehat{\omega}(X_1^c + Y_1^v, X_2^c + Y_2^v) = -\widehat{\omega}(\mathbf{A}, \mathbf{B}).\end{aligned}$$

□

Let η be defined by

$$\eta(\cdot, \cdot) := \widehat{\omega}(K\cdot, \cdot). \tag{3.2.6}$$

(3.2.1) and (3.2.6) immediately imply

$$\begin{aligned}&\eta(X^c, Y^c) = \eta(X^v, Y^v) = 0, \\ &\eta(X^c, Y^v) = \eta(Y^v, X^c) = \omega(X, Y), \quad \forall\ X, Y \in \mathfrak{g}.\end{aligned} \tag{3.2.7}$$

(3.2.6), (3.2.7), and Proposition 3.3 imply that η is a left invariant para-Kähler metric for (TG, K). It only remains to show that η is also flat. As a preliminary step in doing this, let us first calculate the Levi-Civita connection of η with respect to the above basis on $\mathrm{Lie}(TG)$. We denote the Levi Civitia connection by ∇^0, and we adopt the following notation for the Christoffel symbols:

$$\begin{aligned}\nabla^0_{X_i^{\mathbf{c}}} X_j^{\mathbf{c}} &= \Gamma_{ij}^k X_k^{\mathbf{c}} + \Gamma_{ij}^{\bar{k}} X_k^{\mathbf{v}}, \\ \nabla^0_{X_i^{\mathbf{c}}} X_j^{\mathbf{v}} &= \Gamma_{i\bar{j}}^k X_k^{\mathbf{c}} + \Gamma_{i\bar{j}}^{\bar{k}} X_k^{\mathbf{v}},\end{aligned}$$

$$\nabla^0_{X_i^{\mathbf{v}}} X_j^{\mathbf{c}} = \Gamma^k_{\bar{i}j} X_k^{\mathbf{c}} + \Gamma^{\bar{k}}_{\bar{i}j} X_k^{\mathbf{v}},$$
$$\nabla^0_{X_i^{\mathbf{v}}} X_j^{\mathbf{v}} = \Gamma^k_{\bar{i}\bar{j}} X_k^{\mathbf{c}} + \Gamma^{\bar{k}}_{\bar{i}\bar{j}} X_k^{\mathbf{v}}.$$

Lemma 3.4. *The nonzero Christoffel symbols of the Levi-Civita connection of η with respect to the basis $X_1^c, \ldots, X_n^c,\ X_1^v, \ldots, X_n^v$ are given by*

$$\Gamma^m_{ij} = c^l_{ik}\omega_{lj}\omega^{km}, \qquad \Gamma^{\bar{m}}_{i\bar{j}} = c^m_{ij}.$$

Proof. Let $\mathbf{A},\ \mathbf{B},\ \mathbf{C} \in \mathrm{Lie}(TG)$. Since η is left invariant, the Koszul formula reduces to

$$\eta(\nabla^0_{\mathbf{A}}\mathbf{B}, \mathbf{C}) = \frac{1}{2}\left(\eta([\mathbf{A}, \mathbf{B}], \mathbf{C}) + \eta([\mathbf{C}, \mathbf{A}], \mathbf{B}) + \eta([\mathbf{C}, \mathbf{B}], \mathbf{A})\right).$$

Let $\mathbf{A} = X_i^c$, $\mathbf{B} = X_j^c$, and $\mathbf{C} = X_k^c$. Then the Koszul formula along with (3.2.7) and (2.3.4) imply

$$\eta(\nabla^0_{X_i^c} X_j^c, X_k^c) = 0. \tag{3.2.8}$$

(3.2.8) and (3.2.7) imply that $\Gamma^{\overline{m}}_{ij} = 0$. Taking $\mathbf{C} = X_k^v$ gives

(3.2.9)
$$\begin{aligned}
\eta(\nabla^0_{X_i^c} X_j^c, X_k^v) &= \frac{1}{2}\left(\eta([X_i^c, X_j^c], X_k^v) + \eta([X_k^v, X_i^c], X_j^c) + \eta([X_k^v, X_j^c], X_i^c)\right),\\
\Gamma^l_{ij}\eta(X_l^c, X_k^v) &= \frac{1}{2}\left(\eta([X_i, X_j]^c, X_k^v) + \eta([X_k, X_i]^v, X_j^c) + \eta([X_k, X_j]^v, X_i^c)\right),\\
\Gamma^l_{ij}\widehat{\omega}(X_l, X_k) &= \frac{1}{2}\left(\omega([X_i, X_j], X_k) - \omega([X_k, X_i], X_j) - \omega([X_k, X_j], X_i)\right),\\
\Gamma^l_{ij}\omega_{lk} &= \frac{1}{2}(c^l_{ij}\omega_{lk} - c^l_{ki}\omega_{lj} - c^l_{kj}\omega_{li}).
\end{aligned}$$

(3.2.9) implies

$$\Gamma^m_{ij} = \frac{1}{2}(c^m_{ij} + c^l_{ik}\omega_{lj}\omega^{km} + c^l_{jk}\omega_{li}\omega^{km}). \tag{3.2.10}$$

Applying (2.1.2) to (3.2.10) gives

$$\Gamma^m_{ij} = c^l_{ik}\omega_{lj}\omega^{km}. \tag{3.2.11}$$

Next substituting $\mathbf{A} = X_i^c$, $\mathbf{B} = X_j^v$, and $\mathbf{C} = X_k^c$ into the Koszul formula (and simplifying) gives

(3.2.12)
$$\begin{aligned}
\eta(\nabla^0_{X_i^c} X_j^v, X_k^c) &= \frac{1}{2}(-\omega([X_i, X_j], X_k) + \omega([X_k, X_i], X_j) - \omega([X_k, X_j], X_i)\\
-\Gamma^{\bar{l}}_{i\bar{j}}\omega_{lk} &= \frac{1}{2}(-c^l_{ij}\omega_{lk} + c^l_{ki}\omega_{lj} - c^l_{kj}\omega_{li}).
\end{aligned}$$

(3.2.12) implies

$$\Gamma^{\bar{m}}_{i\bar{j}} = \frac{1}{2}(c^m_{ij} + c^l_{ik}\omega_{lj}\omega^{km} - c^l_{jk}\omega_{li}\omega^{km}). \tag{3.2.13}$$

Applying (2.1.2) to (3.2.13) gives

$$\Gamma^{\bar{m}}_{i\bar{j}} = c^m_{ij}. \tag{3.2.14}$$

If we use X_k^v in place of X_k^c in the previous expression, the Koszul formula gives

$$\eta(\nabla^0_{X_i^c} X_j^v, X_k^v) = 0$$

which implies $\Gamma^{m}_{i\bar{j}} = 0$.

For the next case, substitute $\mathbf{A} = X_i^v$, $\mathbf{B} = X_j^c$, and $\mathbf{C} = X_k^c$ into the Koszul formula (and simplify):

$$\begin{aligned}\eta(\nabla^0_{X_i^v} X_j^c, X_k^c) &= \frac{1}{2}(-\omega([X_i, X_j], X_k) - \omega([X_k, X_i], X_j) + \omega([X_k, X_j], X_i)) \\ &= \frac{1}{2}\mathrm{d}\omega(X_i, X_j, X_k) = 0.\end{aligned}$$

This implies $\Gamma^{\bar{m}}_{\bar{i}j} = 0$. The remaining Christoffel symbols $\Gamma^{m}_{\bar{i}j}$, $\Gamma^{m}_{\bar{i}\bar{j}}$, $\Gamma^{\bar{m}}_{\bar{i}\bar{j}}$ are shown to be zero by a similar application of the Koszul formula and (2.3.4). □

Lemma 3.4 has the following immediate corollary.

Corollary 3.5. *For* $\mathbf{A} \in \mathrm{Lie}(TG)$*, let* $\mathrm{ad}_{\mathbf{A}}\mathbf{B} := [\mathbf{A}, \mathbf{B}]$ *for* $\mathbf{B} \in \mathrm{Lie}(TG)$ *denote the adjoint action on* $\mathrm{Lie}(TG)$*. Let* $\mathrm{ad}^\dagger_{\mathbf{A}} : \mathrm{Lie}(TG) \to \mathrm{Lie}(TG)$ *denote the adjoint of* $\mathrm{ad}_{\mathbf{A}}$ *with respect to* η*, that is,*

$$\eta(\mathrm{ad}_{\mathbf{A}}\mathbf{B}, \mathbf{C}) = \eta(\mathbf{B}, \mathrm{ad}^\dagger_{\mathbf{A}}\mathbf{C}).$$

Then

$$\nabla^0_{X^c} Y^c = -\mathrm{ad}^\dagger_{X^c} Y^c, \qquad \nabla^0_{X^c} Y^v = [X, Y]^v, \qquad \nabla^0_{X^v}\mathbf{A} = 0$$

for $X, Y \in \mathfrak{g}$ *and* $\mathbf{A} \in \mathrm{Lie}(TG)$*.*

Proof. $\nabla^0_{X^c} Y^v = [X, Y]^v$ follows from $\Gamma^{m}_{i\bar{j}} = 0$ and $\Gamma^{\bar{m}}_{i\bar{j}} = c^{m}_{ij}$. $\nabla^0_{X^v}\mathbf{A} = 0$ follows from

$$\Gamma^{m}_{\bar{i}j} = \Gamma^{\bar{m}}_{\bar{i}j} = \Gamma^{m}_{\bar{i}\bar{j}} = \Gamma^{\bar{m}}_{\bar{i}\bar{j}} = 0.$$

To verify the remaining equality, observe that

$$\eta(\mathrm{ad}^\dagger_{X^c} Y^c, Z^c) = \eta(Y^c, \mathrm{ad}_{X^c} Z^c) = \eta(Y^c, [X, Z]^c) = 0$$

for all $X, Y, Z \in \mathfrak{g}$. This implies that $\mathrm{ad}^\dagger_{X_i^c} X_j^c$ is of the form $\mathrm{ad}^\dagger_{X_i^c} X_j^c = A^k_{ij} X_k^c$. So

$$\begin{aligned}\eta(\mathrm{ad}^\dagger_{X_i^c} X_j^c, Z_k^v) &= \eta(X_j^c, \mathrm{ad}_{X_i^c} X_k^v), \\ A^l_{ij}\eta(X_l^c, X_k^v) &= c^l_{ik}\eta(X_j^c, X_l^v), \\ A^l_{ij}\omega_{lk} &= c^l_{ik}\omega_{jl}.\end{aligned}$$

Hence,

$$A^m_{ij} = c^l_{ik}\omega_{jl}\omega^{km} = -\Gamma^m_{ij}.$$

This in turn implies $\nabla^0_{X^c} Y^c = -\mathrm{ad}^\dagger_{X^c} Y^c$. □

Theorem 3.6. *Let* (G, ω) *be a symplectic Lie group and let* η *be defined by* (3.2.6)*. Then* η *is a flat left invariant para-Kähler metric on* (TG, K)*.*

Proof. As noted previously, the definition of η and Proposition 3.3 immediately imply that η is a left invariant para-Kähler metric. To prove flatness, we need to verify that

$$R(\mathbf{A},\mathbf{B})\mathbf{C} := \nabla^0_{\mathbf{A}}\nabla^0_{\mathbf{B}}\mathbf{C} - \nabla^0_{\mathbf{B}}\nabla^0_{\mathbf{A}}\mathbf{C} - \nabla^0_{[\mathbf{A},\mathbf{B}]}\mathbf{C} = 0$$

for all $\mathbf{A},\mathbf{B},\mathbf{C} \in \mathrm{Lie}(TG)$. Since every $\mathbf{A} \in \mathrm{Lie}(TG)$ is of the form $\mathbf{A} = X^c + Y^v$ for some $X, Y \in \mathfrak{g}$ and $\nabla^0_{X^v} \equiv 0$ by Lemma 3.4 and Corollary 3.5, we only need to check that

$$R(X_i^c, X_j^c)X_k^c = 0, \qquad R(X_i^c, X_j^c)X_k^v = 0,$$

where $X_1, \ldots, X_n$ is (again) a basis of $\mathfrak{g}$. Expand $R(X_i^c, X_j^c)X_k^c$ by

$$R(X_i^c, X_j^c)X_k^c = \left[\Gamma^l_{jk}\Gamma^m_{il} - \Gamma^l_{ik}\Gamma^m_{jl} - c^l_{ij}\Gamma^m_{lk}\right] X^c_m.$$

Using Lemma 3.4, we compute the coefficients:

$$\begin{aligned}
&\Gamma^l_{jk}\Gamma^m_{il} - \Gamma^l_{ik}\Gamma^m_{jl} - c^l_{ij}\Gamma^m_{lk} \\
&= c^a_{jb}\omega_{ak}\omega^{bl}c^p_{iq}\omega_{pl}\omega^{qm} - c^a_{ib}\omega_{ak}\omega^{bl}c^p_{jq}\omega_{pl}\omega^{qm} - c^l_{ij}c^a_{lb}\omega_{ak}\omega^{bm} \\
&= -c^a_{jb}\omega_{ak}c^b_{iq}\omega^{qm} + c^a_{ib}\omega_{ak}c^b_{jq}\omega^{qm} - c^l_{ij}c^a_{lq}\omega_{ak}\omega^{qm} \\
&= [-c^a_{jb}c^b_{iq} + c^a_{ib}c^b_{jq} - c^l_{ij}c^a_{lq}]\omega_{ak}\omega^{qm} \\
&= -[c^b_{qi}c^a_{bj} + c^b_{jq}c^a_{bi} + c^l_{ij}c^a_{lq}]\omega_{ak}\omega^{qm} = 0,
\end{aligned}$$

where the last equality follows from the Jacobi identity expressed in terms of the structure constants of the basis $X_1, \ldots, X_n$ of $\mathfrak{g}$. This shows that $R(X_i^c, X_j^c)X_k^c = 0$.

For $R(X_i^c, X_j^c)X_k^v$, we have

$$\begin{aligned}
R(X_i^c, X_j^c)X_k^v &= \left[\Gamma^{\bar{l}}_{j\bar{k}}\Gamma^{\bar{m}}_{i\bar{l}} - \Gamma^{\bar{l}}_{i\bar{k}}\Gamma^{\bar{m}}_{j\bar{l}} - c^l_{ij}\Gamma^{\bar{m}}_{l\bar{k}}\right] X^v_m = [c^l_{jk}c^m_{il} - c^l_{ik}c^m_{jl} - c^l_{ij}c^m_{lk}]X^v_m \\
&= -[c^l_{jk}c^m_{li} + c^l_{ki}c^m_{lj} + c^l_{ij}c^m_{lk}]X^v_m = 0.
\end{aligned}$$

This completes the proof for flatness. □

Remark 3.7. In [**17**], families of *hypersymplectic structures* were constructed on the tangent bundle of connected, simply connected *special* symplectic Lie groups. For hypersymplectic geometry, the natural Lie group structure on the tangent bundle is not sufficient. Special symplectic Lie groups carry the additional structure needed to deform the natural Lie group structure of the tangent bundle so that hypersymplectic structures become possible. Formally, a special symplectic Lie group is a triple (G, ω, ∇) such that (G, ω) is a symplectic Lie group and ∇ is a left invariant flat torsion free connection on G such that $\nabla\omega = 0$. With G connected and simply connected, the Lie group structure on TG is defined so that its Lie algebra structure is given by

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2], \nabla_{X_1}Y_2 - \nabla_{X_2}Y_1)$$

for $X_i, Y_i \in \mathfrak{g}$. From this, one sees that for G nonabelian, the deformed Lie group structure on TG defined by ∇ never coincides with the natural Lie group structure on TG (which is the Lie group structure assumed throughout this paper).

4. Relation to the cotangent bundle and the standard symplectic form

Let (G, ω) be a symplectic Lie group. Once again, let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$, $\omega_{ij} := \omega(X_i, X_j)$, $(\omega^{ij}) := (\omega_{ij})^{-1}$, and write $[X_i, X_j] = c_{ij}^k X_k$. It is a well known fact that every symplectic Lie group admits a left invariant flat torsion free connection (cf. [**3**]). On (G, ω), this connection (which we denote as ∇^ω) is given by

$$\omega(\nabla_X^\omega Y, Z) = -\omega(Y, [X, Z]), \qquad X, Y, Z \in \mathfrak{g}.$$

If the Christoffel symbols are defined by $\nabla_{X_i}^\omega X_j = \Gamma_{ij}^k X_k$, a direct calculation shows that

$$\Gamma_{ij}^m = c_{ik}^l \omega_{lj} \omega^{km}.$$

Proposition 3.2 implies that the paracomplex structure on TG defined by ∇^ω cannot be left invariant.

It is another well known fact that any manifold which admits a flat torsion free connection also admits an affine structure [**2**]. It follows as a special case of a result of Bejan[2] [**4, 6**] that the cotangent bundle of any affine manifold admits a flat para-Kähler structure such that the para-Kähler form is the canonical symplectic form on the cotangent bundle.

The above discussion shows that a symplectic Lie group is also an affine manifold. Hence, its cotangent bundle admits a flat para-Kähler structure. Consequently, it's natural that we compare the flat para-Kähler structure on T^*G with the flat para-Kähler structure on TG given by Theorem 3.6. We will show that the flat para-Kähler structures on TG and T^*G are in fact equivalent. From this equivalence, we gain something quite interesting: *a Lie group structure on T^*G for which its flat para-Kähler structure (in particular, the standard symplectic form) is left invariant.*

Before establishing this equivalence, let us first identify T^*G (as a manifold) with $G \times \mathfrak{g}^*$. For $\alpha \in \mathfrak{g}^*$, let

$$\alpha_g := (l_{g^{-1}})^* \alpha \in T_g^* G.$$

Then every element of T^*G is of the form α_g for some $\alpha \in \mathfrak{g}^*$ and some $g \in G$. Hence, T^*G is naturally identified with $G \times \mathfrak{g}^*$ via

$$G \times \mathfrak{g}^* \xrightarrow{\sim} T^*G, \quad (g, \alpha) \mapsto \alpha_g.$$

From this point forth, we set $T^*G = G \times \mathfrak{g}^*$ (as a manifold). Define

$$\varphi_\omega \colon TG \xrightarrow{\sim} T^*G, \qquad (g, X) \mapsto (g, \flat_\omega(X)) \qquad \text{for all } X \in \mathfrak{g},\ g \in G$$

where $\flat_\omega \colon \mathfrak{g} \to \mathfrak{g}^*$ is given by $\flat_\omega(X) := \omega(X, \cdot)$. We now endow T^*G with a Lie group structure by declaring φ_ω to be a Lie group isomorphism. Explicitly, one

[2]An elementary proof of Bejan's result for the special case of affine manifolds is given in Appendix A for the convenience of the reader.

finds that the group structure on T^*G is given by

$$(g,\alpha)\cdot(h,\beta) = (gh, \flat_\omega \circ \mathrm{Ad}_{h^{-1}} \circ \sharp_\omega(\alpha) + \beta),$$
$$(g,\alpha)^{-1} = (g^{-1}, -\flat_\omega \circ \mathrm{Ad}_g \circ \sharp_\omega(\alpha)), \tag{4.0.1}$$

where $\sharp_\omega = \flat_\omega^{-1}$.

Proposition 4.1. *Let Ω be the standard symplectic form on T^*G and let $\widehat{\omega}$ be the para-Kähler form on TG given by Proposition 3.3. Then $\varphi_\omega^*\Omega = \widehat{\omega}$. In particular, Ω is a left invariant symplectic form on T^*G when T^*G is equipped with the group law given by* (4.0.1).

Proof. Let θ denote the Liouville one-form on T^*G. Then $\Omega = d\theta$. Let $(g,\alpha) \in T^*G$ and

$$(X_g, \beta) \in T_{(g,\alpha)}(T^*G) \simeq T_gG \times \mathfrak{g}^*, \qquad X \in \mathfrak{g}.$$

One finds that

$$\theta_{(g,\alpha)}(X_g,\beta) = \alpha(X). \tag{4.0.2}$$

Let $\theta' = \varphi_\omega^*\theta$. For $(g,Y) \in TG$ and $(X_g, Z) \in T_{(g,Y)}(TG) \simeq T_gG \times \mathfrak{g}$, we have

$$\theta'_{(g,Y)}(X_g, Z) = \omega(Y, X). \tag{4.0.3}$$

Let $\Omega' := \varphi_\omega^*\Omega = \varphi_\omega^* d\theta$. Using (4.0.3), we have

$$\begin{aligned}
\Omega'(X_i^c, X_j^c)\Big|_{(g,Y)} &= (\varphi_\omega^*\mathrm{d}\theta)(X_i^c, X_j^c)\Big|_{(g,Y)} \\
&= (\mathrm{d}\theta')(X_i^c, X_j^c)\Big|_{(g,Y)} \\
&= (X_i^c)_{(g,Y)}(\theta'(X_j^c)) - (X_j^c)_{(g,Y)}(\theta'(X_i^c)) - \theta'_{(g,Y)}([X_i^c, X_j^c]) \\
&= -\omega([X_i, Y], X_j) + \omega([X_j, Y], X_i) - \omega(Y, [X_i, X_j]) \\
&= \omega([Y, X_i], X_j) + \omega([X_j, Y], X_i) + \omega([X_i, X_j], Y) \\
&= -\mathrm{d}\omega(X_i, X_j, Y) = 0,
\end{aligned}$$

$$\begin{aligned}
\Omega'(X_i^v, X_j^v)\Big|_{(g,Y)} &= (\varphi_\omega^*\mathrm{d}\theta)(X_i^v, X_j^v)\Big|_{(g,Y)} \\
&= (\mathrm{d}\theta')(X_i^v, X_j^v)\Big|_{(g,Y)} \\
&= (X_i^v)_{(g,Y)}(\theta'(X_j^v)) - (X_j^v)_{(g,Y)}(\theta'(X_i^v)) - \theta'_{(g,Y)}([X_i^v, X_j^v]) \\
&= \omega(X_i, 0) - \omega(X_j, 0) - \omega(Y, 0) = 0.
\end{aligned}$$

$$\begin{aligned}
\Omega'(X_i^c, X_j^v)\Big|_{(g,Y)} &= (\varphi_\omega^*\mathrm{d}\theta)(X_i^c, X_j^v)\Big|_{(g,Y)} \\
&= (\mathrm{d}\theta')(X_i^c, X_j^v)\Big|_{(g,Y)} \\
&= (X_i^c)_{(g,Y)}(\theta'(X_j^v)) - (X_j^v)_{(g,Y)}(\theta'(X_i^c)) - \theta'_{(g,Y)}([X_i^c, X_j^v]) \\
&= -\omega([X_i, Y], 0) - \omega(X_j, X_i) - \omega(Y, 0) = \omega(X_i, X_j).
\end{aligned}$$

From the definition of $\widehat{\omega}$ in (3.2.1), we have $\Omega' = \widehat{\omega}$. $\square$

Let

$$\bar{X}_i^c := (\varphi_\omega)_* X_i^c, \qquad \bar{X}_i^v := (\varphi_\omega)_* X_i^v. \tag{4.0.4}$$

For $(g,\alpha) \in T^*G$, we have

$$(\bar{X}_i^c)_{(g,\alpha)} = ((X_i)_g, -\flat_\omega([X_i, \sharp_\omega(\alpha)])), \qquad (\bar{X}_i^v)_{(g,\alpha)} = (0_g, \flat_\omega(X_i)). \tag{4.0.5}$$

Equip T^*G with the group law given by (4.0.1). Then $\{\bar{X}_i^c, \bar{X}_j^v\}_{i,j=1,\ldots,n}$ is a basis of $\mathrm{Lie}(T^*G)$. Let

$$\bar{K} \colon T(T^*G) \to T(T^*G)$$

denote the paracomplex structure on T^*G induced by the affine structure of G. From the proof of Theorem A.1, the -1 eigenbundle of $\bar{K}$ is given by the vertical distribution $\ker \pi_*$, where $\pi \colon T^*G \to G$ is the natural projection. (4.0.5) implies that $\{\bar{X}_i^v\}_{i=1,\ldots,n}$ is a global frame for the vertical distribution $\ker \pi_*$. Since $\widehat{\omega} = \varphi_\omega^* \Omega$, we have

$$\Omega(\bar{X}_i^c, \bar{X}_j^c) = \Omega(\bar{X}_i^v, \bar{X}_j^v) = 0, \qquad \Omega(\bar{X}_i^c, \bar{X}_j^v) = \widehat{\omega}(X_i^c, X_j^v) = \omega(X_i, X_j). \tag{4.0.6}$$

Since Ω is the para-Kähler form associated to $\bar{K}$ and $\bar{X}_i^v$ are sections of the -1 eigenbundle of $\bar{K}$, we have

$$\begin{aligned} \Omega(\bar{K}\bar{X}_i^c, \bar{K}\bar{X}_j^v) &= -\Omega(\bar{X}_i^c, \bar{X}_j^v), \\ \Omega(\bar{K}\bar{X}_i^c, -\bar{X}_j^v) &= \Omega(\bar{X}_i^c, -\bar{X}_j^v). \end{aligned} \tag{4.0.7}$$

(4.0.6), (4.0.7), and the nondegeneracy of Ω implies

$$\bar{K}\bar{X}_i^c = \bar{X}_i^c. \tag{4.0.8}$$

Hence, $\{\bar{X}_i^c\}_{i=1,\ldots,n}$ is a global frame for the $+1$ eigenbundle of $\bar{K}$. This implies

$$(\varphi_\omega)_* \circ K = \bar{K} \circ (\varphi_\omega)_*. \tag{4.0.9}$$

Now let $\bar{\eta}(\cdot,\cdot) := \Omega(\bar{K}\cdot,\cdot)$ denote the para-Kähler metric on T^*G. (4.0.6) implies

$$\varphi_\omega^* \bar{\eta} = \eta. \tag{4.0.10}$$

Putting everything together, we have proved the following:

Theorem 4.2. *$(\bar{K}, \bar{\eta})$ is a left invariant flat para-Kähler structure for T^*G, where T^*G is equipped with the group law (4.0.1), and $\Omega(\cdot,\cdot) = \bar{\eta}(\bar{K}\cdot,\cdot)$ is the standard symplectic form on T^*G. In addition, $\varphi_\omega \colon (TG, K, \eta) \xrightarrow{\sim} (T^*G, \bar{K}, \bar{\eta})$ is an isomorphism of para-Kähler manifolds which preserves the Lie group structures.*

Remark 4.3. Theorem 4.2 shows that the existing flat para-Kähler structure on T^*G (which uses the standard symplectic form as its para-Kähler form − see Appendix A) is left invariant when T^*G is equipped with the group law (4.0.1). At the same time, Theorem 4.2 also establishes an equivalence with the left invariant flat para-Kähler structure on TG given by Theorem 3.6.

Note that the group law on T^*G given by (4.0.1) does **not**, in general, coincide with the standard group law given by

$$(g,\alpha) \cdot (h,\beta) = (gh, \mathrm{Ad}_{h^{-1}}^*(\alpha) + \beta). \tag{4.0.11}$$

Equation (4.0.11) is equivalent to the statement that $\flat_\omega \circ \mathrm{Ad}_g \circ \sharp_\omega = \mathrm{Ad}_g^*$ for all $g \in G$, which does **not** hold in general for symplectic Lie groups. Indeed, one does not have to look far to find a counterexample, where neither $\flat_\omega \circ \mathrm{Ad}_g \circ \sharp_\omega$ nor $\flat_\omega \circ \mathrm{Ad}_{g^{-1}} \circ \sharp_\omega$ coincide with Ad_g^*. For a counterexample, consider the 2-dimensional Lie group of affine transformations on $\mathbb{R}$.

$$\mathrm{Aff}(2) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \;\middle|\; a \neq 0,\ b \in \mathbb{R} \right\}.$$

The Lie algebra of Aff(2) (denoted $\mathfrak{aff}(2)$) is spanned by

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Equip Aff(2) with the left invariant nondegenerate 2-form ω defined by the condition $\omega(e_1, e_2) = 1$. Since $\dim \mathrm{Aff}(2) = 2$, it follows that ω is necessarily closed. Hence, $(\mathrm{Aff}(2), \omega)$ is a symplectic Lie group.

Let θ^1, θ^2 denote the dual basis to e_1, e_2 and let

$$g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \mathrm{Aff}(2).$$

By direct calculation, one finds

$$\begin{aligned} \flat_\omega \circ \mathrm{Ad}_g(e_1) &= b\theta^1 + \theta^2, \\ \flat_\omega \circ \mathrm{Ad}_{g^{-1}}(e_1) &= -\frac{b}{a}\theta^1 + \theta^2, \\ \mathrm{Ad}_g^* \circ \flat_\omega(e_1) &= \frac{b}{a}\theta^1 + \frac{1}{a}\theta^2, \end{aligned}$$

where we recall that $\mathrm{Ad}_g^*(f)(x) := f(\mathrm{Ad}_{g^{-1}}(x))$ for $f \in \mathfrak{aff}(2)^*$, $x \in \mathfrak{aff}(2)$. It follows immediately that

$$\flat_\omega \circ \mathrm{Ad}_g \circ \sharp_\omega \neq \mathrm{Ad}_g^* \neq \flat_\omega \circ \mathrm{Ad}_{g^{-1}} \circ \sharp_\omega.$$

Hence, the group law on $T^*\mathrm{Aff}(2)$ given by (4.0.1) does not coincide with the standard form of (4.0.11).

When one compares the group law on TG (2.3.1) and the form of its left invariant vector fields (2.3.2), (2.3.3) to its counterparts on T^*G given by (4.0.1) and (4.0.5), one certainly seems simpler than the other. For this reason, one can argue that TG rather than T^*G is the more natural space to work with when uniting the Lie group structure with the para-Kähler structure.

5. Some double field theory

5.1. Review of the generalized metric

As mentioned earlier, in double field theory, one works with a 'double manifold' which Vaisman [**22**] later identified to be a flat para-Kähler manifold. Let (M, K, η) be a flat para-Kähler manifold. In addition to the neutral metric η, the double manifold is equipped with a symmetric tensor field g and an antisymmetric tensor field B. On a flat para-Kähler manifold, one can choose flat paracomplex coordinates $(U, x^i, \tilde{x}_j)$ so that locally

$$\eta = \mathrm{d}x^i \otimes \mathrm{d}\tilde{x}_i + \mathrm{d}\tilde{x}_i \otimes \mathrm{d}x^i, \qquad K\frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i}, \qquad K\frac{\partial}{\partial \tilde{x}_i} = -\frac{\partial}{\partial \tilde{x}_i}.$$

In these 'distinguished coordinates' (using the terminology of [**22**]), g and B locally take the form

$$g = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j, \qquad B = \frac{1}{2}B_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j, \qquad \det(g_{ij}) \neq 0.$$

Hence, g is a (vector bundle) metric on the $+1$ eigenbundle of K (denoted L) and $B \in \Gamma(\wedge^2 L^*)$. Note that g can be taken to be Riemannian or pseudo-Riemannian.

From a physics standpoint, we are to think of g as the spacetime metric and B as the Kalb-Ramond field B, the latter is the string analog of the Maxwell 1-form for a point particle (cf. [**20**]). With the pair (g, B) of 'background fields' (to use the physics terminology), one defines a second metric $\mathcal{H}(g, B)$ on the double manifold (M, K, η). The metric $\mathcal{H}(g, B)$ is called the generalized metric associated to (g, B). With respect to local distinguished coordinates $(x^i, \tilde{x}_j)$, the (local) matrix representation of $\mathcal{H}(g, B)$ is given by

$$\mathcal{H}(g, B) = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}, \tag{5.1.1}$$

where we have abused notation by setting $g = (g_{ij})$, $B = (B_{ij})$, and identifying $\mathcal{H}(g, B)$ with its local matrix representation. This is the matrix which appears in the double field theory literature[3] and encodes the Hamiltonian density for the physics of the theory (see, e.g., [**10**]). By inspecting the local form given by (5.1.1), one can show that the metric $\mathcal{H}(g, B)$ is Riemannian if and only if g is Riemannian.

Vaisman [**22, 23**] provided the following invariant definition of the generalized metric.

Definition 5.1. Let (M, K, η) be a flat para-Kähler manifold. A metric $\mathcal{H}$ on M is a generalized metric if it satisfies the following conditions:

1. $\sharp_{\mathcal{H}} \circ \flat_\eta = \sharp_\eta \circ \flat_{\mathcal{H}}$
2. $\mathcal{H}\Big|_{\tilde{L}}$ is nondegenerate

where $\tilde{L}$ is the -1 eigenbundle of K, $\flat_{\mathcal{H}}(X) := \mathcal{H}(X, \cdot)$, and $\sharp_{\mathcal{H}} := \flat_{\mathcal{H}}^{-1}$; $\flat_\eta$ and $\sharp_\eta$ are defined similarly.

One can show that there is a one-to-one correspondence between generalized metrics $\mathcal{H}$ and pairs (g, B). Suppose then that the generalized metric $\mathcal{H}$ is associated to the pair (g, B). In other words, $\mathcal{H} = \mathcal{H}(g, B)$ has the local form given by

[3] In the physics literature, the coordinates are usually ordered as $(\tilde{x}_i, x^j)$ which has the effect of swapping the (1,1) and (2,2) blocks as well as the (1,2) and (2,1) blocks in (5.1.1).

(5.1.1). We sketch how to recover $\mathcal{H}(g,B)$ from (g,B). Let[4]

$$S_+ := \{(X, \flat_{-B+g}(X)) \mid X \in L\} \subset L \oplus L^*, \tag{5.1.2}$$

$$S_- := \{(X, \flat_{-B-g}(X)) \mid X \in L\} \subset L \oplus L^*. \tag{5.1.3}$$

One can verify that

$$S_+ \oplus S_- = L \oplus L^*. \tag{5.1.4}$$

Define

$$\psi_\eta : TM = L \oplus \tilde{L} \xrightarrow{\sim} L \oplus L^*, \qquad (X,Y) \mapsto (X, \flat_\eta(Y)). \tag{5.1.5}$$

Let

$$\widehat{S}_+ := \psi_\eta^{-1} S_+ = \{X + \sharp_\eta \circ \flat_{-B+g}(X) \mid X \in L\}, \tag{5.1.6}$$

$$\widehat{S}_- := \psi_\eta^{-1} S_- = \{X + \sharp_\eta \circ \flat_{-B-g}(X) \mid X \in L\}. \tag{5.1.7}$$

From (5.1.4), we have

$$TM = \widehat{S}_+ \oplus \widehat{S}_-. \tag{5.1.8}$$

Vaisman showed [**22, 23**] that $\mathcal{H} = \mathcal{H}(g,B)$ is equivalent to the conditions

(a) $\mathcal{H}(g,B)|_{\widehat{S}_+} = \eta$,
(b) $\mathcal{H}(g,B)|_{\widehat{S}_-} = -\eta$,
(c) $\mathcal{H}(g,B)(\widehat{S}_+, \widehat{S}_-) = 0$, that is, $\widehat{S}_+$ and $\widehat{S}_-$ are orthogonal with respect to $\mathcal{H}(g,B)$.

Let $\Phi := \sharp_{\mathcal{H}} \circ \flat_\eta$. By a short calculation, one can show [**22, 23**]

$$\eta(\widehat{S}_+, \widehat{S}_-) = 0, \qquad \mathcal{H}(X,Y) = \eta(\Phi(X), Y),$$

$\Phi|_{\widehat{S}_+} = \mathrm{id}$ and $\Phi|_{\widehat{S}_-} = -\mathrm{id}$. In particular, Φ is an almost paracomplex structure on M.

Motivated by [**15**], we adopt the following global definition for T-duality (cf. [**10**]).

Definition 5.2. Let (M,K,η) be a double manifold. A global T-duality transformation is a bundle automorphism $\mathcal{A}\colon TM \to TM$ such that $\mathcal{A}^*\eta = \eta$. The group of all global T-duality transformations is denoted as $O(n,n)(M)$.

The following result relates Definition 5.2 to Vaisman's definition of the generalized metric.

Proposition 5.3. *Let (M,K,η) be a double manifold and let μ be any metric on M. Let $\mathcal{A}_\mu := \sharp_\eta \circ \flat_\mu$. The following statements are equivalent:*

(i) $\mathcal{A}_\mu \in O(n,n)(M)$,
(ii) $\sharp_\eta \circ \flat_\mu = \sharp_\mu \circ \flat_\eta$,
(iii) $\mathcal{A}_\mu^2 = \mathrm{id}$.

In particular, if $\mathcal{H}$ is a generalized metric, then $\mathcal{A}_{\mathcal{H}}$ is a global T-duality transformation.

[4]In [**22**], Vaisman uses $B \pm g$ in the definitions of $S_\pm$ as opposed to $-B \pm g$. The choice used in [**22**] gives a matrix representation for $\mathcal{H}(g,B)$ whose signs in the off-diagonal blocks are opposite that of the physics matrix given by (5.1.1).

Proof. (i) $\Leftrightarrow$ (ii). Let $p \in M$, $X, Y \in T_pM$ and let $\langle \cdot, \cdot \rangle$ be the natural pairing between TM and T^*M. Then

$$\begin{aligned}(\mathcal{A}_\mu^* \eta)(X,Y) &= \eta(\mathcal{A}_\mu X, \mathcal{A}_\mu Y) = \langle \flat_\eta \circ \mathcal{A}_\mu X, \mathcal{A}_\mu Y \rangle \\ &= \langle \flat_\mu(X), \mathcal{A}_\mu Y \rangle = \mu(\mathcal{A}_\mu Y, X) \\ &= \langle \flat_\mu \circ \mathcal{A}_\mu Y, X \rangle.\end{aligned}$$

From this, we see that $(\mathcal{A}_\mu^* \eta)(X,Y) = \eta(X,Y)$ if and only if $\flat_\mu \circ \mathcal{A}_\mu = \flat_\eta$ which in turn is equivalent to $\sharp_\eta \circ \flat_\mu = \sharp_\mu \circ \flat_\eta$.

(ii) $\Leftrightarrow$ (iii). This follows immediately from the fact that $\mathcal{A}_\mu^{-1} = \sharp_\mu \circ \flat_\eta$. □

In addition, we also have the following observation:

Proposition 5.4. *Let (M, K, η) be a double manifold with pair (g, B) where g is a Riemannian metric on the $+1$ eigenbundle L. Then $\mathcal{A}^*\mathcal{H}(g,B)$ is also a generalized metric for all $\mathcal{A} \in O(n,n)(M)$.*

Proof. Set $\mathcal{H} := \mathcal{H}(g,B)$. Note that

$$\flat_{\mathcal{A}^*\mathcal{H}} = \mathcal{A}^* \circ \flat_{\mathcal{H}} \circ \mathcal{A}, \qquad \sharp_{\mathcal{A}^*\mathcal{H}} = \mathcal{A}^{-1} \circ \sharp_{\mathcal{H}} \circ \mathcal{A}^{*-1}.$$

Since $\mathcal{A}^*\eta = \eta$, we also have

$$\flat_\eta = \mathcal{A}^* \circ \flat_\eta \circ \mathcal{A}, \qquad \sharp_\eta = \mathcal{A}^{-1} \circ \sharp_\eta \circ \mathcal{A}^{*-1}.$$

For the first condition of Definition 5.1, we have

$$\begin{aligned}\sharp_{\mathcal{A}^*\mathcal{H}} \circ \flat_\eta &= (\mathcal{A}^{-1} \circ \sharp_{\mathcal{H}} \circ \mathcal{A}^{*-1}) \circ (\mathcal{A}^* \circ \flat_\eta \circ \mathcal{A}) = \mathcal{A}^{-1} \circ \sharp_{\mathcal{H}} \circ \flat_\eta \circ \mathcal{A} \\ &= \mathcal{A}^{-1} \circ \sharp_\eta \circ \flat_{\mathcal{H}} \circ \mathcal{A} = (\mathcal{A}^{-1} \circ \sharp_\eta \circ \mathcal{A}^{*-1}) \circ (\mathcal{A}^* \circ \flat_{\mathcal{H}} \circ \mathcal{A}) \\ &= \sharp_\eta \circ \flat_{\mathcal{A}^*\mathcal{H}}.\end{aligned}$$

For the second condition of Definition 5.1, we use the fact that $\mathcal{H}$ is Riemannian whenever g is positive definite. (This point can be proven by inspecting the local matrix representation of $\mathcal{H}$ in (5.1.1)). Since $\mathcal{H}$ is Riemannian, it follows that $\mathcal{A}^*\mathcal{H}$ is also Riemannian. In particular, $\mathcal{A}^*\mathcal{H}\Big|_{\tilde{L}}$ is nondegenerate which completes the proof. □

5.2. Review of double metric connections

For a double manifold (M, K, η) equipped with fields (g, B), one has two metrics on M: η and $\mathcal{H}(g,B)$ (which we denote simply as $\mathcal{H}$). In double field theory, one is interested in connections ∇ on M satisfying $\nabla\eta = \nabla\mathcal{H} = 0$. These are the so called double metric connections [**22, 23**]. We now give a brief review of their construction. For $X \in \Gamma(TM)$, let X^+ and X^- denote the projection of X onto the vector bundles $\widehat{S}_+ \to M$ and $\widehat{S}_- \to M$, respectively. Let ∇ be a double metric connection on (M, K, η) with fields (g, B) and define

$$\nabla^\pm_X Y := (\nabla_X Y)^\pm \qquad \text{for all } X, Y \in \Gamma(TM). \tag{5.2.1}$$

The condition that $\nabla\eta = \nabla\mathcal{H} = 0$ coupled with the fact that $\widehat{S}_+$ and $\widehat{S}_-$ are orthogonal with respect to both $\mathcal{H}$ and η, imply

$$\nabla^+_X Y^- = \nabla^-_X Y^+ = 0. \tag{5.2.2}$$

Hence,

$$\nabla_X Y = \nabla^+_X Y^+ + \nabla^-_X Y^-. \tag{5.2.3}$$

Equation (5.2.3) implies

$$\nabla^\pm \mathcal{H}|_{\widehat{S}_\pm} = \nabla^\pm \eta|_{\widehat{S}_\pm} = 0.$$

This observation along with (5.2.3) and the fact that $\mathcal{H}|_{\widehat{S}_\pm} = \pm\eta|_{\widehat{S}_\pm}$ implies that ∇ is equivalent to a pair of connections $\nabla^\pm$ on the vector bundles $\widehat{S}_\pm \to M$ satisfying $\nabla^\pm\mathcal{H}|_{\widehat{S}_\pm} = 0$. We can express the connections $\nabla^\pm$ on $\widehat{S}_\pm$ as a pair of connections $D^\pm$ on the vector bundle $L \to M$ using the bundle isomorphisms $\iota_\pm : L \xrightarrow{\sim} \widehat{S}_\pm$ given by

$$\iota_\pm(X) := \psi_\eta^{-1}(X, \flat_{-B\pm g}(X)) \in \widehat{S}_\pm \qquad \text{for all } X \in L. \tag{5.2.4}$$

By direct calculation, one can show that

$$\iota^*_\pm \mathcal{H}|_{\widehat{S}_\pm} = 2g. \tag{5.2.5}$$

Explicitly, $D^\pm$ are defined by

$$\nabla^+_X(\iota_+\sigma) = \iota_+(D^+_X\sigma), \qquad \nabla^-_X(\iota_-\sigma) = \iota_-(D^-_X\sigma) \tag{5.2.6}$$

for $X \in \Gamma(TM)$ and $\sigma \in \Gamma(L)$. The above argument along with (5.2.5) and (5.2.6) yields the correspondence given in [**22, 23**]:

Proposition 5.5. *For a double manifold (M, K, η) with fields (g, B), there is a one-to-one correspondence between double metric connections and pairs of connections $D^\pm$ on the bundle $L \to M$ satisfying $D^\pm g = 0$.*

5.3. Generalized metrics and double metric connections for TG

Let (G, ω) be a symplectic Lie group and let (TG, K, η) be the associated left invariant flat para-Kähler manfiold. Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$. Since we are ultimately working with a Lie group, it is much more natural to calculate the form of the generalized metric for (TG, K, η) with respect to a basis of Lie(TG) as opposed to local distinguished coordinates. For the data (g, B) defining a generalized metric, we limit ourselves to the case where g and B are both left invariant. Given the paracomplex structure $TTG = H \oplus V$, this means that g is a left invariant bundle metric on H and $B \in \Gamma(\wedge^2 H^*)$ is a left invariant 2-form. The result then is that $\mathcal{H}(g, B)$ is a left invariant metric on TG. We compute the matrix representation of $\mathcal{H}(g, B)$ with respect to the global frame

$$X^c_1, \ldots, X^c_n,\ X^v_1, \ldots, X^v_n. \tag{5.3.1}$$

Let

$$\alpha^1, \ldots, \alpha^n,\ \beta^1, \ldots, \beta^n \tag{5.3.2}$$

be the dual frame. As usual, we let $\omega_{ij} := \omega(X_i, X_j)$ and $(\omega^{ij}) = (\omega_{ij})^{-1}$. The para-Kähler metric η is then given by

$$\eta = \omega_{ij}(\alpha^i \otimes \beta^j + \beta^j \otimes \alpha^i).$$

Let $\mathcal{H}$ be any metric on TG (regarded as a manifold). Then $\mathcal{H}$ takes the form

$$\mathcal{H} = a_{ij}\alpha^i \otimes \alpha^j + r_{ij}(\alpha^i \otimes \beta^j + \beta^j \otimes \alpha^i) + c_{ij}\beta^i \otimes \beta^j.$$

Set $A = (a_{ij})$, $R = (r_{ij})$, and $C = (c_{ij})$. We now determine the conditions on A, R, and C so that $\mathcal{H}$ is a generalized metric. The second condition of Definition 5.1 implies that C is invertible. With abuse of notation, we also set $\omega = (\omega_{ij})$. By direct calculation, one finds that the first condition on Definition 5.1 places the following restrictions on the aforementioned matrices:

$$-A\omega^{-1}R^T + R\omega^{-1}A = 0, \tag{5.3.3}$$

$$-A\omega^{-1}C + R\omega^{-1}R = \omega, \tag{5.3.4}$$

$$-R^T\omega^{-1}C + C\omega^{-1}R = 0, \tag{5.3.5}$$

where C is invertible. Setting

$$g = (g_{ij}) = -\omega C^{-1}\omega, \qquad B = (B_{ij}) = -RC^{-1}\omega,$$

the general solution to (5.3.3)–(5.3.5) is then

$$\mathcal{H} = \mathcal{H}(g, B) = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1}\omega \\ \omega g^{-1}B & -\omega g^{-1}\omega \end{pmatrix}, \tag{5.3.6}$$

where we have identified $\mathcal{H}$ with its matrix representation with respect to the global frame (5.3.1). Hence, (5.3.6) is the matrix representation of the generalized metric with respect to the global frame (5.3.1) for the double manifold (TG, K, η) with fields (g, B), where g and B are actually the left invariant tensors

$$g = g_{ij}\alpha^i \otimes \alpha^j, \qquad B = \frac{1}{2}B_{ij}\alpha^i \wedge \alpha^j. \tag{5.3.7}$$

Let $\mathscr{T}$ denote the group of all left invariant global T-duality transformations for the double manifold (TG, K, η). Hence, $\mathcal{A} \in \mathscr{T}$ is a left invariant bundle automorphism $\mathcal{A}\colon TTG \to TTG$ satisfying $\mathcal{A}^*\eta = \eta$. By Proposition 5.4, $\mathcal{A}^*\mathcal{H}(g, B)$ is also a (left invariant) generalized metric when g is positive definite. Hence, $\mathcal{A}^*\mathcal{H}(g, B) = \mathcal{H}(g_{\mathcal{A}}, B_{\mathcal{A}})$ for some unique pair $(g_{\mathcal{A}}, B_{\mathcal{A}})$. We will call the pair $(g_{\mathcal{A}}, B_{\mathcal{A}})$ the T-duality transform of (g, B) by $\mathcal{A}$. For convenience, let us identify $\mathcal{A}$ with its matrix representation with respect to the global frame (5.3.1) by

$$\mathcal{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{5.3.8}$$

where a, b, c, d are $n \times n$ (real) matrices satisfying

$$-c^T\omega b + a^T\omega d = \omega, \qquad -c^T\omega a + a^T\omega c = -d^T\omega b + b^T\omega d = 0. \tag{5.3.9}$$

The next result explicitly gives the T-duality transform of a pair (g, B) by an element $\mathcal{A} \in \mathscr{T}$.

Proposition 5.6. *Let (TG, K, η) be equipped with the left invariant fields (g, B), where g is positive definite. Let $\mathcal{A} \in \mathscr{T}$ and let* (5.3.8) *be the matrix representation of $\mathcal{A}$ with respect to the global frame* (5.3.1)*. Then $\mathcal{A}^*\mathcal{H}(g, B) = \mathcal{H}(g_{\mathcal{A}}, B_{\mathcal{A}})$ where*

$$g_{\mathcal{A}} = -\omega \left[b^T gb + (Bb - \omega d)^T g^{-1}(Bb - \omega d)\right]^{-1} \omega, \tag{5.3.10}$$

$$B_{\mathcal{A}} = \left[a^T gb + (Ba - \omega c)^T g^{-1}(Bb - \omega d)\right] \omega^{-1} g_{\mathcal{A}}, \tag{5.3.11}$$

where g, B, $g_{\mathcal{A}}$, and $B_{\mathcal{A}}$ have been identified with their matrix representations with respect to the frame $X_1^c, \dots, X_n^c$ on H.

Proof. Expressing everything in terms of their matrix representations and applying (5.3.6), we have

$$\mathcal{A}^T \mathcal{H}(g, B)\mathcal{A} = \mathcal{H}(g_{\mathcal{A}}, B_{\mathcal{A}}). \tag{5.3.12}$$

Comparing the $(2, 2)$-blocks of the left and right sides, we obtain (5.3.10). Likewise, comparing the $(2, 1)$-blocks of both sides yields (5.3.11). □

Remark 5.7. From the point of view of physics, the statement $\mathcal{A}^*\mathcal{H}(g, B) = \mathcal{H}(g_{\mathcal{A}}, B_{\mathcal{A}})$ implies that the physics associated to the background fields (g, B) and $(g_{\mathcal{A}}, B_{\mathcal{A}})$ are equivalent (cf. [**10**]).

Let us now consider left invariant double metric connections on the double manifold $(M = TG, K, \eta)$ for left invariant fields (g, B). Since η, g, and B are left invariant, it follows that the total spaces $\widehat{S}_\pm$ given by (5.1.6)–(5.1.7), as well as the bundle isomorphisms

$$\iota_\pm \colon L = H \to \widehat{S}_\pm$$

given by (5.2.4) are also left invariant. Hence, ∇ is a left invariant double metric connection if and only if the associated projected connections $\nabla^\pm$ on the vector bundles $\widehat{S}_\pm \to TG$ (see (5.2.1)) are left invariant. This in turn is equivalent to the condition that the associated connections $D^\pm$ on the vector bundle $L = H \to M = TG$ (see (5.2.6)) are also left invariant. Proposition 5.5 now implies the following corollary.

Corollary 5.8. *For the double manifold (TG, K, η) with left invariant fields (g, B), there is a one-to-one correspondence between left invariant double metric connections and pairs of left invariant connections $D^\pm$ on the bundle $H \to TG$ satisfying $D^\pm g = 0$.*

We conclude this section by computing all left invariant double metric connections for (TG, K, η) with respect to a fixed left invariant pair (g, B). For $\mathbf{A} \in Lie(TG)$, let $\mathbf{A}^\pm$ denote the projection onto $\widehat{S}_\pm$.

Lemma 5.9. *Let g and B be given by* (5.3.7)*. Then*

$$X_i^{c+} = \frac{1}{2}(\delta_i^m + B_{ij}g^{jm})\iota_+(X_m^c), \tag{5.3.13}$$

$$X_i^{c-} = \frac{1}{2}(\delta_i^m - B_{ij}g^{jm})\iota_-(X_m^c), \tag{5.3.14}$$

$$X_i^{v+} = -\frac{1}{2}\omega_{il}g^{lm}\iota_+(X_m^c), \tag{5.3.15}$$

$$X_i^{v-} = \frac{1}{2}\omega_{il}g^{lm}\iota_-(X_m^c), \tag{5.3.16}$$

where $\delta_i^m = 1$ *if* $i = m$ *and zero otherwise.*

Proof. From the definition of $\iota_\pm \colon H \to \widehat{S}_\pm$ (5.2.4), we have

$$\iota_+(X_i^c) := \psi_\eta^{-1}(X_i^c, (-B_{ij} + g_{ij})\alpha^j) = X_i^c + (-B_{ij} + g_{ij})\omega^{kj}X_k^v, \tag{5.3.17}$$

$$\iota_-(X_i^c) := \psi_\eta^{-1}(X_i^c, (-B_{ij} - g_{ij})\alpha^j) = X_i^c + (-B_{ij} - g_{ij})\omega^{kj}X_k^v. \tag{5.3.18}$$

Using (5.3.17) and (5.3.18), one verifies (5.3.13)–(5.3.16). □

Proposition 5.10. *Let* (g, B) *be a left invariant pair for* (TG, K, η) *given by* (5.3.7) *and let*

$$\{\gamma_{ij}^{k+},\ \gamma_{ij}^{k-},\ \gamma_{\bar{i}j}^{k+},\ \gamma_{\bar{i}j}^{k-} \mid i, j, k = 1, \ldots, n\}$$

be any collection of real numbers satisfying

$$\gamma_{ki}^{m\pm}g_{mj} + \gamma_{kj}^{m\pm}g_{im} = 0, \qquad \gamma_{\bar{k}i}^{m\pm}g_{mj} + \gamma_{\bar{k}j}^{m\pm}g_{im} = 0. \tag{5.3.19}$$

Define

$$\Gamma_{ij}^m = \frac{1}{2}\left[\gamma_{ij}^{m+} + \gamma_{ij}^{m-} + B_{jk}g^{kl}(\gamma_{il}^{m+} - \gamma_{il}^{m-})\right], \tag{5.3.20}$$

$$\begin{aligned}\Gamma_{ij}^{\bar{q}} &= \frac{1}{2}(\gamma_{ij}^{m+} + B_{jk}g^{kl}\gamma_{il}^{m+})(-B_{mp} + g_{mp})\omega^{qp}, \\ &\quad + \frac{1}{2}(\gamma_{ij}^{m-} - B_{jk}g^{kl}\gamma_{il}^{m-})(-B_{mp} - g_{mp})\omega^{qp},\end{aligned} \tag{5.3.21}$$

$$\Gamma_{\bar{i}j}^m = \frac{1}{2}[\gamma_{\bar{i}j}^{m+} + \gamma_{\bar{i}j}^{m-} + B_{jk}g^{kl}(\gamma_{\bar{i}l}^{m+} - \gamma_{\bar{i}l}^{m-})], \tag{5.3.22}$$

$$\begin{aligned}\Gamma_{\bar{i}j}^{\bar{q}} &= \frac{1}{2}(\gamma_{\bar{i}j}^{m+} + B_{jk}g^{kl}\gamma_{\bar{i}l}^{m+})(-B_{mp} + g_{mp})\omega^{qp}. \\ &\quad + \frac{1}{2}(\gamma_{\bar{i}j}^{m-} - B_{jk}g^{kl}\gamma_{\bar{i}l}^{m-})(-B_{mp} - g_{mp})\omega^{qp},\end{aligned} \tag{5.3.23}$$

$$\Gamma_{i\bar{j}}^m = -\frac{1}{2}\omega_{jk}g^{kl}(\gamma_{il}^{m+} - \gamma_{il}^{m-}), \tag{5.3.24}$$

$$\Gamma_{i\bar{j}}^{\bar{q}} = -\frac{1}{2}\omega_{jk}g^{kl}\gamma_{il}^{m+}(-B_{mp} + g_{mp})\omega^{qp} + \frac{1}{2}\omega_{jk}g^{kl}\gamma_{il}^{m-}(-B_{mp} - g_{mp})\omega^{qp}, \tag{5.3.25}$$

$$\Gamma_{\bar{i}\bar{j}}^m = -\frac{1}{2}\omega_{jk}g^{kl}(\gamma_{\bar{i}l}^{m+} - \gamma_{\bar{i}l}^{m-}), \tag{5.3.26}$$

$$\Gamma_{\bar{i}\bar{j}}^{\bar{q}} = -\frac{1}{2}\omega_{jk}g^{kl}\gamma_{\bar{i}l}^{m+}(-B_{mp} + g_{mp})\omega^{qp} + \frac{1}{2}\omega_{jk}g^{kl}\gamma_{\bar{i}l}^{m-}(-B_{mp} - g_{mp})\omega^{qp}. \tag{5.3.27}$$

Then (5.3.20)–(5.3.27) *define the Christoffel symbols of a left invariant double metric connection with respect to the global frame* (5.3.1). *Moreover, every left invariant double metric connection for the pair* (g, B) *is of this form.*

Proof. By Corollary 5.8, there is a one-to-one correspondence between left invariant double metric connections for the pair (g, B) and pairs $D^\pm$ of left invariant

g-metric connections on the vector bundle $H \to TG$. Given a choice of frame (5.3.1), any collection of numbers

$$\{\gamma_{ij}^{k+},\ \gamma_{ij}^{k-},\ \gamma_{\bar{i}j}^{k+},\ \gamma_{\bar{i}j}^{k-} \mid i,j,k=1,\dots,n\}$$

satisfying (5.3.19) uniquely defines a pair $D^{\pm}$ of left invariant g-metric connections via

$$D^{\pm}_{X_i^c} X_j^c = \gamma_{ij}^{m\pm} X_m^c, \qquad D^{\pm}_{X_i^v} X_j^c = \gamma_{\bar{i}j}^{m\pm} X_m^c.$$

Let ∇ be the left invariant double metric connection associated to $D^{\pm}$. We only calculate the Christoffel symbols Γ_{ij}^m and $\Gamma_{ij}^{\bar{q}}$ for ∇. The other Christoffel symbols are computed in an entirely similar manner. From Section 5.2, we have

$$\begin{aligned}
\nabla_{X_i^c} X_j^c &= \nabla^+_{X_i^c} X_j^{c+} + \nabla^-_{X_i^c} X_j^{c-}\\
&= \frac{1}{2}(\delta_j^l + B_{jk} g^{kl}) \nabla^+_{X_i^c} \iota_+(X_l^c) + \frac{1}{2}(\delta_j^l - B_{jk} g^{kl}) \nabla^-_{X_i^c} \iota_-(X_l^c)\\
&= \frac{1}{2}(\delta_j^l + B_{jk} g^{kl}) \iota_+ \left(D^+_{X_i^c} X_l^c\right) + \frac{1}{2}(\delta_j^l - B_{jk} g^{kl}) \iota_- \left(D^-_{X_i^c} X_l^c\right)\\
&= \frac{1}{2}(\delta_j^l + B_{jk} g^{kl}) \gamma_{il}^{m+} \iota_+(X_m^c) + \frac{1}{2}(\delta_j^l - B_{jk} g^{kl}) \gamma_{il}^{m-} \iota_-(X_m^c)\\
&= \frac{1}{2}(\delta_j^l + B_{jk} g^{kl}) \gamma_{il}^{m+} (X_m^c + (-B_{mp} + g_{mp}) \omega^{qp} X_q^v)\\
&\quad + \frac{1}{2}(\delta_j^l - B_{jk} g^{kl}) \gamma_{il}^{m-} (X_m^c + (-B_{mp} - g_{mp}) \omega^{qp} X_q^v)\\
&= \frac{1}{2}\left[\gamma_{ij}^{m+} + \gamma_{ij}^{m-} + B_{jk} g^{kl} (\gamma_{il}^{m+} - \gamma_{il}^{m-})\right] X_m^c\\
&\quad + \frac{1}{2}(\gamma_{ij}^{m+} + B_{jk} g^{kl} \gamma_{il}^{m+})(-B_{mp} + g_{mp}) \omega^{qp} X_q^v\\
&\quad + \frac{1}{2}(\gamma_{ij}^{m-} - B_{jk} g^{kl} \gamma_{il}^{m-})(-B_{mp} - g_{mp}) \omega^{qp} X_q^v,
\end{aligned} \tag{5.3.28}$$

where we have used Lemma 5.9 in the second equality and (5.3.17)–(5.3.18) in the fifth equality. Comparing (5.3.28) to

$$\nabla_{X_i^c} X_j^c = \Gamma_{ij}^m X_m^c + \Gamma_{ij}^{\bar{q}} X_q^v$$

gives (5.3.20)–(5.3.21). □

5.4. Associated metric algebroid

In [**22**], Vaisman showed that the tangent bundle of a double manifold admits a structure similar to that of a Courant algebroid [**13**]. Vaisman called this structure a *metric algebroid.*

Definition 5.11. A metric algebroid consists of the following data:

(a) a vector bundle $E \to M$,
(b) a bundle map $\rho\colon E \to TM$ (the anchor map),
(c) a bundle metric η on E,
(d) an $\mathbb{R}$-bilinear product $\bigstar\colon \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ (the metric product)

which satisfies the following axioms:

(i) $\rho(e)\eta(e_1, e_2) = \eta(e \bigstar e_1, e_2) + \eta(e_1, e \bigstar e_2)$ (η-compatibility axiom),
(ii) $e \bigstar e = \partial(\eta(e, e))$ (normalization axiom),

where $\partial \colon C^\infty(M) \to \Gamma(E)$ is defined by $\partial := \frac{1}{2}\sharp_\eta \circ \rho^* \circ d$.

In this section, we describe the metric algebroid structure for the tangent bundle of the double manifold (TG, K, η). For the anchor map, we simply take $\rho = \mathrm{id} \colon TTG \to TTG$. Then the map $\partial \colon C^\infty(TG) \to \Gamma(TTG)$ is

$$\partial = \frac{1}{2}\sharp_\eta \circ d.$$

Let $X_1, \ldots, X_n$ be a basis of left invariant vector fields on G. Again, we write $[X_i, X_j] = c_{ij}^k X_k$, $\omega_{ij} = \omega(X_i, X_j)$ and $(\omega^{ij}) = (\omega_{ij})^{-1}$. Explicitly,

$$\partial f = \frac{1}{2}\omega^{ij}((X_i^v f)X_j^c - (X_i^c f)X_j^v) \in \Gamma(TTG) \qquad \text{for all } f \in C^\infty(TG). \tag{5.4.1}$$

To construct a metric product $\bigstar$ on $\Gamma(TTG)$, let ∇^0 again denote the Levi-Civita connection of η. For vector fields $\mathbf{X}, \mathbf{Y} \in \Gamma(TTG)$ (not necessarily left invariant), define a skew-symmetric product $\mathbf{X} \wedge_{\nabla^0} \mathbf{Y} \in \Gamma(TTG)$ by

$$\eta(\mathbf{Z}, \mathbf{X} \wedge_{\nabla^0} \mathbf{Y}) = \frac{1}{2}[\eta(\mathbf{X}, \nabla^0_{\mathbf{Z}}\mathbf{Y}) - \eta(\mathbf{Y}, \nabla^0_{\mathbf{Z}}\mathbf{X})] \qquad \text{for all } \mathbf{Z} \in \Gamma(TTG). \tag{5.4.2}$$

For $f \in C^\infty(TG)$, one can show that

$$\mathbf{X} \wedge_{\nabla^0} (f\mathbf{Y}) = f(\mathbf{X} \wedge_{\nabla^0} \mathbf{Y}) + \eta(\mathbf{X}, \mathbf{Y})\partial f. \tag{5.4.3}$$

Using Lemma 3.4, one finds

$$X_i^c \wedge_{\nabla^0} X_j^c = X_i^v \wedge_{\nabla^0} X_j^v = 0, \qquad X_i^c \wedge_{\nabla^0} X_j^v = c_{kj}^l \omega_{li}\omega^{km} X_m^v. \tag{5.4.4}$$

For $\mathbf{X}, \mathbf{Y} \in \Gamma(TTG)$, define a bracket on $\Gamma(TTG)$ by

$$[\mathbf{X}, \mathbf{Y}]_{\nabla^0} := [\mathbf{X}, \mathbf{Y}] - \mathbf{X} \wedge_{\nabla^0} \mathbf{Y}. \tag{5.4.5}$$

In [**22**], $[\cdot,\cdot]_{\nabla^0}$ is called the ∇^0-bracket. For $f \in C^\infty(TG)$, we have

$$[\mathbf{X}, f\mathbf{Y}]_{\nabla^0} = f[\mathbf{X}, \mathbf{Y}]_{\nabla^0} + (\mathbf{X}f)\mathbf{Y} - \eta(\mathbf{X}, \mathbf{Y})\partial f. \tag{5.4.6}$$

By direct calculation, one finds

(5.4.7)
$$[X_i^c, X_j^c]_{\nabla^0} = c_{ij}^m X_m^c, \quad [X_i^c, X_j^v]_{\nabla^0} = (c_{ij}^m - c_{kj}^l \omega_{li}\omega^{km})X_m^v, \quad [X_i^v, X_j^v]_{\nabla^0} = 0.$$

For $\mathbf{X}, \mathbf{Y} \in \Gamma(TTG)$, consider the $\mathbb{R}$-bilinear product $\bigstar_{\nabla^0}$ on $\Gamma(TTG)$ defined by

$$\mathbf{X} \bigstar_{\nabla^0} \mathbf{Y} = [\mathbf{X}, \mathbf{Y}]_{\nabla^0} + \partial\eta(\mathbf{X}, \mathbf{Y}). \tag{5.4.8}$$

When $\mathbf{X}$ and $\mathbf{Y}$ are left invariant, we have $\mathbf{X} \bigstar_{\nabla^0} \mathbf{Y} = [\mathbf{X}, \mathbf{Y}]_{\nabla^0}$. For $f \in C^\infty(TG)$, $\mathbf{X}, \mathbf{Y} \in \Gamma(TTG)$, one has

$$\mathbf{X} \bigstar_{\nabla^0} (f\mathbf{Y}) = f(\mathbf{X} \bigstar_{\nabla^0} \mathbf{Y}) + (\mathbf{X}f)\mathbf{Y}.$$

Using the definitions of $[\cdot,\cdot]_{\nabla^0}$ and ∂, one can check that $\bigstar_{\nabla^0}$ is a metric product. It then follows that $TTG \to TG$ is a metric algebroid with anchor $\rho = id$, bundle metric η, and metric product $\bigstar_{\nabla^0}$.

Appendix A. Flat para-Kähler manifolds from affine manifolds

We remind the reader that an affine manifold is a smooth n-dimensional manifold Q which can be covered by charts with the property that the transition maps between overlapping charts are locally affine transformations of $\mathbb{R}^n$. The aforementioned charts are called affine charts. Hence, if (U, x^i) and (V, y^i) are overlapping charts, then locally

$$y^i = a^i_j x^j + b^i, \qquad a^i_j,\ b^i \in \mathbb{R}.$$

This is equivalent to the condition that $\frac{\partial y^i}{\partial x^j}$ be locally constant.

The following result may be well-known to experts. Bejan proves in [**4**] that a flat connection ∇ on a manifold M yields a para-Kähler structure on T^*M. A closer examination of this result shows that if the flat connection is also torsion free, then the para-Kähler structure is also flat. From [**2**], it is a well known fact that the existence of a flat torsion free connection on a manifold is equivalent to an affine structure on the manifold. For the reader's convenience, we present an elementary proof of Bejan's result for the special case of affine manifolds.

Theorem A.1. *Let Q be an affine manifold. Then T^*Q admits a flat para-Kähler structure (K, η) such that $\Omega(\cdot, \cdot) := \eta(K\cdot, \cdot)$ is the canonical symplectic form on T^*Q.*

Proof. Let (U, x^i) be an affine chart on Q and let $(T^*U, x^i, \tilde{x}_j)$ be the induced coordinates on $T^*U \subset T^*Q$. Define $K_U \colon T(T^*U) \to T(T^*U)$ by

$$K_U := \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^i - \frac{\partial}{\partial \tilde{x}_i} \otimes \mathrm{d}\tilde{x}_i.$$

Let (V, y^i) be another affine chart on Q such that $U \cap V \neq \emptyset$ and let $(T^*V, y^i, \tilde{y}_j)$ be the induced coordinates on T^*V. Define $K_V \colon T(T^*V) \to T(T^*V)$ by

$$K_V := \frac{\partial}{\partial y^i} \otimes \mathrm{d}y^i - \frac{\partial}{\partial \tilde{y}_i} \otimes \mathrm{d}\tilde{y}_i.$$

Note that

$$x^i = x^i(y), \qquad \tilde{x}_i = \tilde{y}_j \frac{\partial y^j}{\partial x^i}.$$

Since (U, x^i) and (V, y^i) are affine charts, $\frac{\partial y^j}{\partial x^i}$ is locally constant. Hence, $\tilde{x}_j = \tilde{x}_j(\tilde{y})$ locally. So on $T^*U \cap T^*V$, we have

$$\begin{aligned} K_U &= \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^i - \frac{\partial}{\partial \tilde{x}_i} \otimes \mathrm{d}\tilde{x}_i = \frac{\partial y^j}{\partial x^i}\frac{\partial x^i}{\partial y^k}\frac{\partial}{\partial y^j} \otimes \mathrm{d}y^k - \frac{\partial \tilde{y}_j}{\partial \tilde{x}_i}\frac{\partial y^k}{\partial x^i}\frac{\partial}{\partial \tilde{y}_j} \otimes d\tilde{y}_k \\ &= \delta^j_k \frac{\partial}{\partial y^j} \otimes \mathrm{d}y^k - \frac{\partial x^i}{\partial y^j}\frac{\partial y^k}{\partial x^i}\frac{\partial}{\partial \tilde{y}_j} \otimes d\tilde{y}_k = \frac{\partial}{\partial y^j} \otimes \mathrm{d}y^j - \delta^k_j \frac{\partial}{\partial \tilde{y}_j} \otimes d\tilde{y}_k \\ &= \frac{\partial}{\partial y^j} \otimes \mathrm{d}y^j - \frac{\partial}{\partial \tilde{y}_j} \otimes d\tilde{y}_j = K_V. \end{aligned}$$

Hence, the K_U's glue together to form a global bundle map $K\colon T(T^*Q) \to T(T^*Q)$. In other words, K is defined via

$$K|_{T(T^*U)} = K_U.$$

Since $K_U^2 = \mathrm{id}$ on $T(T^*U)$, we have that $K^2 = \mathrm{id}$. From the coordinate system $(T^*U, x^i, \tilde{x}_i)$, it is clear that the $+1$ eigenbundle L of $T(T^*Q)$ has a local frame given by

$$\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n},$$

and the -1 eigenbundle $\tilde{L}$ has a local frame given by

$$\frac{\partial}{\partial \tilde{x}_1}, \ldots, \frac{\partial}{\partial \tilde{x}_n}.$$

This immediately implies that L and $\tilde{L}$ are involutive distributions. Hence, K is a paracomplex structure and the local coordinate systems of the form $(T^*U, x^i, \tilde{x}_i)$ are paracomplex coordinates, where (U, x^i) are local affine coordinates on Q.

Now let Ω denote the canonical symplectic form on T^*Q. With respect to the paracomplex coordinates $(T^*U, x^i, \tilde{x}_i)$, Ω is locally given by

$$\Omega = \mathrm{d}x^i \wedge \mathrm{d}\tilde{x}_i.$$

Let $\eta(\cdot,\cdot) := \Omega(K\cdot,\cdot)$. Then locally

$$\eta = \mathrm{d}x^i \otimes \mathrm{d}\tilde{x}_i + \mathrm{d}\tilde{x}_i \otimes \mathrm{d}x^i.$$

Clearly, $\eta(K\cdot, K\cdot) = -\eta(\cdot,\cdot)$ and $\Omega(\cdot,\cdot) = \eta(K\cdot,\cdot)$. Hence, (T^*Q, K, η) is para-Kähler. Moreover, since the components of η are locally constant, it follows that η is also flat. □

From the proof of Theorem A.1, we see that if Q is an affine manifold, then T^*Q is also affine. This has the following immediate corollary.

Corollary A.2. *Let Q be an affine manifold and let $T^{*,k+1}Q := T^*(T^{*,k}Q)$, where $T^{*,0}Q := Q$. Then $T^{*,k+1}Q$ admits a flat para-Kähler structure.*

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