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STRONG CONVERGENCE THEOREMS USING THREE-STEP MEAN ITERATION FOR ZAMFIRESCU MAPPINGS IN BANACH SPACES

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Abstract. This paper addresses the approximation problem for fixed points of Zamfirescu mappings (Z-mapping) [Arch. Math. **23**(1) (1972), 292–298]. We use a three-step mean iteration that combines Noor's iteration as well as mean-valued iteration, and we prove a general theorem that extends Berinde's strong convergence theorem [Acta Math. Univ. Comenianae **73**(1) (2004), 119–126]. Our results are obtained in arbitrary real Banach space setting. Furthermore, an application to a variational inequality problem is presented in a framework of real Hilbert spaces.

1. Introduction

Let X be a complete metric space with a metric d. A mapping $T\colon X \to X$ is called a *contraction* if there exists $a \in (0,1)$ such that

$$d(Tx, Ty) \le a d(x, y) \tag{1.1}$$

for all $x, y \in X$. One of the most famous fixed point theorem is known as the Banach contraction principle:

Theorem 1.1 ([**4**])**.** *Let X be a complete metric space and let $T\colon X \to X$ be a contraction. Then, T has a unique fixed point p, and a sequence $\{x_n\}$ defined by*

$$x_{n+1} = Tx_n \quad \text{for all } n \in \mathbb{N} \tag{1.2}$$

converges to the fixed point p for any initial point $x_1 \in X$.

In Theorem 1.1, $\mathbb{N}$ stands for the set of natural numbers. This theorem has been extended in various directions; see, for example, [**2, 6, 15, 31, 37, 39, 40**]. In 1968, Kannan [**14**] defined a mapping that satisfies the following condition: there exists $b \in \left(0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \le b\left(d(x, Tx) + d(y, Ty)\right) \tag{1.3}$$

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for all $x, y \in X$. In 1972, Chatterjea [**9**] introduced a type of mappings defined by the following condition: there exists $c \in \left(0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq c(d(x, Ty) + d(Tx, y)) \tag{1.4}$$

for all $x, y \in X$. Kannan [**14**] and Chatterjea [**9**] proved the same conclusion as expressed by Theorem 1.1. It is remarkable that Kannan mappings (1.3) and Chatterjea mappings (1.4) are not necessarily continuous; see, for instance, Berinde [**8**]. In 1972, Zamfirescu [**43**] unified these conditions (1.1), (1.3), and (1.4) and defined a class of mappings:

Definition 1.1 ([**43**]). Let X be a metric space. A mapping $T\colon X \to X$ is called a Zamfirescu mapping, or simply Z-mapping, if there exist $a \in (0, 1)$ and $b, c \in \left(0, \frac{1}{2}\right)$ such that for any $x, y \in X$, at least one of the following three conditions holds:

$(Z1)$ $d(Tx, Ty) \leq ad(x, y)$;

$(Z2)$ $d(Tx, Ty) \leq b(d(x, Tx) + d(y, Ty))$;

$(Z3)$ $d(Tx, Ty) \leq c(d(x, Ty) + d(Tx, y))$.

This class of Z-mappings simultaneously contains contraction mappings (1.1), Kannan mappings (1.3), and Chatterjea mappings (1.4). Zamfirescu demonstrated the following theorem, which generalizes Theorem 1.1:

Theorem 1.2 ([**43**]). *Let X be a complete metric space and let $T\colon X \to X$ be a Z-mapping. Then, there exists a unique fixed point p of T. Furthermore, T is a Picard mapping; in other words, $T^n x \to p$ for every $x \in X$.*

The iteration procedure (1.2), which also appears in Theorem 1.2, is called the *Picard iteration*. In the literature of fixed point theory on Banach or Hilbert spaces, many other approximation methods have been established. The following iteration is called the Mann's type [**26**]:

$$\begin{aligned} x_1 &= x \in C \text{ is given,} \\ x_{n+1} &= \lambda_n x_n + (1 - \lambda_n) T x_n \end{aligned} \tag{1.5}$$

for all $n \in \mathbb{N}$, where $\lambda_n \in [0, 1]$ under certain conditions. Sequences generated by this rule (1.5) converge weakly to a fixed point of T; see, for example, Reich [**32**]. If $\lambda_n = 0$, then the Mann iteration (1.5) coincides with the Picard iteration (1.2). In 1974, Ishikawa [**13**] proposed a more general two-step iteration procedure:

$$\begin{aligned} x_1 &= x \in C \text{ is given,} \\ y_n &= A_n x_n + (1 - A_n) T x_n, \text{ and} \\ x_{n+1} &= \lambda_n x_n + (1 - \lambda_n) T y_n \end{aligned} \tag{1.6}$$

for all $n \in \mathbb{N}$, where $\lambda_n, A_n \in [0, 1]$ under certain conditions. If $A_n = 1$, then the Ishikawa iteration (1.6) is equivalent to the Mann's iteration (1.5). In a uniformly convex Banach space, Rhoades [**34**] obtained a strong convergence of a sequence defined by (1.6) for a Z-mapping T; see [**34**, Theorem 8]. Berinde [**5**] then later proved the following theorem in an arbitrary real Banach space setting.

Theorem 1.3 ([**5**]). *Let C be a nonempty, closed, and convex subset of a real Banach space E and let T be a Z-mapping from C into itself. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0,1]$ such that $\sum_{n=1}^{\infty}(1-\lambda_n)=\infty$. Let $\{A_n\}$ be a sequence of real numbers in $[0,1]$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned} x_1 &= x \in C \text{ is given,} \\ y_n &= A_n x_n + (1-A_n)\, T x_n, \\ x_{n+1} &= \lambda_n x_n + (1-\lambda_n)\, T y_n \end{aligned}$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a unique fixed point p of T.

For related results, see also Rhoades [**33**] and Berinde [**7**]. The Ishikawa iteration has been further extended to a three-step version. Noor [**29**] introduced the following algorithm:

$$\begin{aligned} x_1 &= x \in C \text{ is given,} \\ v_n &= \zeta_n x_n + (1-\zeta_n)\, T x_n, \\ y_n &= A_n x_n + (1-A_n)\, T v_n, \text{ and} \\ x_{n+1} &= \lambda_n x_n + (1-\lambda_n)\, T y_n \end{aligned} \tag{1.7}$$

for all $n \in \mathbb{N}$. Obviously, if $\zeta_n = 1$ in (1.7), it coincides with the Ishikawa iteration. For other variations of three-step iteration, see also Dashputre and Diwan [**11**], Phuengrattana and Suantai [**30**], Chugh et al. [**10**], and Kondo [**21, 22**].

On the other hand, the following type of mean-valued iteration is often used to approximate fixed points:

$$x_{n+1} = \lambda_n x_n + (1-\lambda_n)\, \frac{1}{n} \sum_{i=1}^{n} T^i x_n \tag{1.8}$$

for all $n \in \mathbb{N}$; see Shimizu and Takahashi [**35**] and Atsushiba and Takahashi [**1**]. This type of mean iteration has its roots in Baillon [**3**]. For more recent works concerning the mean iteration method, see [**12, 16, 18, 20**] and the papers cited therein. According to Maruyama et al. [**27**] and Kondo and Takahashi [**23, 24, 25**], the following iteration is effective for some classes of nonlinear mappings:

$$x_{n+1} = \lambda_n x_n + \mu_n T x_n + \nu_n T^2 x_n \tag{1.9}$$

for all $n \in \mathbb{N}$. For this type of iteration, see also Kondo [**17, 19**] and Singh et al. [**36**].

This paper addresses the fixed point approximation problem for Zamfirescu mapping (Z-mapping) in arbitrary real Banach spaces. Berinde's theorem (Theorem 1.3) and the Noor's iteration are extended by incorporating the mean iteration (1.8) and the "2-iteration" (1.9). After preparing some lemmas in Section 2, we demonstrate a theorem that jointly employs these iterations in Section 3. Berinde's theorem (Theorem 1.3) is derived from our theorem. Our main theorem applies to classes of mappings, such as contraction mappings (1.1), Kannan mappings (1.3), and Chatterjea mappings (1.4). Furthermore, we apply a result to a variational inequality problem (VIP) and present an approximation method for finding a solution to the VIP in Section 4. Such an attempt implies that our result can be applied to optimization problems because, as is well-known, VIPs directly connect with optimization problems.

2. Lemmas

In this section, three lemmas are prepared.

Lemma 2.1. *Let C be a nonempty subset of a normed linear space E and let $T\colon C \to C$ be a Z-mapping. Then, there exists $\rho \in (0,1)$ such that*

$$(1) \qquad \left\| T^H x - p \right\| \leq \rho \left\| x - p \right\| \text{ and}$$

$$(2) \qquad \left\| \frac{1}{n} \sum_{i=h}^{n+h-1} T^i x - p \right\| \leq \rho \left\| x - p \right\|$$

for any $x \in C$ and a fixed point p of T, where $H, n, h \in \mathbb{N}$.

Proof. Note that a fixed point of a Z-mapping T is unique (if it exists). First, we prove (1). Let $x \in C$ and let $p \in C$ be a unique fixed point of T. As T is a Z-mapping, at least one of the conditions ($Z1$), ($Z2$), or ($Z3$) in Definition 1.1 holds. If ($Z1$) holds, then it follows from $p = Tp$ that

$$\|Tx - p\| = \|Tx - Tp\| \leq a \|x - p\|, \tag{2.1}$$

where $a \in (0,1)$. If ($Z2$) holds true, then

$$\begin{aligned} \|Tx - p\| = \|Tx - Tp\| &\leq b \left(\|x - Tx\| + \|p - Tp\| \right) \\ &= b \|x - Tx\| \leq b \|x - p\| + b \|p - Tx\|, \end{aligned}$$

where $b \in \left(0, \frac{1}{2}\right)$. This implies that

$$(1 - b) \|Tx - p\| \leq b \|x - p\|,$$

and hence, we have

$$\|Tx - p\| \leq \frac{b}{1 - b} \|x - p\|. \tag{2.2}$$

If ($Z3$) holds, then there exists $c \in \left(0, \frac{1}{2}\right)$ such that

$$\begin{aligned} \|Tx - p\| &= \|Tx - Tp\| \\ &\leq c \left(\|x - Tp\| + \|Tx - p\| \right) \\ &= c \left(\|x - p\| + \|Tx - p\| \right). \end{aligned}$$

This expression yields

$$\|Tx - p\| \leq \frac{c}{1 - c} \|x - p\|. \tag{2.3}$$

Define $\rho = \max\left\{a, \ \frac{b}{1-b}, \ \frac{c}{1-c}\right\}$. As $a \in (0,1)$ and $b, c \in \left(0, \frac{1}{2}\right)$, we have $\rho \in (0,1)$.

From (2.1)–(2.3), we obtain $\|Tx-p\| \le \rho\|x-p\|$ for any $x \in C$ and a fixed point p of T. As $\rho \in (0,1)$, we can express the desired result as follows:

$$\left\|T^H x-p\right\| \le \rho\left\|T^{H-1}x-p\right\| \le \cdots \le \rho^H\|x-p\| \le \rho\|x-p\|.$$

Next, we prove (2). From (1), it holds that

$$\begin{aligned}\left\|\frac{1}{n}\sum_{i=h}^{n+h-1} T^i x-p\right\| &= \frac{1}{n}\left\|\sum_{i=h}^{n+h-1} T^i x-np\right\| = \frac{1}{n}\left\|\sum_{i=h}^{n+h-1}\left(T^i x-p\right)\right\| \\ &\le \frac{1}{n}\sum_{i=h}^{n+h-1}\left\|T^i x-p\right\| \\ &= \frac{1}{n}\Big(\left\|T^h x-p\right\|+\left\|T^{h+1}x-p\right\|+\cdots+\left\|T^{n+h-1}x-p\right\|\Big) \\ &\le \frac{1}{n}\left(\rho\|x-p\|+\rho\|x-p\|+\cdots+\rho\|x-p\|\right) \\ &= \rho\|x-p\|.\end{aligned}$$

This completes the proof. □

It is clear that (1) in Lemma 2.1 holds true in a setting of a metric space. The next lemma is a slightly generalized version of an inequality used in Berinde [**5**].

Lemma 2.2. *Let* $A, a, \alpha, \rho \in [0,1]$ *and let* $\lambda, \mu, \nu, \xi \in [0,1]$ *with* $\lambda+\mu+\nu+\xi = 1$. *Then,*

$$\lambda+\mu\rho A+\mu\rho^2(1-A)+\nu\rho a+\nu\rho^2(1-a)+\xi\rho\alpha+\xi\rho^2(1-\alpha) \le 1-(1-\rho)(1-\lambda).$$

Proof. Easy calculation yields

$$\begin{aligned}&\lambda+\mu\rho A+\mu\rho^2(1-A)+\nu\rho a+\nu\rho^2(1-a)+\xi\rho\alpha+\xi\rho^2(1-\alpha) \\ =\ &\lambda+\mu\rho(A+\rho(1-A))+\nu\rho(a+\rho(1-a))+\xi\rho(\alpha+\rho(1-\alpha)) \\ =\ &\lambda+\mu\rho((1-\rho)A+\rho)+\nu\rho((1-\rho)a+\rho)+\xi\rho((1-\rho)\alpha+\rho).\end{aligned}$$

Since $\mu\rho(1-\rho) \ge 0$, $\nu\rho(1-\rho)$, $\xi\rho(1-\rho) \ge 0$, substituting $A = a = \alpha = 1$, we obtain the desired result as follows:

$$\begin{aligned}&\lambda+\mu\rho A+\mu\rho^2(1-A)+\nu\rho a+\nu\rho^2(1-a)+\xi\rho\alpha+\xi\rho^2(1-\alpha) \\ \le\ &\lambda+\mu\rho+\nu\rho+\xi\rho \\ =\ &\lambda-1+1+\rho(1-\lambda) \\ =\ &1-(1-\lambda)(1-\rho).\end{aligned}$$

□

Although the next lemma is well-known in the literature, we provide a proof for completeness.

Lemma 2.3. *Let* $\{\lambda_n\}$ *be a sequence of real numbers in the interval* $[0,1]$ *such that* $\sum_{n=1}^{\infty}(1-\lambda_n) = \infty$. *Let* $\Lambda > 0$ *be a positive real number such that* $\Lambda(1-\lambda_n) < 1$ *for all* $n \in \mathbb{N}$. *Then,* $\prod_{n=1}^{\infty}(1-\Lambda(1-\lambda_n)) = 0$.

Proof. Define $P_n = \prod_{i=1}^{n} (1 - \Lambda (1 - \lambda_i))$. As $\Lambda (1 - \lambda_i) < 1$, it holds that $P_n > 0$. Our goal is to show that $P_n \to 0$. In the inequality $\log (1 - x) \leq -x$ (for all $x < 1$), letting $x = \Lambda (1 - \lambda_i) (< 1)$ yields

$$\log P_n = \sum_{i=1}^{n} \log (1 - \Lambda (1 - \lambda_i)) \leq -\Lambda \sum_{i=1}^{n} (1 - \lambda_i) .$$

This result implies that

$$0 < P_n \leq \exp \biggl(- \Lambda \sum_{i=1}^{n} (1 - \lambda_i) \biggr).$$

From $\sum_{i=1}^{\infty} (1 - \lambda_i) = \infty$ and $\Lambda > 0$, we obtain $P_n \to 0$ as n tends to infinity. This outcome completes the proof. □

3. Primary results

In this section, we prove a strong convergence theorem, which extends a three-step iteration (1.7) by incorporating the mean iteration (1.8) and the "2-iteration" (1.9). The result generalizes Theorem 1.3 and complements Theorem 1.2, which is related to the Picard iteration.

Theorem 3.1. *Let C be a nonempty, closed, and convex subset of a real Banach space E and let T be a Z-mapping from C into itself. Let $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{\xi_n\}$, $\{A_n\}$, $\{B_n\}$, $\{C_n\}$, $\{D_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\zeta_n\}$, $\{\eta_n\}$, $\{\theta_n\}$, $\{\iota_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that*

$$\begin{aligned} &\lambda_n + \mu_n + \nu_n + \xi_n = 1, \quad A_n + B_n + C_n + D_n = 1, \\ &a_n + b_n + c_n + d_n = 1, \quad \alpha_n + \beta_n + \gamma_n + \delta_n = 1, \text{ and} \\ &\zeta_n + \eta_n + \theta_n + \iota_n = 1 \end{aligned}$$

for all $n \in \mathbb{N}$. Assume that $\sum_{n=1}^{\infty} (1 - \lambda_n) = \infty$. Define a sequence $\{x_n\}$ in C as follows:

$$\begin{aligned} x_1 &= x \in C \text{ is given}, \\ v_n &= \zeta_n x_n + \eta_n T^{H_1} x_n + \theta_n T^{H_2} x_n + \iota_n \frac{1}{n} \sum_{i=h}^{n+h-1} T^i x_n, \\ w_n &= \alpha_n x_n + \beta_n T^{M_1} v_n + \gamma_n T^{M_2} v_n + \delta_n \frac{1}{n} \sum_{i=m}^{n+m-1} T^i v_n, \\ z_n &= a_n x_n + b_n T^{L_1} v_n + c_n T^{L_2} v_n + d_n \frac{1}{n} \sum_{i=l}^{n+l-1} T^i v_n, \\ y_n &= A_n x_n + B_n T^{K_1} v_n + C_n T^{K_2} v_n + D_n \frac{1}{n} \sum_{i=k}^{n+k-1} T^i v_n, \\ x_{n+1} &= \lambda_n x_n + \mu_n T^{J_1} y_n + \nu_n T^{J_2} z_n + \xi_n \frac{1}{n} \sum_{i=j}^{n+j-1} T^i w_n \end{aligned}$$

for all $n \in \mathbb{N}$*, where* $J_1, K_1, L_1, M_1, H_1, J_2, K_2, L_2, M_2, H_2, j, k, l, m, h \in \mathbb{N}$*. Then,* $\{x_n\}$ *converges strongly to a unique fixed point* p *of* T*.*

Proof. As C is closed in a Banach space, it is complete. Thus, from Theorem 1.2, T has a unique fixed point $p \in C$. Define

$$\rho = \max\left\{a,\ \frac{b}{1-b},\ \frac{c}{1-c}\right\} \in (0,1)\,,$$

where $a \in (0,1)$ and $b, c \in \left(0, \frac{1}{2}\right)$ are those in Definition 1.1. As T is a Z-mapping, we can verify that

$$\|v_n - p\| \leq \|x_n - p\| \tag{3.1}$$

for all $n \in \mathbb{N}$. Since $\rho \in (0,1)$, it holds from Lemma 2.1 that

$$\begin{aligned}
\|v_n - p\| &= \left\|\zeta_n x_n + \eta_n T^{H_1} x_n + \theta_n T^{H_2} x_n + \iota_n \frac{1}{n} \sum_{i=h}^{n+h-1} T^i x_n - p\right\| \\
&\leq \zeta_n \|x_n - p\| + \eta_n \left\|T^{H_1} x_n - p\right\| + \theta_n \left\|T^{H_2} x_n - p\right\| \\
&\quad + \iota_n \left\|\frac{1}{n} \sum_{i=h}^{n+h-1} T^i x_n - p\right\| \\
&\leq \zeta_n \|x_n - p\| + \eta_n \rho \|x_n - p\| + \theta_n \rho \|x_n - p\| + \iota_n \rho \|x_n - p\| \\
&\leq \zeta_n \|x_n - p\| + \eta_n \|x_n - p\| + \theta_n \|x_n - p\| + \iota_n \|x_n - p\| \\
&= \|x_n - p\|
\end{aligned}$$

as claimed. Furthermore, using Lemma 2.1 and (3.1), we can show that

$$\|y_n - p\| \leq A_n \|x_n - p\| + (1 - A_n)\,\rho\,\|x_n - p\| \tag{3.2}$$

for all $n \in \mathbb{N}$ as follows:

$$\begin{aligned}
\|y_n - p\| &= \left\|A_n x_n + B_n T^{K_1} v_n + C_n T^{K_2} v_n + D_n \frac{1}{n} \sum_{i=k}^{n+k-1} T^i v_n - p\right\| \\
&\leq A_n \|x_n - p\| + B_n \left\|T^{K_1} v_n - p\right\| + C_n \left\|T^{K_2} v_n - p\right\| \\
&\quad + D_n \left\|\frac{1}{n} \sum_{i=k}^{n+k-1} T^i v_n - p\right\| \\
&\leq A_n \|x_n - p\| + B_n \rho \|v_n - p\| + C_n \rho \|v_n - p\| + D_n \rho \|v_n - p\| \\
&= A_n \|x_n - p\| + (1 - A_n)\,\rho\,\|v_n - p\| \\
&\leq A_n \|x_n - p\| + (1 - A_n)\,\rho\,\|x_n - p\|\,.
\end{aligned}$$

Similarly, we can demonstrate that

$$\|z_n - p\| \leq a_n \|x_n - p\| + (1 - a_n)\,\rho\,\|x_n - p\| \ \text{ and} \tag{3.3}$$

$$\|w_n - p\| \leq \alpha_n \|x_n - p\| + (1 - \alpha_n)\,\rho\,\|x_n - p\| \tag{3.4}$$

for all $n \in \mathbb{N}$. From Lemma 2.1 and (3.2)–(3.4), the following holds:

$$\begin{aligned}
\|x_{n+1} - p\| &= \left\| \lambda_n x_n + \mu_n T^{J_1} y_n + \nu_n T^{J_2} z_n + \xi_n \frac{1}{n} \sum_{i=j}^{n+j-1} T^i w_n - p \right\| \\
&\leq \lambda_n \|x_n - p\| + \mu_n \left\| T^{J_1} y_n - p \right\| + \nu_n \left\| T^{J_2} z_n - p \right\| \\
&\quad + \xi_n \left\| \frac{1}{n} \sum_{i=j}^{n+j-1} T^i w_n - p \right\| \\
&\leq \lambda_n \|x_n - p\| + \mu_n \rho \|y_n - p\| + \nu_n \rho \|z_n - p\| + \xi_n \rho \|w_n - p\| \\
&\leq \lambda_n \|x_n - p\| + \mu_n \rho \left(A_n \|x_n - p\| + (1 - A_n) \rho \|x_n - p\| \right) \\
&\quad + \nu_n \rho \left(a_n \|x_n - p\| + (1 - a_n) \rho \|x_n - p\| \right) \\
&\quad + \xi_n \rho \left(\alpha_n \|x_n - p\| + (1 - \alpha_n) \rho \|x_n - p\| \right) \\
&= \{ \lambda_n + \mu_n \rho A_n + \mu_n \rho^2 (1 - A_n) + \nu_n \rho a_n + \nu_n \rho^2 (1 - a_n) \\
&\quad + \xi_n \rho \alpha_n + \xi_n \rho^2 (1 - \alpha_n) \} \|x_n - p\| .
\end{aligned}$$

Using Lemma 2.2, we obtain

$$\begin{aligned}
\|x_{n+1} - p\| &\leq (1 - (1 - \rho)(1 - \lambda_n)) \|x_n - p\| \\
&\leq \cdots \\
&\leq \prod_{i=1}^{n} (1 - (1 - \rho)(1 - \lambda_i)) \|x_1 - p\| .
\end{aligned}$$

Because $1 - \rho > 0$, $(1 - \rho)(1 - \lambda_i) < 1$, and $\sum_{i=1}^{\infty} (1 - \lambda_i) = \infty$, from Lemma 2.3, we obtain $\prod_{i=1}^{n} (1 - (1 - \rho)(1 - \lambda_i)) \to 0$ in the limit as $n \to \infty$. This result indicates that $x_n \to p$, which completes the proof. □

Remark. Theorem 3.1 generates Theorem 1.3. Indeed, let $\eta_n = \theta_n = \iota_n = 0$ in Theorem 3.1. Then, $\zeta_n = 1$, and we have $v_n = x_n$. Furthermore, let $\nu_n = \xi_n = 0$. Then, z_n and w_n do not affect x_{n+1}. Finally, putting $C_n = D_n = 0$ and $J_1 = K_1 = 1$, we obtain Theorem 1.3.

Theorem 3.1 also generates the following corollary, which combines the Ishikawa iteration (1.6) and the mean iteration (1.8).

Corollary 3.1. *Let C be a nonempty, closed, and convex subset of a real Banach space E and let T be a Z-mapping from C into itself. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0, 1]$ such that $\sum_{n=1}^{\infty} (1 - \lambda_n) = \infty$. Let $\{\alpha_n\}$ be a sequence of real numbers in $[0, 1]$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned}
&x_1 = x \in C \text{ is given,} \\
&y_n = \alpha_n x_n + (1 - \alpha_n) \frac{1}{n} \sum_{i=1}^{n} T^i x_n, \\
&x_{n+1} = \lambda_n x_n + (1 - \lambda_n) \frac{1}{n} \sum_{i=1}^{n} T^i y_n
\end{aligned} \tag{3.5}$$

for all $n \in \mathbb{N}$. *Then,* $\{x_n\}$ *converges strongly to a unique fixed point* p *of* T.

Proof. Let $\eta_n = \theta_n = \iota_n = 0$ in Theorem 3.1. By this operation, we have $v_n = x_n$. Next, set $\mu_n = \nu_n = 0$. Then, y_n and z_n do not affect x_{n+1}. Furthermore, substitute $\beta_n = \gamma_n = 0$ and $j = m = 1$. Finally, replacing the notation w_n by y_n, we obtain the desired result. □

If $\alpha_n = 1$ in Corollary 3.1, the iteration procedure (3.5) coincides with the mean iteration (1.8) in Introduction. Also, (3.5) in Corollary 3.1 can be replaced by

$$\begin{aligned} x_1 &= x \in C \text{ is given,} \\ y_n &= A_n x_n + (1 - A_n)\,\frac{1}{n}\sum_{i=1}^{n} T^i x_n, \\ x_{n+1} &= \lambda_n x_n + (1-\lambda_n)\,T y_n, \end{aligned}$$

and other various versions. Further, we can derive the following corollary.

Corollary 3.2. *Let* C *be a nonempty, closed, and convex subset of a real Banach space* E *and let* T *be a* Z*-mapping from* C *into itself. Let* $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{A_n\}$, $\{B_n\}$, $\{C_n\}$ *be sequences of real numbers in the interval* $[0,1]$ *such that* $\lambda_n + \mu_n + \nu_n = 1$ *and* $A_n + B_n + C_n = 1$. *Assume that* $\sum_{n=1}^{\infty}(1-\lambda_n) = \infty$. *Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$\begin{aligned} x_1 &= x \in C \textit{ is given,} \\ y_n &= A_n x_n + B_n T x_n + C_n T^2 x_n, \\ x_{n+1} &= \lambda_n x_n + \mu_n T y_n + \nu_n T^2 y_n \end{aligned}$$

for all $n \in \mathbb{N}$. *Then,* $\{x_n\}$ *converges strongly to a unique fixed point* p *of* T.

Proof. First, substitute $\eta_n = \theta_n = \iota_n = 0$ in Theorem 3.1. Then, $v_n = x_n$. Furthermore, let $\xi_n = D_n = d_n = 0$, $A_n = a_n$, $B_n = b_n$, $C_n = c_n$, $K_1 = L_1$, and $K_2 = L_2$. This yields $y_n = z_n$. Finally, putting $J_1 = K_1 = 1$ and $J_2 = K_2 = 2$, we obtain the desired result. □

If $A_n = 1$ in Corollary 3.2, the "2-iteration" (1.9) is obtained. The Noor's type (1.7) three-step iteration is also derived from Theorem 3.1.

Corollary 3.3. *Let* C *be a nonempty, closed, and convex subset of a real Banach space* E *and let* T *be a* Z*-mapping from* C *into itself. Let* $\{\lambda_n\}$, $\{A_n\}$, $\{\zeta_n\}$ *be sequences of real numbers in the interval* $[0,1]$. *Assume that* $\sum_{n=1}^{\infty}(1-\lambda_n) = \infty$. *Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$\begin{aligned} x_1 &= x \in C \textit{ is given,} \\ v_n &= \zeta_n x_n + (1-\zeta_n)\,T x_n, \\ y_n &= A_n x_n + (1-A_n)\,T v_n, \\ x_{n+1} &= \lambda_n x_n + (1-\lambda_n)\,T y_n \end{aligned}$$

for all $n \in \mathbb{N}$. *Then,* $\{x_n\}$ *converges strongly to a unique fixed point* p *of* T.

Proof. Let $\nu_n = \xi_n = C_n = D_n = \theta_n = \iota_n = 0$ and $J_1 = K_1 = H_1 = 1$ in Theorem 3.1. Then, the desired result follows. □

4. Application

In this section, we present a strong convergence theorem that approximates a solution to a variational inequality problem (VIP) as an application of a result in the previous section. Let C be a nonempty subset of a real Hilbert space H. For a mapping $A\colon C \to H$, the set of solutions to a variational inequality problem (VIP) is denoted by

$$VI(C, A) = \{x \in C : \langle y - x,\ Ax\rangle \geq 0 \text{ for all } y \in C\}.$$

For VIPs, the following types of mappings have been used in the literature. A mapping $A\colon C \to H$ is called *K-Lipschitz continuous* if there exists $K > 0$ such that

$$\|Ax - Ay\| \leq K\|x - y\| \tag{4.1}$$

for all $x, y \in C$. If $K < 1$, A is a contraction mapping. A mapping $A\colon C \to H$ is said to be *monotone* if

$$0 \leq \langle x - y,\ Ax - Ay\rangle \tag{4.2}$$

for all $x, y \in C$. A mapping $A\colon C \to H$ is called *strongly monotone* if

$$0 < \langle x - y,\ Ax - Ay\rangle \tag{4.3}$$

for $x, y \in C$ such that $x \neq y$. A mapping $A\colon C \to H$ is called *η-strongly monotone* if there exists $\eta > 0$ such that

$$\eta\|x - y\|^2 \leq \langle x - y,\ Ax - Ay\rangle \tag{4.4}$$

for all $x, y \in C$. An η-strongly monotone mapping is strongly monotone.

For a mapping $T\colon C \to H$, the set of fixed points is denoted by

$$F(T) = \{x \in C : x = Tx\}.$$

To solve VIPs by applying the fixed point theory, the following two lemmas are crucial.

Lemma 4.1. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H, let P_C be the metric projection from H onto C, and let A be a mapping from C into H. Then, it holds that $VI(C, A) = F(P_C(I - \mu A))$ for all $\mu > 0$.*

Proof. It follows that

$$\begin{aligned} x \in F(P_C(I - \mu A)) &\iff x = P_C(x - \mu Ax) \\ &\iff \langle (x - \mu Ax) - x,\ x - y\rangle \geq 0 \text{ for all } y \in C \\ &\iff \langle Ax,\ x - y\rangle \leq 0 \text{ for all } y \in C \\ &\iff x \in VI(C, A). \end{aligned}$$

Thus, we obtain the desired result. □

Lemma 4.2. *Let $A\colon C \to H$ be an η-strongly monotone and K-Lipschitz continuous mapping, where C is a nonempty subset of H and $0 < \eta \leq K$. Then, for $\mu \in \left(0, \frac{2\eta}{K^2}\right)$, $I - \mu A$ is a contraction mapping from C into H.*

Proof. Let $x, y \in C$. As A is η-strongly monotone and K-Lipschitz continuous, it holds that

$$\begin{aligned} & \|(I-\mu A)\,x - (I-\mu A)\,y\|^2 \\ = \ & \|x - y - \mu\,(Ax - Ay)\|^2 \\ = \ & \|x-y\|^2 - 2\mu\,\langle x-y,\ Ax-Ay\rangle + \mu^2\,\|Ax-Ay\|^2 \\ \leq \ & \|x-y\|^2 - 2\mu\eta\,\|x-y\|^2 + \mu^2 K^2\,\|x-y\|^2 \\ = \ & \left\{1 - \mu\left(2\eta - \mu K^2\right)\right\}\|x-y\|^2 . \end{aligned}$$

Using the conditions $0 < \mu < 2\eta/K^2$ and $0 < \eta \leq K$, we have

$$0 \leq 1 - \mu\left(2\eta - \mu k^2\right) < 1.$$

This implies that $I - \mu A$ is a contraction mapping. □

As the metric projection is nonexpansive, the self-mapping $P_C\,(I - \mu A)$ on C is a contraction mapping under the conditions in Lemma 4.2. Consequently, the set $VI\,(C, A)\,(= F\,(P_C\,(I - \mu A)))$ consists of only one element and Picard iteration effectively works to approximate the unique element of $VI\,(C, A)$.

Beyond the Picard iteration, more general types of iteration schemes introduced in Section 3 are applicable to approximate the solution to a VIP. We present a three-step iteration as an application of Corollary 3.3.

Theorem 4.1. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H and let P_C be the metric projection from H onto C. Let $A\colon C \to H$ be an η-strongly monotone and K-Lipschitz continuous mapping, where $0 < \eta \leq K$. For $\mu \in \left(0, \frac{2\eta}{K^2}\right)$, define $T = P_C\,(I - \mu A)$, where I is the identity mapping defined on C. Let $\{\lambda_n\}$, $\{A_n\}$, $\{\zeta_n\}$ be sequences of real numbers in the interval $[0, 1]$. Assume that $\sum_{n=1}^{\infty}(1 - \lambda_n) = \infty$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned} x_1 &= x \in C \text{ is given,} \\ v_n &= \zeta_n x_n + (1 - \zeta_n)\,T x_n, \\ y_n &= A_n x_n + (1 - A_n)\,T v_n, \\ x_{n+1} &= \lambda_n x_n + (1 - \lambda_n)\,T y_n \end{aligned} \tag{4.5}$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a unique element of $VI(C, A)$.

Proof. From Lemma 4.2, $T\,(= P_C\,(I - \mu A))$ is a contraction mapping from C into itself. Hence, T is a Z-mapping and it has a unique fixed point $x^* \in F\,(T)$. From Lemma 4.1, $x^* \in F\,(T) = VI(C, A)$. As $\sum_{n=1}^{\infty}(1 - \lambda_n) = \infty$ is assumed, from Corollary 3.3, the sequence $\{x_n\}$ defined by (4.5) converges strongly to $x^* \in F\,(T) = VI(C, A)$. This completes the proof. □

Setting $\zeta_n = A_n = 1$ and $\lambda_n = 0$ in (4.5), Picard iteration is deduced. Therefore, the iteration (4.5) is more general than Picard iteration. For related researches concerning VIPs, see also Yamada [**42**], Xu and Kim [**41**], Muangchoo [**28**], and Truong et al. [**38**].

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