Acta Math. Univ. Comenianae
Vol. XCI, 3 (2022), pp. 205–223

BIVARIATE ABSTRACT FRACTIONAL MONOTONE CONSTRAINED APPROXIMATION BY POLYNOMIALS

G. A. ANASTASSIOU

Abstract. Let $f \in C^{r,p}\left([0,1]^2\right)$, $r, p \in \mathbb{N}$, and let L^* be an abstract linear left or right fractional mixed partial differential operator such that $L^*(f) \geq 0$ for all (x, y) in a critical region of $[0,1]^2$ that depends on L^*. Then there exists a sequentce of two-dimensional polynomials $Q_{\overline{m_1},\overline{m_2}}(x,y)$ with $L^*\left(Q_{\overline{m_1},\overline{m_2}}(x,y)\right) \geq 0$ there, where $\overline{m_1}, \overline{m_2} \in \mathbb{N}$ such that $\overline{m_1} > r$, $\overline{m_2} > p$, so that f is approximated left or right abstract fractionally simultaneously and uniformly by $Q_{\overline{m_1},\overline{m_2}}$ on $[0,1]^2$. This restricted left or right abstract fractional approximation is achieved quantitatively by the use of a suitable integer partial derivatives two-dimensional first modulus of continuity.

This monotone constrained fractional approximation applies to a wide range of Caputo type fractional calculi of singular or non-singular kernels.

1. Introduction

The topic of monotone approximation started in [**7**] has become a major trend in approximation theory. A typical problem in this subject is: given a positive integer k, approximate a given function whose kth derivative is ≥ 0 by polynomials having this property.

In [**4**], the authors replaced the kth derivative with a linear differential operator of order k. We mention this motivating result.

Theorem 1. *Let h, k, p be integers, $0 \leq h \leq k \leq p$, and let f be a real function, $f^{(p)}$ continuous in $[-1,1]$ with first modulus of continuity $\omega_1(f^{(p)}, x)$ there. Let $a_j(x)$, $j = h, h+1, \ldots, k$, be real functions defined and bounded on $[-1,1]$, and assume $a_h(x)$ is either $\geq$ some number $\alpha > 0$ or $\leq$ some number $\beta < 0$ throughout $[-1,1]$. Consider the operator*

$$L = \sum_{j=h}^{k} a_j(x) \left[\frac{\mathrm{d}^j}{\mathrm{d}x^j}\right]$$

Received September 20, 2021; revised February 19, 2022.
2020 *Mathematics Subject Classification.* Primary 26A33, 41A10, 41A17, 41A25, 41A28, 41A29, 41A63.
Key words and phrases. Monotone constrained approximation; Caputo type abstract fractional derivative; abstract bivariate fractional linear differential operator; two-dimensional modulus of continuity.

and throughout $[-1,1]$, *suppose*

$$L(f) \geq 0. \tag{1}$$

Then, for every integer $n \geq 1$, *there is a real polynomial* $Q_n(x)$ *of degree* $\leq n$ *such that*

$$L(Q_n) \geq 0 \quad \text{throughout } [-1,1],$$

and

$$\max_{-1\leq x\leq 1} |f(x) - Q_n(x)| \leq Cn^{k-p} \omega_1\Big(f^{(p)}, \frac{1}{n}\Big),$$

where C *is independent of* n *or* f.

We need the following definitions.

Definition 2 (D. D. Stancu [**8**])**.** Let $f \in C([0,1]^2)$, $[0,1]^2 = [0,1] \times [0,1]$, where (x_1, y_1), $(x_2, y_2) \in [0,1]^2$, and $\delta_1, \delta_2 \geq 0$. The first modulus of continuity of f is defined as follows:

$$\omega_1(f, \delta_1, \delta_2) = \sup_{\substack{|x_1-x_2|\leq\delta_1 \\ |y_1-y_2|\leq\delta_2}} |f(x_1, y_1) - f(x_2, y_2)|.$$

Definition 3. Let f be a real-valued function defined on $[0,1]^2$ and let m, n be two positive integers. Let $B_{m,n}$ be the Bernstein (polynomial) operator of order (m, n) given by

$$B_{m,n}(f; x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} f\Big(\frac{i}{m}, \frac{j}{n}\Big) \cdot \binom{m}{i} \cdot \binom{n}{j} \cdot x^i (1-x)^{m-i} y^j (1-y)^{n-j}. \tag{2}$$

For integers $r, s \geq 0$, we denote by $f^{(r,s)}$ the differential operator of order (r, s) given by

$$f^{(r,s)}(x, y) = \frac{\partial^{r+s} f x, y)}{\partial x^r \partial y^s}.$$

We use next theorems.

Theorem 4 (I. Badea, C. Badea [**5**])**.** *It holds that*

$$\begin{aligned} &\|f^{(k,l)} - (B_{m,n}f)^{(k,l)}\|_\infty \\ &\leq t(k,l) \cdot \omega_1\left(f^{(k,l)}; \frac{1}{\sqrt{m-k}}, \frac{1}{\sqrt{n-l}}\right) + \max\left\{\frac{k(k-1)}{m}, \frac{l(l-1)}{n}\right\} \cdot \|f^{(k,l)}\|_\infty, \end{aligned} \tag{3}$$

where $m > k \geq 0$, $n > l \geq 0$ *are integers,* f *is a real-valued function on* $[0,1]^2$ *such that* $f^{(k,l)}$ *is continuous, and* t *is a positive real-valued function on* $\mathbb{Z}_+^2 = \{0, 1, 2, \dots\}^2$. *Here* $\|\cdot\|_\infty$ *is the supremum norm on* $[0,1]^2$.

Denote $C^{r,p}([0,1]^2) := \{f : [0,1]^2 \to \mathbb{R};\ f^{(k,l)}$ is continuous for $0 \leq k \leq r$, $0 \leq l \leq p\}$.

In [**1**], the author proved the following motivational result.

Theorem 5. *Let h_1, h_2, v_1, v_2, r, p be integers, $0 \le h_1 \le v_1 \le r$, $0 \le h_2 \le v_2 \le p$ and let $f \in C^{r,p}\left([0,1]^2\right)$. Let $\alpha_{i,j}(x,y)$, $i = h_1, h_1+1, \ldots, v_1$, $j = h_2, h_2+1, \ldots, v_2$ be real-valued functions, defined and bounded in $[0,1]^2$ and suppose $\alpha_{h_1 h_2}$ is either $\ge \alpha > 0$ or $\le \beta < 0$ throughout $[0,1]^2$. Consider the operator*

$$L = \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{ij}(x,y) \frac{\partial^{i+j}}{\partial x^i \partial y^j} \tag{4}$$

and assume that throughout $[0,1]^2$,

$$L(f) \ge 0.$$

Then for integers m, n with $m > r$, $n > p$, there exists a polynomial $Q_{m,n}(x,y)$ of degree (m,n) such that $L(Q_{m,n}(x,y)) \ge 0$ throughout $[0,1]^2$, and

$$\|f^{(k,l)} - Q_{m,n}^{(k,l)}\|_\infty \le \frac{P_{m,n}(L,f)}{(h_1-k)!\,(h_2-l)!} + M_{m,n}^{k,l}(f) \tag{5}$$

for all $(0,0) \le (k,l) \le (h_1,h_2)$. Furthermore we get

$$\|f^{(k,l)} - Q_{m,n}^{(k,l)}\|_\infty \le M_{m,n}^{k,l}(f) \tag{6}$$

for all $(h_1+1, h_2+1) \le (k,l) \le (r,p)$. Also (6) is true whenever $0 \le k \le h_1$, $h_2+1 \le l \le p$ or $h_1+1 \le k \le r$, $0 \le l \le h_2$. Here

$$M_{m,n}^{k,l} \equiv M_{m,n}^{k,l}(f) \equiv t(k,l) \cdot \omega_1\left(f^{(k,l)}; \frac{1}{\sqrt{m-k}}, \frac{1}{\sqrt{n-l}}\right) + \max\left\{\frac{k(k-1)}{m}, \frac{l(l-1)}{n}\right\} \cdot \|f^{(k,l)}\|_\infty \tag{7}$$

and

$$P_{m,n} \equiv P_{m,n}(L,f) \equiv \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} l_{ij} \cdot M_{m,n}^{i,j}, \tag{8}$$

where t is a positive real-valued function on $\mathbb{Z}_+^2$ and

$$l_{ij} \equiv \sup_{(x,y)\in[0,1]^2} |\alpha_{h_1 h_2}^{-1}(x,y) \cdot \alpha_{ij}(x,y)| < \infty. \tag{9}$$

In [**2**], we extended Theorem 5 to the fractional level. Indeed there L is replaced by L^*, a linear left Caputo fractional mixed partial differential operator. Now the monotonicity property is only true on a critical region of $[0,1]^2$ that depends on L^* parameters. Simultaneous fractional convergence remains true on all of $[0,1]^2$.

We need the following definitions.

Definition 6. Let $\alpha_1, \alpha_2 > 0$, $\alpha = (\alpha_1, \alpha_2)$, $f \in C\left([0,1]^2\right)$, and let $x = (x_1, x_2)$ $(t_1, t_2) \in [0,1]^2$. We define the left mixed Riemann-Liouville fractional two dimensional integral of order α (see also [**6**]):

$$\left(I_{0+}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_1}\int_0^{x_2} (x_1-t_1)^{\alpha_1-1}(x_2-t_2)^{\alpha_2-1} f(t_1,t_2)\mathrm{d}t_1\mathrm{d}t_2, \tag{10}$$

with $x_1, x_2 > 0$.

Notice here that $I_{0+}^{\alpha}(|f|) < \infty$.

Definition 7 ([**2**])**.** Let $\alpha_1, \alpha_2 > 0$ with $\lceil \alpha_1 \rceil = m_1$, $\lceil \alpha_2 \rceil = m_2$, ($\lceil \cdot \rceil$ ceiling of the number). Let here $f \in C^{m_1,m_2}\left([0,1]^2\right)$. We consider the left (Caputo type) fractional partial derivative

$$D_{*0}^{(\alpha_1,\alpha_2)} f(x) := \frac{1}{\Gamma(m_1-\alpha_1)\,\Gamma(m_2-\alpha_2)} \cdot \int_0^{x_1}\int_0^{x_2} (x_1-t_1)^{m_1-\alpha_1-1}(x_2-t_2)^{m_2-\alpha_2-1} \frac{\partial^{m_1+m_2} f(t_1,t_2)}{\partial t_1^{m_1}\partial t_2^{m_2}} \mathrm{d}t_1 \mathrm{d}t_2, \tag{11}$$

for all $x = (x_1, x_2) \in [0,1]^2$, where Γ is the gamma function

$$\Gamma(\nu) = \int_0^\infty \mathrm{e}^{-t} t^{\nu-1} \mathrm{d}t, \qquad \nu > 0. \tag{12}$$

We set

$$D_{*0}^{(0,0)} f(x) := f(x) \qquad \text{for all } x \in [0,1]^2. \tag{13}$$

$$D_{*0}^{(m_1,m_2)} f(x) := \frac{\partial^{m_1+m_2} f(x)}{\partial x_1^{m_1}\partial x_2^{m_2}} \qquad \text{for all } x \in [0,1]^2. \tag{14}$$

Definition 8 ([**2**])**.** We also set

$$D_{*0}^{(0,\alpha_2)} f(x) := \frac{1}{\Gamma(m_2-\alpha_2)} \int_0^{x_2} (x_2-t_2)^{m_2-\alpha_2-1} \frac{\partial^{m_2} f(x_1,t_2)}{\partial t_2^{m_2}} \mathrm{d}t_2, \tag{15}$$

$$D_{*0}^{(\alpha_1,0)} f(x) := \frac{1}{\Gamma(m_1-\alpha_1)} \int_0^{x_1} (x_1-t_1)^{m_1-\alpha_1-1} \frac{\partial^{m_1} f(t_1,x_2)}{\partial t_1^{m_1}} \mathrm{d}t_1, \tag{16}$$

and

$$D_{*0}^{(m_1,\alpha_2)} f(x) := \frac{1}{\Gamma(m_2-\alpha_2)} \int_0^{x_2} (x_2-t_2)^{m_2-\alpha_2-1} \frac{\partial^{m_1+m_2} f(x_1,t_2)}{\partial x_1^{m_1}\partial t_2^{m_2}} \mathrm{d}t_2, \tag{17}$$

$$D_{*0}^{(\alpha_1,m_2)} f(x) := \frac{1}{\Gamma(m_1-\alpha_1)} \int_0^{x_1} (x_1-t_1)^{m_1-\alpha_1-1} \frac{\partial^{m_1+m_2} f(t_1,x_2)}{\partial t_1^{m_1}\partial x_2^{m_2}} \mathrm{d}t_1. \tag{18}$$

The following result is the main motivation for this work.

Theorem 9 ([**2**])**.** *Let h_1, h_2, v_1, v_2, r, p be integers, $0 \le h_1 \le v_1 \le r$, $0 \le h_2 \le v_2 \le p$, and let $f \in C^{r,p}\left([0,1]^2\right)$. Let $\alpha_{ij}(x,y)$, $i = h_1, h_1+1, \ldots, v_1$; $j = h_2, h_2+1, \ldots, v_2$ be real valued functions, defined and bounded in $[0,1]^2$, and assume $\alpha_{h_1h_2}$ is either $\ge \alpha > 0$ or $\le \beta < 0$ throughout $[0,1]^2$. Let*

$$0 \le \alpha_{1h_1} \le h_1 < \alpha_{11} \le h_1+1 < \alpha_{12} \le h_1+2 < \alpha_{13} \le h_1+3 < \cdots < h_{1v_1} \le v_1 < \cdots < \alpha_{1r} \le r$$

with $\lceil \alpha_{1h_1} \rceil = h_1$;

$$0 \le \alpha_{2h_2} \le h_2 < \alpha_{21} \le h_2+1 < \alpha_{22} \le h_2+2 < \alpha_{23} \le h_2+3 < \cdots < \alpha_{2v_2} \le v_2 < \cdots < \alpha_{2p} \le p$$

with $\lceil \alpha_{2h_2} \rceil = h_2$. Consider the left fractional differential bivariate operator

$$L^* := \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{ij}(x,y)\, D_{*0}^{(\alpha_{1i},\alpha_{2j})}. \tag{19}$$

Let integers $\overline{m_1}, \overline{m_2}$ *with* $\overline{m_1} > r$, $\overline{m_2} > p$. *Set*

$$l_{ij} := \sup_{(x,y)\in[0,1]^2} \left|\alpha_{h_1 h_2}^{-1}(x,y) \cdot \alpha_{ij}(x,y)\right| < \infty.$$

Also set ($\lceil \alpha_{1i} \rceil = i$, $\lceil \alpha_{2j} \rceil = j$, $\lceil \cdot \rceil$ *ceiling of number*)

$$(20) \quad M^{i,j}_{\overline{m_1},\overline{m_2}} := M^{i,j}_{\overline{m_1},\overline{m_2}}(f)$$

$$:= \frac{1}{\Gamma(i-\alpha_{1i}+1)\,\Gamma(j-\alpha_{2j}+1)} \left\{ t(i,j)\,\omega_1\!\left(f^{(i,j)}; \frac{1}{\sqrt{\overline{m_1}-i}}, \frac{1}{\sqrt{\overline{m_2}-j}}\right) + \max\left\{ \frac{i(i-1)}{\overline{m_1}}, \frac{j(j-1)}{\overline{m_2}} \right\} \cdot |f^{(i,j)}|_\infty \right\},$$

$i = h_1, \ldots, v_1$, $j = h_2, \ldots, v_2$.

Here t *is a positive real-valued function on* $\mathbb{Z}_+^2$, $\|\cdot\|_\infty$ *is the supremum norm on* $[0,1]^2$. *Call*

$$P_{\overline{m_1},\overline{m_2}} := P_{\overline{m_1},\overline{m_2}}(f) = \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} l_{ij} \cdot M^{i,j}_{\overline{m_1},\overline{m_2}}. \tag{21}$$

Then there exists a polynomial $Q_{\overline{m_1},\overline{m_2}}(x,y)$ *of degree* $(\overline{m_1},\overline{m_2})$ *on* $[0,1]^2$ *such that*

$$(22) \quad \left\| D_{*0}^{(\alpha_{1k},\alpha_{2l})}(f) - D_{*0}^{(\alpha_{1k},\alpha_{2l})}(Q_{\overline{m_1},\overline{m_2}}) \right\|_\infty \le \frac{\Gamma(h_1-k+1)\,\Gamma(h_2-l+1)\,P_{\overline{m_1},\overline{m_2}}}{\Gamma(h_1-\alpha_{1k}+1)\,\Gamma(h_2-\alpha_{2l}+1)\,(h_1-k)!\,(h_2-l)!} + M^{k,l}_{\overline{m_1},\overline{m_2}}$$

for $(0,0) \le (k,l) \le (h_1,h_2)$.

If $(h_1+1, h_2+1) \le (k,l) \le (r,p)$, *or* $0 \le k \le h_1$, $h_2+1 \le l \le p$, *or* $h_1+1 \le k \le r$, $0 \le l \le h_2$, *we get*

$$\left\| D_{*0}^{(\alpha_{1k},\alpha_{2l})}(f) - D_{*0}^{(\alpha_{1k},\alpha_{2l})}(Q_{\overline{m_1},\overline{m_2}}) \right\|_\infty \le M^{k,l}_{\overline{m_1},\overline{m_2}}. \tag{23}$$

By assuming $L^*(f(1,1)) \ge 0$, *we get* $L^*(Q_{\overline{m_1},\overline{m_2}}(1,1)) \ge 0$.

Let $1 \ge x, y > 0$ *with* $\alpha_{1h_1} \ne h_1$ *and* $\alpha_{2h_2} \ne h_2$ *such that*

$$x \ge \left(\Gamma(h_1-\alpha_{1h_1}+1)\right)^{\frac{1}{\left(h_1-\alpha_{1h_1}\right)}}, \qquad y \ge \left(\Gamma(h_2-\alpha_{2h_2}+1)\right)^{\frac{1}{\left(h_2-\alpha_{2h_2}\right)}}, \tag{24}$$

and

$$L^*(f(x,y)) \ge 0.$$

Then

$$L^*(Q_{\overline{m_1},\overline{m_2}}(x,y)) \ge 0.$$

Some notation follows.

Definition 10. Let f be a real-valued function defined on $[0,1]^2$ and let $\overline{m_1}, \overline{m_2} \in \mathbb{N}$. Let $B_{\overline{m_1},\overline{m_2}}$ be the Bernstein (polynomial) operator of order $(\overline{m_1},\overline{m_2})$

given by

$$B_{\overline{m_1},\overline{m_2}}\left(f;x_1,x_2\right) := \sum_{i_1=0}^{\overline{m_1}}\sum_{i_2=0}^{\overline{m_2}} f\left(\frac{i_1}{\overline{m_1}},\frac{i_2}{\overline{m_2}}\right)\binom{\overline{m_1}}{i_1}\binom{\overline{m_2}}{i_2} x_1^{i_1}\left(1-x_1\right)^{\overline{m_1}-i_1} x_2^{i_2}\left(1-x_2\right)^{\overline{m_2}-i_2}. \tag{25}$$

In this work, we generalize Theorem 9 to abstract kernels that can be singular or non-singular, again bivariate constrained monotonicity takes place over a critical region of $[0,1]^2$, however bivariate abstract fractional simultaneous approximation is true over the whole of $[0,1]^2$. We cover both the left and right sides of this bivariate fractional approximation.

We need the following abstract fractional background.

2. About bivariate abstract fractional calculus

Let h_1,h_2,v_1,v_2,r,p be integers, $0 \le h_1 \le v_1 \le r$, $0 \le h_2 \le v_2 \le p$, and let $f \in C^{r,p}([0,1]^2)$. Here $h_1 \le i \le v_1$, $h_2 \le j \le v_2$. Let $\alpha_{1i},\alpha_{2j} \ge 0$, $\alpha_{1i},\alpha_{2j} \notin \mathbb{Z}_+$: $\lceil \alpha_{1i} \rceil = i$, $\lceil \alpha_{2j} \rceil = j$, $i=0,1,\ldots,r$, $j=0,1,\ldots,p$, ($\lceil \cdot \rceil$ is the ceiling of number), $\alpha_{10}=0$, $\alpha_{20}=0$.

Consider also the integrable functions $k_{1i} := K_{\alpha_{1i}}, k_{2j} := K_{\alpha_{2j}} : [0,1] \to \mathbb{R}_+$, $i=0,1,\ldots,r$, $j=0,1,\ldots,p$.

I) We first consider the abstract left Caputo type bivariate fractional partial derivative of orders $(\alpha_{1i},\alpha_{2j})$

$${}_{k_{2j}}^{k_{1i}}D_{*0}^{(\alpha_{1i},\alpha_{2j})} f(x) := \int_0^{x_1}\int_0^{x_2} k_{1i}\left(x_1-t_1\right) k_{2j}\left(x_2-t_2\right) \frac{\partial^{i+j} f(t_1,t_2)}{\partial t_1^i \partial t_2^j} \mathrm{d}t_1 \mathrm{d}t_2 \tag{26}$$

for all $x=(x_1,x_2)\in[0,1]^2$. We set

$$\begin{aligned} &{}_{k_{20}}^{k_{10}}D_{*0}^{(0,0)} f(x) := f(x), \\ &{}_{k_{2j}}^{k_{1i}}D_{*0}^{(i,j)} f(x) := \frac{\partial^{i+j} f(x_1,x_2)}{\partial x_1^i \partial x_2^j} \qquad \text{for all } x=(x_1,x_2)\in[0,1]^2. \end{aligned} \tag{27}$$

We also set

$${}_{k_{2j}}^{k_{1i}}D_{*0}^{(i,\alpha_{2j})} f(x) := \int_0^{x_2} k_{2j}\left(x_2-t_2\right) \frac{\partial^{i+j} f\left(x_1,t_2\right)}{\partial x_1^i \partial t_2^j} \mathrm{d}t_2, \tag{28}$$

$${}_{k_{2j}}^{k_{1i}}D_{*0}^{(\alpha_{1i},j)} f(x) := \int_0^{x_1} k_{1i}(x_1-t_1) \frac{\partial^{i+j} f\left(t_1,x_2\right)}{\partial t_1^i \partial x_2^j} \mathrm{d}t_1, \tag{29}$$

and in particular, we define:

$${}_{k_{2j}}^{k_{10}}D_{*0}^{(0,\alpha_{2j})} f(x) := \int_0^{x_2} k_{2j}\left(x_2-t_2\right) \frac{\partial^{j} f\left(x_1,t_2\right)}{\partial t_2^j} \mathrm{d}t_2, \tag{30}$$

$${}_{k_{20}}^{k_{1i}}D_{*0}^{(\alpha_{1i},0)} f(x) := \int_0^{x_1} k_{1i}(x_1-t_1) \frac{\partial^{i} f\left(t_1,x_2\right)}{\partial t_1^i} \mathrm{d}t_1, \tag{31}$$

for all $x = (x_1, x_2) \in [0,1]^2$.

We assume that there exists a critical region $\emptyset \neq \Phi \subseteq [0,1]^2$ such that

(32)
$$\Phi := \left\{ (x,y) \in [0,1]^2 : \int_0^x k_{1h_1}(z)\,\mathrm{d}z,\ \int_0^y k_{2h_2}(z)\,\mathrm{d}z \geq 1, \text{for any of } h_1, h_2 \neq 0 \right\}.$$

In [**2**], we got that $0 < \Gamma(h_1 - \alpha_{1h_1} + 1)$, $\Gamma(h_2 - \alpha_{2h_2} + 1) \leq 1$, where Γ is the gamma function, and there is

$$k_{ih_i}(z) = \frac{z^{h_i - \alpha_{ih_i} - 1}}{\Gamma(h_i - \alpha_{ih_i})}, \qquad i = 1, 2, \text{ for all } z \in [0,1], \tag{33}$$

and it holds

$$\int_0^1 k_{ih_i}(z)\,\mathrm{d}z = \frac{1}{\Gamma(h_i - \alpha_{ih_i} + 1)} \geq 1, \tag{34}$$

proving there that $\Phi \neq \emptyset$ by $x = y = 1$.

Also in [**2**], when $\alpha_{1h_1} \neq h_1$ and $\alpha_{2h_2} \neq h_2$, and $0 < x, y < 1$, the critical region Φ contains all (x, y) such that

$$\begin{cases} 1 > x \geq (\Gamma(h_1 - \alpha_{1h_1} + 1))^{\frac{1}{(h_1 - \alpha_{1h_1})}}, \\ 1 > y \geq (\Gamma(h_2 - \alpha_{2h_2} + 1))^{\frac{1}{(h_2 - \alpha_{2h_2})}}, \end{cases} \tag{35}$$

so again $\Phi \neq \emptyset$, non-trivially.

II) We also consider the abstract right side Caputo type bivariate fractional partial derivative of orders $(\alpha_{1i}, \alpha_{2j})$

(36)
$${}^{k_{1i}}_{k_{2j}}D^{(\alpha_{1i}, \alpha_{2j})}_{1-} f(x := (-1)^{i+j} \int_{x_1}^1 \int_{x_2}^1 k_{1i}(t_1 - x_1)\, k_{2j}(t_2 - x_2)\, \frac{\partial^{i+j} f(t_1, t_2)}{\partial t_1^i \partial t_2^j} \mathrm{d}t_1 \mathrm{d}t_2$$

for all $x = (x_1, x_2) \in [0,1]^2$.

We set

$$\begin{aligned} &{}^{k_{10}}_{k_{20}}D^{(0,0)}_{1-} f(x) := f(x), \\ &{}^{k_{1i}}_{k_{2j}}D^{(i,j)}_{1-} f(x) := (-1)^{i+j} \frac{\partial^{i+j} f(x_1, x_2)}{\partial x_1^i \partial x_2^j} \qquad \text{for all } x = (x_1, x_2) \in [0,1]^2. \end{aligned} \tag{37}$$

We also set

$${}^{k_{1i}}_{k_{2j}}D^{(i, \alpha_{2j})}_{1-} f(x) := (-1)^j \int_{x_2}^1 k_{2j}(t_2 - x_2) \frac{\partial^{i+j} f(x_1, t_2)}{\partial x_1^i \partial t_2^j} \mathrm{d}t_2, \tag{38}$$

$${}^{k_{1i}}_{k_{2j}}D^{(\alpha_{1i}, j)}_{1-} f(x) := (-1)^i \int_{x_1}^1 k_{1i}(t_1 - x_1) \frac{\partial^{i+j} f(t_1, x_2)}{\partial t_1^i \partial x_2^j} \mathrm{d}t_1, \tag{39}$$

and in particular, we define:

$${}^{k_{10}}_{k_{2j}}D^{(0, \alpha_{2j})}_{1-} f(x) := (-1)^j \int_{x_2}^1 k_{2j}(t_2 - x_2) \frac{\partial^{j} f(x_1, t_2)}{\partial t_2^j} \mathrm{d}t_2, \tag{40}$$

$$ {}_{k_{20}}^{k_{1i}}D_{1-}^{(\alpha_{1i},0)}f(x) := (-1)^i \int_{x_1}^{1} k_{1i}(t_1 - x_1) \frac{\partial^i f(t_1, x_2)}{\partial t_1^i} \mathrm{d}t_1 \quad (41) $$

for all $x = (x_1, x_2) \in [0,1]^2$.

We assume that there exists a critical region $\varnothing \neq \Psi \subseteq [0,1]^2$ such that

(42)

$$ \Psi := \left\{ (x,y) \in [0,1]^2 : \int_0^{1-x} k_{1h_1}(z)\,\mathrm{d}z, \int_0^{1-y} k_{2h_2}(z)\mathrm{d}z \geq 1, \text{ for any of } h_1, h_2 \neq 0 \right\}. $$

When $x = y = 0$, we have the conditions in (42) fulfilled by (33), (34), so there $\Psi \neq \emptyset$, see [**3**].

Also in [**3**], when $\alpha_{1h_1} \neq h_1$ and $\alpha_{2h_2} \neq h_2$, and $0 < x, y < 1$, the critical region Ψ contains all (x,y) such that

$$ \begin{cases} 1 - x \geq \left(\Gamma\left(h_1 - \alpha_{1h_1} + 1\right)\right)^{\frac{1}{\left(h_1 - \alpha_{1h_1}\right)}}, \\ 1 - y \geq \left(\Gamma\left(h_2 - \alpha_{2h_2} + 1\right)\right)^{\frac{1}{\left(h_2 - \alpha_{2h_2}\right)}}, \end{cases} \quad (43) $$

equivalently,

$$ \begin{cases} 0 < x \leq 1 - \left(\Gamma\left(h_1 - \alpha_{1h_1} + 1\right)\right)^{\frac{1}{\left(h_1 - \alpha_{1h_1}\right)}}, \\ 0 < y \leq 1 - \left(\Gamma\left(h_2 - \alpha_{2h_2} + 1\right)\right)^{\frac{1}{\left(h_2 - \alpha_{2h_2}\right)}}. \end{cases} \quad (44) $$

so again $\Psi \neq \emptyset$, non-trivially.

3. Main results

We present our left side result.

Theorem 11. *Let h_1, h_2, v_1, v_2, r, p be integers, $0 \leq h_1 \leq v_1 \leq r$, $0 \leq h_2 \leq v_2 \leq p$, and let $f \in C^{r,p}\left([0,1]^2\right)$. Let $\alpha_{ij}(x,y)$, $i = h_1, h_1+1, \ldots, v_1$, $j = h_2, h_2+1, \ldots, v_2$ be real valued functions defined and bounded in $[0,1]^2$. We follow the terminology of Section 2. We assume that $\alpha_{h_1 h_2}$ is either $\geq \alpha > 0$ or $\leq \beta < 0$ throughout the critical region Φ, see (32). Let also $\alpha_{1i}, \alpha_{2j} \geq 0$, $\alpha_{1i}, \alpha_{2j} \notin \mathbb{Z}_+ : \lceil \alpha_{1i} \rceil = i$, $\lceil \alpha_{2j} \rceil = j$, $i = 0, 1, \ldots, r$, $j = 0, 1, \ldots, p$, $\alpha_{10} = 0$, $\alpha_{20} = 0$. Consider the bivariate left fractional differential operator*

$$ L^* := \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{ij}(x,y) D_{*0}^{(\alpha_{1i}, \alpha_{2j})}. \quad (45) $$

Let the integers $\overline{m_1}$, $\overline{m_2}$ with $\overline{m_1} > r$, $\overline{m_2} > p$. Set

$$ l_{ij} := \left|\alpha_{h_1 h_2}^{-1}(x,y)\alpha_{ij}(x,y)\right|_{\infty,[0,1]^2} < \infty. \quad (46) $$

Also set

$$M^{i,j}_{\overline{m_1},\overline{m_2}} := M^{i,j}_{\overline{m_1},\overline{m_2}}(f) := \lambda_{1i}\lambda_{2j}\left[t(i,j)\omega_1\left(f^{(i,j)};\frac{1}{\sqrt{\overline{m_1}-i}},\frac{1}{\sqrt{\overline{m_2}-j}}\right) + \max\left\{\frac{i\,(i-1)}{\overline{m_1}},\frac{j\,(j-1)}{\overline{m_2}}\right\}\left\|f^{(i,j)}_{\infty,[0,1]^2}\right\|\right] \tag{47}$$

for $i=0,1,\ldots,r,\ j=0,1,\ldots,p$.

Above it is $t\colon \mathbb{Z}^2_+ \to \mathbb{R}_+$, *and*

$$\lambda_{1i} := \int_0^1 k_{1i}\,(z)\,\mathrm{d}z, \qquad \lambda_{2j} := \int_0^1 k_{2j}\,(z)\,\mathrm{d}z \tag{48}$$

for $i=1,\ldots,r,\ j=1,\ldots,p$, *and* $\lambda_{10}:=\lambda_{20}:=1$. *Call*

$$P_{\overline{m_1},\overline{m_2}} := P_{\overline{m_1},\overline{m_2}}(f) := \sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2} l_{ij}M^{i,j}_{\overline{m_1},\overline{m_2}}. \tag{49}$$

Then there exists a polynomial $Q_{\overline{m_1},\overline{m_2}}(x,y)$ *of degree* $(\overline{m_1},\overline{m_2})$ *on* $[0,1]^2$ *such that*

i)

$$\begin{aligned}&\|{}^{k_{1i}}_{k_{2j}}D^{(\alpha_{1i},\alpha_{2j})}_{*0}(f) - {}^{k_{1i}}_{k_{2j}}D^{(\alpha_{1i},\alpha_{2j})}_{*0}\left(Q_{\overline{m_1},\overline{m_2}}\right)\|_{\infty,[0,1]^2}\\ &\le \frac{P_{\overline{m_1},\overline{m_2}}}{h_1!h_2!}\|{}^{k_{1i}}_{k_{2j}}D^{(\alpha_{1i},\alpha_{2j})}_{*0}\left(x^{h_1}y^{h_2}\right)\|_{\infty,[0,1]^2} + M^{i,j}_{\overline{m_1},\overline{m_2}},\end{aligned} \tag{50}$$

for $(0,0)\le(i,j)\le(h_1,h_2)$, *and*

ii) *if* $(h_1+1,h_2+1)\le(i,j)\le(r,p)$, *or* $0\le i\le h_1$, $h_2+1\le j\le p$, *or* $h_1+1\le i\le r$, $0\le j\le h_2$,

$$\|{}^{k_{1i}}_{k_{2j}}D^{(\alpha_{1i},\alpha_{2j})}_{*0}(f) - {}^{k_{1i}}_{k_{2j}}D^{(\alpha_{1i},\alpha_{2j})}_{*0}\left(Q_{\overline{m_1},\overline{m_2}}\right)\|_{\infty,[0,1]^2} \le M^{i,j}_{\overline{m_1},\overline{m_2}}. \tag{51}$$

By assuming $L^*f(x,y)\ge 0$ *for all* $(x,y)\in\Phi$, *we obtain* $L^*\left(Q_{\overline{m_1},\overline{m_2}}(x,y)\right)\ge 0$ *for all* $(x,y)\in\Phi$.

Proof. We observe that $(\overline{m_1},\overline{m_2}\in\mathbb{N}:\overline{m_1}>i,\ \overline{m_2}>j)$:

$$\begin{aligned}&\left|{}^{k_{1i}}_{k_{2j}}D^{(\alpha_{1i},\alpha_{2j})}_{*0}f(x_1,x_2) - {}^{k_{1i}}_{k_{2j}}D^{(\alpha_{1i},\alpha_{2j})}_{*0}\left(B_{\overline{m_1},\overline{m_2}}f\right)(x_1,x_2)\right|\\ &=\left|\int_0^{x_1}\int_0^{x_2}k_{1i}(x_1-t_1)k_{2j}\,(x_2-t_2)\,\frac{\partial^{i+j}f\,(t_1,t_2)}{\partial t_1^i\partial t_2^j}\mathrm{d}t_1\mathrm{d}t_2\right.\\ &\left.-\int_0^{x_1}\int_0^{x_2}k_{1i}(x_1-t_1)k_{2j}\,(x_2-t_2)\,\frac{\partial^{i+j}\left(B_{\overline{m_1},\overline{m_2}}f\right)(t_1,t_2)}{\partial t_1^i\partial t_2^j}\mathrm{d}t_1\mathrm{d}t_2\right|\\ &=\left|\int_0^{x_1}\int_0^{x_2}k_{1i}(x_1-t_1)k_{2j}\,(x_2-t_2)\left(\frac{\partial^{i+j}f\,(t_1,t_2)}{\partial t_1^i\partial t_2^j}-\frac{\partial^{i+j}\left(B_{\overline{m_1},\overline{m_2}}f\right)(t_1,t_2)}{\partial t_1^i\partial t_2^j}\right)\mathrm{d}t_1\mathrm{d}t_2\right|\\ &\le\int_0^{x_1}\int_0^{x_2}k_{1i}(x_1-t_1)k_{2j}\,(x_2-t_2)\left|\frac{\partial^{i+j}f\,(t_1,t_2)}{\partial t_1^i\partial t_2^j}-\frac{\partial^{i+j}\left(B_{\overline{m_1},\overline{m_2}}f\right)(t_1,t_2)}{\partial t_1^i\partial t_2^j}\right|\mathrm{d}t_1\mathrm{d}t_2\end{aligned} \tag{52}$$

(53)

$$\overset{(3)}{\leq}\left(\int_0^{x_1}\int_0^{x_2}k_{1i}(x_1-t_1)k_{2j}\left(x_2-t_2\right)\mathrm{d}t_1\mathrm{d}t_2\right)$$

$$\left[t(i,j)\omega_1\left(f^{(i,j)};\frac{1}{\sqrt{\overline{m_1}-i}},\frac{1}{\sqrt{\overline{m_2}-j}}\right)+\max\left\{\frac{i\,(i-1)}{\overline{m_1}},\frac{j\,(j-1)}{\overline{m_2}}\right\}\|f^{(i,j)}\|_{\infty,[0,1]^2}\right]$$

$$=\left(\int_0^{x_1}k_{1i}(x_1-t_1)\mathrm{d}t_1\right)\left(\int_0^{x_2}k_{2j}\left(x_2-t_2\right)\mathrm{d}t_2\right)$$

$$\left[t(i,j)\omega_1\left(f^{(i,j)};\frac{1}{\sqrt{\overline{m_1}-i}},\frac{1}{\sqrt{\overline{m_2}-j}}\right)+\max\left\{\frac{i\,(i-1)}{\overline{m_1}},\frac{j\,(j-1)}{\overline{m_2}}\right\}\|f^{(i,j)}\|_{\infty,[0,1]^2}\right]$$

(54)

$$=\left(\int_0^{x_1}k_{1i}\left(z\right)\mathrm{d}z\right)\left(\int_0^{x_2}k_{2j}\left(z\right)\mathrm{d}z\right)$$

$$\left[t(i,j)\omega_1\left(f^{(i,j)};\frac{1}{\sqrt{\overline{m_1}-i}},\frac{1}{\sqrt{\overline{m_2}-j}}\right)+\max\left\{\frac{i\,(i-1)}{\overline{m_1}},\frac{j\,(j-1)}{\overline{m_2}}\right\}\|f^{(i,j)}\|_{\infty,[0,1]^2}\right]$$

$$\leq\left(\int_0^{1}k_{1i}\left(z\right)\mathrm{d}z\right)\left(\int_0^{1}k_{2j}\left(z\right)\mathrm{d}z\right)$$

$$\left[t(i,j)\omega_1\left(f^{(i,j)};\frac{1}{\sqrt{\overline{m_1}-i}},\frac{1}{\sqrt{\overline{m_2}-j}}\right)+\max\left\{\frac{i\,(i-1)}{\overline{m_1}},\frac{j\,(j-1)}{\overline{m_2}}\right\}\|f^{(i,j)}\|_{\infty,[0,1]^2}\right].$$

We have proved that

(55)

$$\left|{}^{k_{1i}}_{k_{2j}}D_{*0}^{\left(\alpha_{1i},\alpha_{2j}\right)}f(x_1,x_2)-{}^{k_{1i}}_{k_{2j}}D_{*0}^{\left(\alpha_{1i},\alpha_{2j}\right)}\left(B_{\overline{m_1},\overline{m_2}}f\right)(x_1,x_2)\right|$$

$$\leq\left(\int_0^{1}k_{1i}\left(z\right)\mathrm{d}z\right)\left(\int_0^{1}k_{2j}\left(z\right)\mathrm{d}z\right)$$

$$\left[t(i,j)\omega_1\left(f^{(i,j)};\frac{1}{\sqrt{\overline{m_1}-i}},\frac{1}{\sqrt{\overline{m_2}-j}}\right)+\max\left\{\frac{i\,(i-1)}{\overline{m_1}},\frac{j\,(j-1)}{\overline{m_2}}\right\}\|f^{(i,j)}\|_{\infty,[0,1]^2}\right]$$

for all $(x_1,x_2)\in[0,1]^2$, $i=1,\dots,r$, $j=1,\dots,p$, where $\overline{m_1},\overline{m_2}\in\mathbb{N}$: $\overline{m_1}>r$, $\overline{m_2}>p$.

So we have established that

(56)

$$\left\|{}^{k_{1i}}_{k_{2j}}D_{*0}^{\left(\alpha_{1i},\alpha_{2j}\right)}f(x_1,x_2)-{}^{k_{1i}}_{k_{2j}}D_{*0}^{\left(\alpha_{1i},\alpha_{2j}\right)}\left(B_{\overline{m_1},\overline{m_2}}f\right)(x_1,x_2)\right\|_{\infty,[0,1]^2}$$

$$\leq\left(\int_0^{1}k_{1i}\left(z\right)\mathrm{d}z\right)\left(\int_0^{1}k_{2j}\left(z\right)\mathrm{d}z\right)$$

$$\left[t(i,j)\omega_1\left(f^{(i,j)};\frac{1}{\sqrt{\overline{m_1}-i}},\frac{1}{\sqrt{\overline{m_2}-j}}\right)+\max\left\{\frac{i\,(i-1)}{\overline{m_1}},\frac{j\,(j-1)}{\overline{m_2}}\right\}\|f^{(i,j)}\|_{\infty,[0,1]^2}\right]$$

for $i=1,\dots,r$, $j=1,\dots,p$, $\overline{m_1},\overline{m_2}\in\mathbb{N}$: $\overline{m_1}>r$, $\overline{m_2}>p$.

We call

$$\lambda_{1i}:=\int_0^1k_{1i}\left(z\right)\mathrm{d}z,\qquad\lambda_{2j}:=\int_0^1k_{2j}\left(z\right)\mathrm{d}z\tag{57}$$

for $i=1,\dots,r$, $j=1,\dots,p$, as in (48).

We also set $\lambda_{10}:=\lambda_{20}:=1$.

Thus, the following inequality is valid in general:

$$\left\| {}^{k_{1i}}_{k_{2j}} D^{(\alpha_{1i},\alpha_{2j})}_{*0} f(x_1,x_2) - {}^{k_{1i}}_{k_{2j}} D^{(\alpha_{1i},\alpha_{2j})}_{*0} \left(B_{\overline{m_1},\overline{m_2}} f\right)(x_1,x_2) \right\|_{\infty,[0,1]^2} \tag{58}$$
$$\leq \lambda_{1i}\lambda_{2j} \left[t(i,j)\omega_1 \left(f^{(i,j)}; \frac{1}{\sqrt{\overline{m_1}-i}}, \frac{1}{\sqrt{\overline{m_2}-j}} \right) + \max\left\{ \frac{i(i-1)}{\overline{m_1}}, \frac{j(j-1)}{\overline{m_2}} \right\} \|f^{(i,j)}\|_{\infty,[0,1]^2} \right]$$
$$= M^{i,j}_{\overline{m_1},\overline{m_2}}$$

for $i = 0,1,\ldots,r$, $j = 0,1,\ldots,p$; $\overline{m_1},\overline{m_2} \in \mathbb{N} : \overline{m_1} > r$, $\overline{m_2} > p$, case of Theorem 4.

<u>*Case* (i)</u>: Assume throughout Φ that $\alpha_{h_1h_2}(x,y) \geq \alpha > 0$. Call

$$Q_{\overline{m_1},\overline{m_2}}(x,y) := B_{\overline{m_1},\overline{m_2}}(f;x,y) + P_{\overline{m_1},\overline{m_2}} \frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!} \tag{59}$$

for all $(x,y) \in [0,1]^2$.

To remind, here it is

$$l_{ij} := \|\alpha^{-1}_{h_1h_2}(x,y)\alpha_{ij}(x,y)\|_{\infty,[0,1]^2} \tag{60}$$

and

$$P_{\overline{m_1},\overline{m_2}} := \sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2} l_{ij} M^{i,j}_{\overline{m_1},\overline{m_2}}.$$

Therefore, by (58), we obtain

$$\left\| {}^{k_{1i}}_{k_{2j}} D^{(\alpha_{1i},\alpha_{2j})}_{*0} \left(f + P_{\overline{m_1},\overline{m_2}} \frac{x^{h_1}y^{h_2}}{h_1!h_2!} \right) - {}^{k_{1i}}_{k_{2j}} D^{(\alpha_{1i},\alpha_{2j})}_{*0} \left(Q_{\overline{m_1},\overline{m_2}}\right) \right\|_{\infty,[0,1]^2} \leq M^{i,j}_{\overline{m_1},\overline{m_2}}, \tag{61}$$

for all $0 \leq i \leq r$, $0 \leq j \leq p$.

Let $(0,0) \leq (i,j) \leq (h_1,h_2)$, by (61), we get

$$\left\| {}^{k_{1i}}_{k_{2j}} D^{(\alpha_{1i},\alpha_{2j})}_{*0} (f) - {}^{k_{1i}}_{k_{2j}} D^{(\alpha_{1i},\alpha_{2j})}_{*0} \left(Q_{\overline{m_1},\overline{m_2}}\right) \right\|_{\infty,[0,1]^2} \tag{62}$$
$$\leq \frac{P_{\overline{m_1},\overline{m_2}}}{h_1!h_2!} \left\| {}^{k_{1i}}_{k_{2j}} D^{(\alpha_{1i},\alpha_{2j})}_{*0} \left(x^{h_1}y^{h_2}\right) \right\|_{\infty,[0,1]^2} + M^{i,j}_{\overline{m_1},\overline{m_2}},$$

proving (50).

If $(h_1+1,h_2+1) \leq (i,j) \leq (r,p)$, or $0 \leq i \leq h_1$, $h_2+1 \leq j \leq p$, or $h_1+1 \leq i \leq r$, $0 \leq j \leq h_2$ by (61), we get

$$\left\| {}^{k_{1i}}_{k_{2j}} D^{(\alpha_{1i},\alpha_{2j})}_{*0} (f) - {}^{k_{1i}}_{k_{2j}} D^{(\alpha_{1i},\alpha_{2j})}_{*0} \left(Q_{\overline{m_1},\overline{m_2}}\right) \right\|_{\infty,[0,1]^2} \leq M^{i,j}_{\overline{m_1},\overline{m_2}}, \tag{63}$$

proving (51).

For (x, y) in the critical region Φ, we can write

$$
\begin{aligned}
&\alpha_{h_1h_2}^{-1}(x,y)L^*\left(Q_{\overline{m_1},\overline{m_2}}(x,y)\right)\\
&= \alpha_{h_1h_2}^{-1}(x,y)L^*\left(f(x,y)\right) + \frac{P_{\overline{m_1},\overline{m_2}}}{h_1!h_2!}\left({}_{k_{2h_2}}^{k_{1h_1}}D_{*0}^{\left(\alpha_{1h_1},\alpha_{2h_2}\right)}\left(x^{h_1}y^{h_2}\right)\right)\\
&+ \sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2}\alpha_{h_1h_2}^{-1}(x,y)\alpha_{ij}(x,y)\ {}_{k_{2j}}^{k_{1i}}D_{*0}^{\left(\alpha_{1i},\alpha_{2j}\right)}
\end{aligned}
\tag{64}
$$

$$
\left[Q_{\overline{m_1},\overline{m_2}}(x,y) - f(x,y) - P_{\overline{m_1},\overline{m_2}}\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}\right] \qquad (\text{by } L^*f \geq 0)
$$

$$
\begin{aligned}
&\overset{(61)}{\geq} \frac{P_{\overline{m_1},\overline{m_2}}}{h_1!h_2!}\left({}_{k_{2h_2}}^{k_{1h_1}}D_{*0}^{\left(\alpha_{1h_1},\alpha_{2h_2}\right)}\left(x^{h_1}y^{h_2}\right)\right) - \sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2} l_{ij}M_{\overline{m_1},\overline{m_2}}^{i,j}\\
&= P_{\overline{m_1},\overline{m_2}}\left[\frac{{}_{k_{2h_2}}^{k_{1h_1}}D_{*0}^{\left(\alpha_{1h_1},\alpha_{2h_2}\right)}\left(x^{h_1}y^{h_2}\right)}{h_1!h_2!} - 1\right]\\
&= P_{\overline{m_1},\overline{m_2}}\left[\frac{\left({}_{k_{20}}^{k_{1h_1}}D_{*0}^{\left(\alpha_{1h_1},0\right)}x^{h_1}\right)\left({}_{k_{2h_2}}^{k_{10}}D_{*0}^{\left(0,\alpha_{2h_2}\right)}y^{h_2}\right)}{h_1!h_2!} - 1\right] =: \varphi.
\end{aligned}
\tag{65}
$$

If $h_1 = h_2 = 0$, then $\alpha_{1h_1} = \alpha_{2h_2} = 0$ and $\varphi = 0$. If $h_1 = 0$ and $h_2 \neq 0$, then

$$
\begin{aligned}
\varphi &= P_{\overline{m_1},\overline{m_2}}\left[\frac{{}_{k_{2h_2}}^{k_{10}}D_{*0}^{\left(0,\alpha_{2h_2}\right)}y^{h_2}}{h_2!} - 1\right] = P_{\overline{m_1},\overline{m_2}}\left[\int_0^y k_{2h_2}(y-t)\mathrm{d}t - 1\right]\\
&= P_{\overline{m_1},\overline{m_2}}\left(\int_0^y k_{2h_2}(z)\mathrm{d}z - 1\right) \geq 0.
\end{aligned}
\tag{66}
$$

Similarly, we treat the case $h_1 \neq 0$, $h_2 = 0$.

When $h_1, h_2 \neq 0$, then we have

$$
\begin{aligned}
\varphi &= P_{\overline{m_1},\overline{m_2}}\left[\left(\int_0^x k_{1h_1}(x-t)\,\mathrm{d}t\right)\left(\int_0^y k_{2h_2}(y-t)\,\mathrm{d}t\right) - 1\right]\\
&= P_{\overline{m_1},\overline{m_2}}\left[\left(\int_0^x k_{1h_1}(z)\mathrm{d}z\right)\left(\int_0^y k_{2h_2}(z)\mathrm{d}z\right) - 1\right] \geq 0.
\end{aligned}
\tag{67}
$$

So in all four cases we have proved that

$$
L^*\left(Q_{\overline{m_1},\overline{m_2}}\right)(x,y) \geq 0 \qquad \text{for all } (x,y) \in \Phi.
\tag{68}
$$

Case (ii): Assume throughout Φ that $\alpha_{h_1h_2} \leq \beta < 0$. Consider

$$
\overline{Q}_{\overline{m_1},\overline{m_2}}(x,y) := B_{\overline{m_1},\overline{m_2}}(f;x,y) - P_{\overline{m_1},\overline{m_2}}\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}
$$

for all $(x,y) \in [0,1]^2$.

Therefore by (58), we obtain

(69)
$$\left\| {}_{k_{2j}}^{k_{1i}}D_{*0}^{(\alpha_{1i},\alpha_{2j})}\left(f - P_{\overline{m_1},\overline{m_2}}\frac{x^{h_1}y^{h_2}}{h_1!h_2!}\right) - {}_{k_{2j}}^{k_{1i}}D_{*0}^{(\alpha_{1i},\alpha_{2j})}\left(\overline{Q}_{\overline{m_1},\overline{m_2}}\right)\right\|_{\infty,[0,1]^2} \leq M_{\overline{m_1},\overline{m_2}}^{i,j}$$

for all $0 \leq i \leq r$, $0 \leq j \leq p$.

Also both (50) and (51) are valid, just replace $Q_{\overline{m_1},\overline{m_2}}$ by $\overline{Q}_{\overline{m_1},\overline{m_2}}$ in (62), (63). For (x, y) in the critical region Φ, we can write

$$\begin{aligned}
&\alpha_{h_1h_2}^{-1}(x,y)L^*\left(\overline{Q}_{\overline{m_1},\overline{m_2}}(x,y)\right) \\
&= \alpha_{h_1h_2}^{-1}(x,y)L^*\left(f(x,y)\right) - \frac{P_{\overline{m_1},\overline{m_2}}}{h_1!h_2!}\left({}_{k_{2h_2}}^{k_{1h_1}}D_{*0}^{(\alpha_{1h_1},\alpha_{2h_2})}\left(x^{h_1}y^{h_2}\right)\right) \\
&\quad + \sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2}\alpha_{h_1h_2}^{-1}(x,y)\alpha_{ij}(x,y)\; {}_{k_{2j}}^{k_{1i}}D_{*0}^{(\alpha_{1i},\alpha_{2j})}
\end{aligned}$$

$$\left[\overline{Q}_{\overline{m_1},\overline{m_2}}(x,y) - f(x,y) + P_{\overline{m_1},\overline{m_2}}\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}\right] \qquad (\text{by } L^*f \geq 0) \tag{70}$$

$$\begin{aligned}
&\overset{(69)}{\leq} -\frac{P_{\overline{m_1},\overline{m_2}}}{h_1!h_2!}\left({}_{k_{2h_2}}^{k_{1h_1}}D_{*0}^{(\alpha_{1h_1},\alpha_{2h_2})}\left(x^{h_1}y^{h_2}\right)\right) + \sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2} l_{ij}M_{\overline{m_1},\overline{m_2}}^{i,j} \\
&= P_{\overline{m_1},\overline{m_2}}\left[1 - \frac{{}_{k_{2h_2}}^{k_{1h_1}}D_{*0}^{(\alpha_{1h_1},\alpha_{2h_2})}\left(x^{h_1}y^{h_2}\right)}{h_1!h_2!}\right]
\end{aligned}$$

$$= P_{\overline{m_1},\overline{m_2}}\left[1 - \frac{\left({}_{k_{20}}^{k_{1h_1}}D_{*0}^{(\alpha_{1h_1},0)}x^{h_1}\right)\left({}_{k_{2h_2}}^{k_{10}}D_{*0}^{(0,\alpha_{2h_2})}y^{h_2}\right)}{h_1!h_2!}\right] =: \psi. \tag{71}$$

If $h_1 = h_2 = 0$, then $\alpha_{1h_1} = \alpha_{2h_2} = 0$ and $\psi = 0$.
If $h_1 = 0$ and $h_2 \neq 0$, then

$$\begin{aligned}
\varphi &= P_{\overline{m_1},\overline{m_2}}\left[1 - \frac{{}_{k_{2h_2}}^{k_{10}}D_{*0}^{(0,\alpha_{2h_2})}y^{h_2}}{h_2!}\right] \qquad (72)\\
&= P_{\overline{m_1},\overline{m_2}}\left[1 - \int_0^y k_{2h_2}(y-t)\,\mathrm{d}t\right] = P_{\overline{m_1},\overline{m_2}}\left[1 - \int_0^y k_{2h_2}(z)\mathrm{d}z\right] \leq 0.
\end{aligned}$$

Similarly, we treat the case $h_1 \neq 0$, $h_2 = 0$.
When $h_1, h_2 \neq 0$, then we have

$$\begin{aligned}
\varphi &= P_{\overline{m_1},\overline{m_2}}\left[1 - \left(\int_0^x k_{1h_1}(x-t)\,\mathrm{d}t\right)\left(\int_0^y k_{2h_2}(y-t)\,\mathrm{d}t\right)\right] \qquad (73)\\
&= P_{\overline{m_1},\overline{m_2}}\left[1 - \left(\int_0^x k_{1h_1}(z)\mathrm{d}z\right)\left(\int_0^y k_{2h_2}(z)\mathrm{d}z\right)\right] \leq 0.
\end{aligned}$$

So in all four cases we have proved that

$$L^*\left(\overline{Q}_{\overline{m_1},\overline{m_2}}\right)(x,y) \geq 0 \qquad \text{for all } (x,y) \in \Phi. \tag{74}$$

□

Next we give our right side result.

Theorem 12. *Let h_1, h_2 be even and v_1, v_2, r, p be integers, $0 \le h_1 \le v_1 \le r$, $0 \le h_2 \le v_2 \le p$, and let $f \in C^{r,p}([0,1]^2)$. Let $\alpha_{ij}(x,y)$, $i = h_1, h_1+1, \ldots, v_1$, $j = h_2, h_2+1, \ldots, v_2$ be real valued functions, defined and bounded in $[0,1]^2$. We follow the terminology of Section 2. We assume that $\alpha_{h_1 h_2}$ is either $\ge \alpha > 0$ or $\le \beta < 0$ throughout the critical region Ψ, see (42). Let also $\alpha_{1i}, \alpha_{2j} \ge 0$, $\alpha_{1i}, \alpha_{2j} \notin \mathbb{Z}_+ : \lceil \alpha_{1i} \rceil = i$, $\lceil \alpha_{2j} \rceil = j$, $i = 0,1,\ldots,r$, $j = 0,1,\ldots,p$, $\alpha_{10} = 0$, $\alpha_{20} = 0$. Consider the bivariate right fractional differential operator*

$$L^* := \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{ij}(x,y) D_{1-}^{(\alpha_{1i}, \alpha_{2j})}. \tag{75}$$

Let the integers $\overline{m_1}$, $\overline{m_2}$ with $\overline{m_1} > r$, $\overline{m_2} > p$. Set

$$l_{ij} := \| \alpha_{h_1 h_2}^{-1}(x,y) \alpha_{ij}(x,y) \|_{\infty,[0,1]^2} < \infty. \tag{76}$$

Also set

$$M_{\overline{m_1},\overline{m_2}}^{i,j} := M_{\overline{m_1},\overline{m_2}}^{i,j}(f) := \lambda_{1i}\lambda_{2j} \Bigg[t(i,j)\, \omega_1 \left(f^{(i,j)}; \frac{1}{\sqrt{\overline{m_1} - i}}, \frac{1}{\sqrt{\overline{m_2} - j}} \right) + \max \left\{ \frac{i(i-1)}{\overline{m_1}}, \frac{j(j-1)}{\overline{m_2}} \right\} \| f_{\infty,[0,1]^2}^{(i,j)} \| \Bigg] \tag{77}$$

for $i = 0,1,\ldots,r$; $j = 0,1,\ldots,p$.
Above it is $t : \mathbb{Z}_+^2 \to \mathbb{R}_+$ and

$$\lambda_{1i} := \int_0^1 k_{1i}(z)\mathrm{d}z, \qquad \lambda_{2j} := \int_0^1 k_{2j}(z)\mathrm{d}z \tag{78}$$

for $i = 1,\ldots,r$, $j = 1,\ldots,p$ and $\lambda_{10} := \lambda_{20} := 1$.

Call

$$P_{\overline{m_1},\overline{m_2}} := P_{\overline{m_1},\overline{m_2}}(f) := \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} l_{ij} M_{\overline{m_1},\overline{m_2}}^{i,j}. \tag{79}$$

Then there exists a polynomial $Q_{\overline{m_1},\overline{m_2}}(x,y)$ of degree $(\overline{m_1}, \overline{m_2})$ on $[0,1]^2$, such that

i)

$$\begin{aligned} &\| {}^{k_{1i}}_{k_{2j}} D_{1-}^{(\alpha_{1i},\alpha_{2j})}(f) - {}^{k_{1i}}_{k_{2j}} D_{1-}^{(\alpha_{1i},\alpha_{2j})}(Q_{\overline{m_1},\overline{m_2}}) \|_{\infty,[0,1]^2} \\ &\le \frac{P_{\overline{m_1},\overline{m_2}}}{h_1! h_2!} \| {}^{k_{1i}}_{k_{2j}} D_{1-}^{(\alpha_{1i},\alpha_{2j})} \left(x^{h_1} y^{h_2} \right) \|_{\infty,[0,1]^2} + M_{\overline{m_1},\overline{m_2}}^{i,j} \end{aligned} \tag{80}$$

for $(0,0) \le (i,j) \le (h_1,h_2)$, and

ii)

if $(h_1+1, h_2+1) \le (i,j) \le (r,p)$, or $0 \le i \le h_1$, $h_2+1 \le j \le p$, or $h_1+1 \le i \le r$, $0 \le j \le h_2$,

$$\| {}^{k_{1i}}_{k_{2j}} D_{1-}^{(\alpha_{1i},\alpha_{2j})}(f) - {}^{k_{1i}}_{k_{2j}} D_{1-}^{(\alpha_{1i},\alpha_{2j})}(Q_{\overline{m_1},\overline{m_2}}) \|_{\infty,[0,1]^2} \le M_{\overline{m_1},\overline{m_2}}^{i,j}. \tag{81}$$

By assuming $L^* f(x,y) \ge 0$ for all $(x,y) \in \Psi$, we obtain $L^*\left(Q_{\overline{m_1},\overline{m_2}}(x,y)\right) \ge 0$ for all $(x,y) \in \Psi$.

Proof. We observe that ($\overline{m_1}, \overline{m_2} \in \mathbb{N} : \overline{m_1} > i$, $\overline{m_2} > j$):

$$\left| {}_{k_{2j}}^{k_{1i}} D_{1-}^{\left(\alpha_{1i},\alpha_{2j}\right)} f(x_1,x_2) - {}_{k_{2j}}^{k_{1i}} D_{1-}^{\left(\alpha_{1i},\alpha_{2j}\right)} \left(B_{\overline{m_1},\overline{m_2}} f\right)(x_1,x_2) \right|$$

(82)

$$= \left| (-1)^{i+j} \int_{x_1}^{1} \int_{x_2}^{1} k_{1i}(t_1 - x_1)\, k_{2j}(t_2 - x_2)\, \frac{\partial^{i+j} f(t_1,t_2)}{\partial t_1^i \partial t_2^j} \mathrm{d}t_1 \mathrm{d}t_2 \right.$$
$$\left. - (-1)^{i+j} \int_{x_1}^{1} \int_{x_2}^{1} k_{1i}(t_1 - x_1)\, k_{2j}(t_2 - x_2)\, \frac{\partial i + j \left(B_{\overline{m_1},\overline{m_2}} f\right)(t_1,t_2)}{\partial t_1^i \partial t_2^j} \mathrm{d}t_1 \mathrm{d}t_2 \right|$$
$$= \left| \int_{x_1}^{1} \int_{x_2}^{1} k_{1i}(t_1 - x_1)\, k_{2j}(t_2 - x_2) \left(\frac{\partial^{i+j} f(t_1,t_2)}{\partial t_1^i \partial t_2^j} - \frac{\partial^{i+j} \left(B_{\overline{m_1},\overline{m_2}} f\right)(t_1,t_2)}{\partial t_1^i \partial t_2^j} \right) \mathrm{d}t_1 \mathrm{d}t_2 \right|$$
$$\leq \int_{x_1}^{1} \int_{x_2}^{1} k_{1i}(t_1 - x_1)\, k_{2j}(t_2 - x_2) \left| \frac{\partial^{i+j} f(t_1,t_2)}{\partial t_1^i \partial t_2^j} - \frac{\partial^{i+j} \left(B_{\overline{m_1},\overline{m_2}} f\right)(t_1,t_2)}{\partial t_1^i \partial t_2^j} \right| \mathrm{d}t_1 \mathrm{d}t_2$$

(83)

$$\overset{(3)}{\leq} \left(\int_{x_1}^{1} \int_{x_2}^{1} k_{1i}(t_1 - x_1)\, k_{2j}(t_2 - x_2)\, \mathrm{d}t_1 \mathrm{d}t_2 \right)$$
$$\left[t(i,j)\omega_1 \left(f^{(i,j)}; \frac{1}{\sqrt{\overline{m_1} - i}}, \frac{1}{\sqrt{\overline{m_2} - j}} \right) + \max\left\{ \frac{i(i-1)}{\overline{m_1}}, \frac{j(j-1)}{\overline{m_2}} \right\} \|f^{(i,j)}\|_{\infty,[0,1]^2} \right]$$
$$= \left(\int_{x_1}^{1} k_{1i}(t_1 - x_1)\, \mathrm{d}t_1 \right) \left(\int_{x_2}^{1} k_{2j}(t_2 - x_2)\, \mathrm{d}t_2 \right)$$
$$\left[t(i,j)\omega_1 \left(f^{(i,j)}; \frac{1}{\sqrt{\overline{m_1} - i}}, \frac{1}{\sqrt{\overline{m_2} - j}} \right) + \max\left\{ \frac{i(i-1)}{\overline{m_1}}, \frac{j(j-1)}{\overline{m_2}} \right\} \|f^{(i,j)}\|_{\infty,[0,1]^2} \right]$$

(84)

$$= \left(\int_{0}^{1-x_1} k_{1i}(z)\mathrm{d}z \right) \left(\int_{0}^{1-x_2} k_{2j}(z)\mathrm{d}z \right)$$
$$\left[t(i,j)\omega_1 \left(f^{(i,j)}; \frac{1}{\sqrt{\overline{m_1} - i}}, \frac{1}{\sqrt{\overline{m_2} - j}} \right) + \max\left\{ \frac{i(i-1)}{\overline{m_1}}, \frac{j(j-1)}{\overline{m_2}} \right\} \|f^{(i,j)}\|_{\infty,[0,1]^2} \right]$$
$$\leq \left(\int_{0}^{1} k_{1i}(z)\mathrm{d}z \right) \left(\int_{0}^{1} k_{2j}(z)\mathrm{d}z \right)$$
$$\left[t(i,j)\omega_1 \left(f^{(i,j)}; \frac{1}{\sqrt{\overline{m_1} - i}}, \frac{1}{\sqrt{\overline{m_2} - j}} \right) + \max\left\{ \frac{i(i-1)}{\overline{m_1}}, \frac{j(j-1)}{\overline{m_2}} \right\} \|f^{(i,j)}\|_{\infty,[0,1]^2} \right].$$

We have proved that

(85)

$$\left| {}_{k_{2j}}^{k_{1i}} D_{1-}^{\left(\alpha_{1i},\alpha_{2j}\right)} f(x_1,x_2) - {}_{k_{2j}}^{k_{1i}} D_{1-}^{\left(\alpha_{1i},\alpha_{2j}\right)} \left(B_{\overline{m_1},\overline{m_2}} f\right)(x_1,x_2) \right|$$
$$\leq \left(\int_{0}^{1} k_{1i}(z)\mathrm{d}z \right) \left(\int_{0}^{1} k_{2j}(z)\mathrm{d}z \right)$$
$$\left[t(i,j)\omega_1 \left(f^{(i,j)}; \frac{1}{\sqrt{\overline{m_1} - i}}, \frac{1}{\sqrt{\overline{m_2} - j}} \right) + \max\left\{ \frac{i(i-1)}{\overline{m_1}}, \frac{j(j-1)}{\overline{m_2}} \right\} \|f^{(i,j)}\|_{\infty,[0,1]^2} \right]$$

for all $(x_1, x_2) \in [0,1]^2$, $i = 1, \ldots, r$, $j = 1, \ldots, p$, where $\overline{m_1}, \overline{m_2} \in \mathbb{N} : \overline{m_1} > r$, $\overline{m_2} > p$.

So we have established that

(86)
$$\left\|{}^{k_{1i}}_{k_{2j}}D^{\left(\alpha_{1i},\alpha_{2j}\right)}_{1-}f(x_1,x_2) - {}^{k_{1i}}_{k_{2j}}D^{\left(\alpha_{1i},\alpha_{2j}\right)}_{1-}\left(B_{\overline{m_1},\overline{m_2}}f\right)(x_1,x_2)\right\|_{\infty,[0,1]^2}$$
$$\leq \left(\int_0^1 k_{1i}(z)\mathrm{d}z\right)\left(\int_0^1 k_{2j}(z)\mathrm{d}z\right)$$
$$\left[t(i,j)\omega_1\left(f^{(i,j)};\frac{1}{\sqrt{\overline{m_1}-i}},\frac{1}{\sqrt{\overline{m_2}-j}}\right) + \max\left\{\frac{i\,(i-1)}{\overline{m_1}},\frac{j\,(j-1)}{\overline{m_2}}\right\}\|f^{(i,j)}\|_{\infty,[0,1]^2}\right]$$

for $i=1,\ldots,r$, $j=1,\ldots,p$, $\overline{m_1},\overline{m_2}\in\mathbb{N}:\overline{m_1}>r$, $\overline{m_2}>p$. We call

(87)
$$\lambda_{1i} := \int_0^1 k_{1i}(z)\mathrm{d}z, \qquad \lambda_{2j} := \int_0^1 k_{2j}(z)\mathrm{d}z$$

for $i=1,\ldots,r$, $j=1,\ldots,p$, as in (78).

We also set $\lambda_{10} := \lambda_{20} := 1$.

Thus, the following inequality is valid in general:

(88)
$$\left\|{}^{k_{1i}}_{k_{2j}}D^{\left(\alpha_{1i},\alpha_{2j}\right)}_{1-}f(x_1,x_2) - {}^{k_{1i}}_{k_{2j}}D^{\left(\alpha_{1i},\alpha_{2j}\right)}_{1-}\left(B_{\overline{m_1},\overline{m_2}}f\right)(x_1,x_2)\right\|_{\infty,[0,1]^2}$$
$$\leq \lambda_{1i}\lambda_{2j}\left[t(i,j)\omega_1\left(f^{(i,j)};\frac{1}{\sqrt{\overline{m_1}-i}},\frac{1}{\sqrt{\overline{m_2}-j}}\right)+\max\left\{\frac{i\,(i-1)}{\overline{m_1}},\frac{j\,(j-1)}{\overline{m_2}}\right\}\|f^{(i,j)}\|_{\infty,[0,1]^2}\right]$$
$$= M^{i,j}_{\overline{m_1},\overline{m_2}}$$

for $i=0,1,\ldots,r$, $j=0,1,\ldots,p$, $\overline{m_1},\overline{m_2}\in\mathbb{N}:\overline{m_1}>r$, $\overline{m_2}>p$, case of Theorem 4.

Case (i): Assume throughout Ψ that $\alpha_{h_1h_2}(x,y)\geq\alpha>0$. Call

(89)
$$Q_{\overline{m_1},\overline{m_2}}(x,y) := B_{\overline{m_1},\overline{m_2}}(f;x,y) + P_{\overline{m_1},\overline{m_2}}\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}$$

for all $(x,y)\in[0,1]^2$. To remind, here it is

(90)
$$l_{ij} := \|\alpha^{-1}_{h_1h_2}(x,y)\alpha_{ij}(x,y)\|_{\infty,[0,1]^2}$$

and

$$P_{\overline{m_1},\overline{m_2}} := \sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2} l_{ij}M^{i,j}_{\overline{m_1},\overline{m_2}}.$$

Therefore, by (88), we obtain

(91)
$$\left\|{}^{k_{1i}}_{k_{2j}}D^{\left(\alpha_{1i},\alpha_{2j}\right)}_{1-}\left(f+P_{\overline{m_1},\overline{m_2}}\frac{x^{h_1}y^{h_2}}{h_1!h_2!}\right) - {}^{k_{1i}}_{k_{2j}}D^{\left(\alpha_{1i},\alpha_{2j}\right)}_{1-}\left(Q_{\overline{m_1},\overline{m_2}}\right)\right\|_{\infty,[0,1]^2} \leq M^{i,j}_{\overline{m_1},\overline{m_2}}$$

for all $0\leq i\leq r$, $0\leq j\leq p$.

Let $(0,0)\leq(i,j)\leq(h_1,h_2)$, by (91), we get

(92)
$$\left\|{}^{k_{1i}}_{k_{2j}}D^{\left(\alpha_{1i},\alpha_{2j}\right)}_{1-}(f) - {}^{k_{1i}}_{k_{2j}}D^{\left(\alpha_{1i},\alpha_{2j}\right)}_{1-}\left(Q_{\overline{m_1},\overline{m_2}}\right)\right\|_{\infty,[0,1]^2}$$
$$\leq \frac{P_{\overline{m_1},\overline{m_2}}}{h_1!h_2!}\|{}^{k_{1i}}_{k_{2j}}D^{\left(\alpha_{1i},\alpha_{2j}\right)}_{1-}\left(x^{h_1}y^{h_2}\right)\|_{\infty,[0,1]^2} + M^{i,j}_{\overline{m_1},\overline{m_2}},$$

proving (80).

If $(h_1+1, h_2+1) \leq (i,j) \leq (r,p)$, or $0 \leq i \leq h_1$, $h_2+1 \leq j \leq p$, or $h_1+1 \leq i \leq r$, $0 \leq j \leq h_2$ by (91), we get

$$\left\| {}_{k_{2j}}^{k_{1i}} D_{1-}^{(\alpha_{1i},\alpha_{2j})}(f) - {}_{k_{2j}}^{k_{1i}} D_{1-}^{(\alpha_{1i},\alpha_{2j})}\left(Q_{\overline{m_1},\overline{m_2}}\right) \right\|_{\infty,[0,1]^2} \leq M_{\overline{m_1},\overline{m_2}}^{i,j}, \tag{93}$$

proving (81).

For (x,y) in the critical region Ψ, we can write

$$\begin{aligned} &\alpha_{h_1h_2}^{-1}(x,y) L^*\left(Q_{\overline{m_1},\overline{m_2}}(x,y)\right) = \alpha_{h_1h_2}^{-1}(x,y) L^*\left(f(x,y)\right) \\ &\quad + \frac{P_{\overline{m_1},\overline{m_2}}}{h_1! h_2!}\left({}_{k_{2h_2}}^{k_{1h_1}} D_{1-}^{(\alpha_{1h_1},\alpha_{2h_2})}\left(x^{h_1} y^{h_2}\right)\right) \\ &\quad + \sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2} \alpha_{h_1h_2}^{-1}(x,y)\alpha_{ij}(x,y)\, {}_{k_{2j}}^{k_{1i}} D_{1-}^{(\alpha_{1i},\alpha_{2j})} \\ &\quad \left[Q_{\overline{m_1},\overline{m_2}}(x,y) - f(x,y) - P_{\overline{m_1},\overline{m_2}}\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}\right] \qquad (\text{by } L^* f \geq 0) \\ &\quad \overset{(91)}{\geq} \frac{P_{\overline{m_1},\overline{m_2}}}{h_1! h_2!}\left({}_{k_{2h_2}}^{k_{1h_1}} D_{1-}^{(\alpha_{1h_1},\alpha_{2h_2})}\left(x^{h_1} y^{h_2}\right)\right) - \sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2} l_{ij} M_{\overline{m_1},\overline{m_2}}^{i,j} \end{aligned} \tag{94}$$

$$= P_{\overline{m_1},\overline{m_2}}\left[\frac{{}_{k_{2h_2}}^{k_{1h_1}} D_{1-}^{(\alpha_{1h_1},\alpha_{2h_2})}\left(x^{h_1} y^{h_2}\right)}{h_1! h_2!} - 1\right]$$

$$= P_{\overline{m_1},\overline{m_2}}\left[\frac{\left({}_{k_{20}}^{k_{1h_1}} D_{1-}^{(\alpha_{1h_1},0)} x^{h_1}\right)\left({}_{k_{2h_2}}^{k_{10}} D_{1-}^{(0,\alpha_{2h_2})} y^{h_2}\right)}{h_1! h_2!} - 1\right] =: \psi. \tag{95}$$

If $h_1 = h_2 = 0$, then $\alpha_{1h_1} = \alpha_{2h_2} = 0$ and $\psi = 0$.

If $h_1 = 0$ and $h_2 \neq 0$, then

$$\begin{aligned} \psi &= P_{\overline{m_1},\overline{m_2}}\left[\frac{{}_{k_{2h_2}}^{k_{10}} D_{1-}^{(0,\alpha_{2h_2})} y^{h_2}}{h_2!} - 1\right] \\ &= P_{\overline{m_1},\overline{m_2}}\left[(-1)^{h_2}\int_y^1 k_{2h_2}(t-y)\,\mathrm{d}t - 1\right] \\ &\overset{(h_2 \text{ is even})}{=} P_{\overline{m_1},\overline{m_2}}\left(\int_0^{1-y} k_{2h_2}(z)\,\mathrm{d}z - 1\right). \end{aligned} \tag{96}$$

Similarly, we treat the case $h_1 \neq 0$, $h_2 = 0$.
When $h_1, h_2 \neq 0$, then we have

$$\begin{aligned} \psi &= P_{\overline{m_1},\overline{m_2}}\left[\left(\int_x^1 k_{1h_1}(t-x)\,\mathrm{d}t\right)\left(\int_y^1 k_{2h_2}(t-y)\,\mathrm{d}t\right) - 1\right] \\ &= P_{\overline{m_1},\overline{m_2}}\left[\left(\int_0^{1-x} k_{1h_1}(z)\mathrm{d}z\right)\left(\int_0^{1-y} k_{2h_2}(z)\mathrm{d}z\right) - 1\right] \geq 0. \end{aligned} \tag{97}$$

So in all four cases we have proved that

$$L^*\left(Q_{\overline{m_1},\overline{m_2}}\right)(x,y) \geq 0 \qquad \text{for all } (x,y) \in \Psi. \tag{98}$$

<u>*Case* (ii)</u>: Assume throughout Ψ that $\alpha_{h_1h_2} \le \beta < 0$. Consider

$$\overline{Q}_{\overline{m_1},\overline{m_2}}(x,y) := B_{\overline{m_1},\overline{m_2}}(f;x,y) - P_{\overline{m_1},\overline{m_2}} \frac{x^{h_1}}{h_1!} \frac{y^{h_2}}{h_2!},$$

for all $(x,y) \in [0,1]^2$.
Therefore, by (88), we obtain

$$\left\| {}_{k_{2j}}^{k_{1i}} D_{1-}^{\left(\alpha_{1i},\alpha_{2j}\right)} \left(f - P_{\overline{m_1},\overline{m_2}} \frac{x^{h_1}y^{h_2}}{h_1!h_2!} \right) - {}_{k_{2j}}^{k_{1i}} D_{1-}^{\left(\alpha_{1i},\alpha_{2j}\right)} \left(\overline{Q}_{\overline{m_1},\overline{m_2}} \right) \right\|_{\infty,[0,1]^2} \le M_{\overline{m_1},\overline{m_2}}^{i,j} \tag{99}$$

for all $0 \le i \le r$, $0 \le j \le p$.
Also both (80) and (81) are valid, just replace $Q_{\overline{m_1},\overline{m_2}}$ by $\overline{Q}_{\overline{m_1},\overline{m_2}}$ in (92), (93).

For (x,y) in the critical region Ψ, we can write

$$\begin{aligned}
&\alpha_{h_1h_2}^{-1}(x,y) L^* \left(\overline{Q}_{\overline{m_1},\overline{m_2}}(x,y) \right) \\
&= \alpha_{h_1h_2}^{-1}(x,y) L^* \left(f(x,y) \right) - \frac{P_{\overline{m_1},\overline{m_2}}}{h_1!h_2!} \left({}_{k_{2h_2}}^{k_{1h_1}} D_{1-}^{\left(\alpha_{1h_1},\alpha_{2h_2}\right)} \left(x^{h_1} y^{h_2} \right) \right) \\
&+ \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{h_1h_2}^{-1}(x,y) \alpha_{ij}(x,y) \, {}_{k_{2j}}^{k_{1i}} D_{1-}^{\left(\alpha_{1i},\alpha_{2j}\right)} \\
&\left[\overline{Q}_{\overline{m_1},\overline{m_2}}(x,y) - f(x,y) + P_{\overline{m_1},\overline{m_2}} \frac{x^{h_1}}{h_1!} \frac{y^{h_2}}{h_2!} \right] \qquad (\text{by } L^* f \ge 0) \qquad (100) \\
&\overset{(99)}{\le} -\frac{P_{\overline{m_1},\overline{m_2}}}{h_1!h_2!} \left({}_{k_{2h_2}}^{k_{1h_1}} D_{1-}^{\left(\alpha_{1h_1},\alpha_{2h_2}\right)} \left(x^{h_1} y^{h_2} \right) \right) + \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} l_{ij} M_{\overline{m_1},\overline{m_2}}^{i,j} \\
&= P_{\overline{m_1},\overline{m_2}} \left[1 - \frac{ {}_{k_{2h_2}}^{k_{1h_1}} D_{1-}^{\left(\alpha_{1h_1},\alpha_{2h_2}\right)} \left(x^{h_1} y^{h_2} \right) }{h_1!h_2!} \right] \\
&= P_{\overline{m_1},\overline{m_2}} \left[1 - \frac{ \left({}_{k_{20}}^{k_{1h_1}} D_{1-}^{(\alpha_{1h_1},0)} x^{h_1} \right) \left({}_{k_{2h_2}}^{k_{10}} D_{1-}^{\left(0,\alpha_{2h_2}\right)} y^{h_2} \right) }{h_1!h_2!} \right] =: \psi. \qquad (101)
\end{aligned}$$

If $h_1 = h_2 = 0$, then $\alpha_{1h_1} = \alpha_{2h_2} = 0$ and $\psi = 0$.
If $h_1 = 0$ and $h_2 \ne 0$, then

$$\begin{aligned}
\psi &= P_{\overline{m_1},\overline{m_2}} \left[1 - \frac{ {}_{k_{2h_2}}^{k_{10}} D_{1-}^{\left(0,\alpha_{2h_2}\right)} y^{h_2} }{h_2!} \right] \qquad (102) \\
&= P_{\overline{m_1},\overline{m_2}} \left[1 - \int_y^1 k_{2h_2}(t-y) \, \mathrm{d}t \right] = P_{\overline{m_1},\overline{m_2}} \left[1 - \int_0^{1-y} k_{2h_2}(z) \mathrm{d}z \right] \le 0.
\end{aligned}$$

Similarly, we treat the case $h_1 \ne 0$, $h_2 = 0$.

When $h_1, h_2 \ne 0$, then we have

$$\begin{aligned}
\psi &= P_{\overline{m_1},\overline{m_2}} \left[1 - \left(\int_x^1 k_{1h_1}(t-x) \, \mathrm{d}t \right) \left(\int_y^1 k_{2h_2}(t-y) \, \mathrm{d}t \right) \right] \qquad (103) \\
&= P_{\overline{m_1},\overline{m_2}} \left[1 - \left(\int_0^{1-x} k_{1h_1}(z) \mathrm{d}z \right) \left(\int_0^{1-y} k_{2h_2}(z) \mathrm{d}z \right) \right] \le 0.
\end{aligned}$$

So in all four cases we have proved that

$$L^*\left(\overline{Q}_{\overline{m_1},\overline{m_2}}\right)(x,y) \geq 0 \qquad \text{for all } (x,y) \in \Psi. \tag{104}$$ □

Conclusion 13. Theorem 11 generalizes greatly Theorem 9, and Theorem 12 generalizes greatly the main Theorem 2.9 of [**3**]. These generalizations involve abstract fractional kernels that can be singular or non-singular kernels. That is, we cover all kinds of left and right Caputo type fractional calculi of singular and non-singular kernel.

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G. A. Anastassiou, Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, U.S.A.,

e-mail: ganastss@memphis.edu