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MATRIX DIFFERENCE EQUATIONS WITH JUMP CONDITIONS AND HYPERBOLIC EIGENPARAMETER

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ABSTRACT. Problems of difference equations with jump (discontinuity) conditions have an important role for many branches of sciences. They can be used to model a wide range of real-world applications such as heating, massing in physics, bursting rhythm models in medicine, optimal control models in economics, and so on. In this paper, we consider some spectral and scattering properties of matrix difference equations with jump conditions and hyperbolic eigenparameter. Using the asymptotic behavior of Jost function, we find eigenvalues, spectral singularities, resolvent operator, and spectrum of this problem. Also, we investigate scattering matrix and get some properties of scattering matrix. Finally, we present an example about the scattering matrix and the existence of eigenvalues in special cases.

1. INTRODUCTION

The difference equations with jump conditions lie in a special important position in the theory of difference equations. Within this theory, scattering and spectral analysis of these equations is an important tool to investigate the qualitative properties of eigenvalues, spectral singularities, scattering solutions of such equations. These equations involve some discontinuities during many evolution processes. At a certain moment, the state may change abruptly and takes a short time compared to the whole duration. These sudden effects are recognized as instantaneous impulses. The conditions involving impulsive effects are called impulsive conditions or jump conditions. Such conditions are also called transmission conditions, point interaction conditions, interface conditions or interior conditions in literature (see [**1, 13, 21, 22, 23**]). It is well-known that the theory of difference equations with jump points takes form under favor of the theory of the differential equations with jump points or with impulses. Because of this, we refer to the monographs (see [**5, 4, 14, 19, 24**]), for the mathematical theory of such equations. In recent years, spectral and scattering analysis of difference equations with jump conditions has received a lot of attention (see [**2, 3, 7, 8, 9, 10, 11, 12, 15**]), and most of the published works have not been related to matrix form except [**6**]. In this study,

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we investigate spectral and scattering properties of a matrix difference operator $\mathcal{L}$ generated by a matrix difference expression and with jump conditions. These spectral and scattering properties contain Jost solution, scattering solutions, scattering matrix, eigenvalues, spectral singularities, resolvent operator and spectrum. Differently from [**6**], this work consists of hyperbolic eigenparameter and it provides a new perspective of the problem. Because analytical region of Jost solution has changed as a result of hyperbolic eigenparameter and renewed the region of the problem. This new approach provide wide applications in physics, economics, and engineering.

Let us introduce the Hilbert space $l_2\left(\mathbb{N},\mathbb{C}^h\right)$ such that

$$l_2\left(\mathbb{N},\mathbb{C}^h\right) := \left\{Y = \{Y_n\}_{n\in\mathbb{N}},\ Y_n \in \mathbb{C}^h,\ \|Y\|^2 = \sum_{m\in\mathbb{N}} \|Y_n\|^2 < \infty\right\},$$

where $\mathbb{C}^h$ is an h-dimensional ($h < \infty$) Euclidian space, $\|\cdot\|$ denotes the matrix norm in $\mathbb{C}^h$. Further, we denote by $\mathcal{L}$ the operator in $l_2\left(\mathbb{N},\mathbb{C}^h\right)$ by the following matrix expression

$$(ly)_n = Y_{n-1} + B_nY_n + Y_{n+1}, \qquad n \in \mathbb{N} \setminus \{m_0-1, m_0, m_0+1\}, \tag{1}$$

and the boundary condition

$$Y_0 = 0 \tag{2}$$

with the jump conditions

$$Y_{m_0+1} = UY_{m_0-1}, \qquad Y_{m_0+2} = VY_{m_0-2}, \tag{3}$$

where B_n, $n \in \mathbb{N}$ are linear operators (matrices) acting in $\mathbb{C}^h$, and m_0 is an arbitrary natural number. Throughout the paper, we assume that $B := \{B_n\}_{n\in\mathbb{N}}$ is a selfadjoint matrix satisfying

$$\sum_{m\in\mathbb{N}} n\|B_n\| < \infty, \tag{4}$$

and U, V are selfadjoint diagonal matrices in $\mathbb{C}^h$ such that all eigenvalues of them are different and nonzero. Since B is selfadjoint matrix, it is evident that if $Y_n(z)$ is the solution of (1), then $Y_n^T(z)$ also will be a solution of (1), where T shows transpose operator.

Related to the operator $\mathcal{L}$, we consider the matrix difference equation

$$Y_{n-1} + B_nY_n + Y_{n+1} = \mu Y_n, \qquad n \in \mathbb{N} \setminus \{m_0-1, m_0, m_0+1\}, \tag{5}$$

with the boundary condition (2) and jump conditions (3), where $\mu = 2\cosh z$ is a spectral parameter.

The remainder of the manuscript organized as follows: In the second Section, we give some definitions and preliminaries to help us for other sections. In Section 3, we present Jost solution, scattering solutions and scattering matrix of (5). Then, we investigate the properties of them. In Section 4, we find the resolvent operator of $\mathcal{L}$ and examine the properties of eigenvalues, spectral singularities and continuous spectrum of $\mathcal{L}$. Also, we present an asymptotic equation for Jost solution. In Section 5, we are interested in an unperturbated discrete impulsive

Sturm-Liouville equation with hyperbolic eigenparameter as a special example of (1)–(3). We determine the eigenvalues and spectral singularities of this example. Finally, we express some conclusions in Section 6.

2. Preliminaries

In this Section, our goal is to present some basic concepts and definitions concerning the main problem.

Definition 2.1. Wronskian of any two solutions $K = \{K_n(z)\}$ and $M = \{M_n(z)\}$ of (1) is defined as

$$W\left[K, M^T\right](n) := M_{n-1}^T K_n - M_n^T K_{n-1}. \tag{6}$$

In this paper, we consider two semi-strips

$$J = \Big\{z \in \mathbb{C} : \operatorname{Re} z < 0, -\frac{\pi}{2} \leq \operatorname{Im} z \leq \frac{3\pi}{2}\Big\}, \qquad J_0 := J \cup J_1,$$

where $J_1 := \big\{z \in \mathbb{C} : \operatorname{Re} z = 0,\ \operatorname{Im} z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]\big\}$. Throughout the paper, we shortly show the set J_1 by $\left[-\frac{\pi}{2}\mathrm{i}, \frac{3\pi}{2}\mathrm{i}\right]$. Assume that $P(z) = \{P_n(z)\}$ and $Q(z) = \{Q_n(z)\}$ are the fundamental solutions of (5) for $z \in J_0$ and $n = 0, 1, \ldots, m_0 - 1$, satisfying the initial conditions:

$$\begin{aligned} P_0(z) &= 0, \qquad & P_1(z) &= I, \\ Q_0(z) &= I, \qquad & Q_1(z) &= 0. \end{aligned}$$

On the other hand, it is well-known that a bounded solution of equation (5) in $\overline{\mathbb{C}}_- := \{z \in \mathbb{C} : \operatorname{Re} z \leq 0\}$ for $\mu = 2\cosh z$ exists. We will show this bounded solution by $E(z) = \{E_n(z)\}$ in this paper. For $z \in \overline{\mathbb{C}}_-$, solution is represented by

$$E_n(z) = \mathrm{e}^{nz}\Big[I + \sum_{m=1}^{\infty} A_{nm}\mathrm{e}^{mz}\Big] \qquad n = m_0 + 1, m_0 + 2, \ldots$$

where A_{nm} is expressed in terms of $\{B_n\}$ and $E(z)$ is called the Jost solution of the equation (5) in [**17**].

Jost solution is analytic with respect to z in $\mathbb{C}_- := \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$, continuous in $\overline{\mathbb{C}}_-$, and for all z in $\overline{\mathbb{C}}_-$, $E_n(z) = E_n(z + 2\pi)$. Furthermore, $E(z)$ provides the following asymptotic equalities for $z \in \overline{\mathbb{C}}_-$,

$$\begin{aligned} E_n(z) &= \mathrm{e}^{nz}\left[I + o(1)\right], \qquad & n &\to \infty, \\ E_n(z) &= \mathrm{e}^{nz}\left[I + o(1)\right], \qquad & \operatorname{Re} z &\to \infty. \end{aligned} \tag{7}$$

Besides Jost solution, equation (5) has an unbounded solution $\hat{E}_n(z)$, which satisfies the following asymptotic equation

$$\hat{E}_n(z) = \mathrm{e}^{-nz}\left[I + o(1)\right], \qquad z \in \overline{\mathbb{C}}_-,\ n \to \infty. \tag{8}$$

3. Scattering solutions and scattering matrix

In this section, we are interested in equation (5) with the conditions (2) and (3). We shortly call this boundary value problem with jump conditions by BVP.

Firstly, we define the following solution of BVP for $z \in J_0$, by using $P(z)$, $Q(z)$, and $E(z)$:

$$F_n(z) = \begin{cases} P_n(z)D_1(z) + Q_n(z)D_2(z), & n \in \{0, 1, \ldots, m_0 - 1\} \\ E_n(z), & n \in \{m_0 + 1, m_0 + 2, \ldots\}, \end{cases} \tag{9}$$

where D_1 and D_2 are z-dependent coefficients. We can write following equalities by using jump conditions (3)

$$U^{-1}E_{m_0+1}(z) = P_{m_0-1}(z)D_1(z) + Q_{m_0-1}(z)D_2(z), \tag{10}$$

$$V^{-1}E_{m_0+2}(z) = P_{m_0-2}(z)D_1(z) + Q_{m_0-2}(z)D_2(z). \tag{11}$$

By means of (6), it is obvious that $W\left[P(z), P^T(z)\right] = 0$, $W\left[Q(z), Q^T(z)\right] = 0$, and $W\left[P(z), Q^T(z)\right] = I$ for all $z \in \overline{\mathbb{C}}_-$. Then by using (10) and (11), we get the coefficients $D_1(z)$ and $D_2(z)$

$$D_1(z) = U^{-1}V^{-1}\left[VQ^T_{m_0-2}E_{m_0+1}(z) - UQ^T_{m_0-1}E_{m_0+2}(z)\right], \tag{12}$$

$$D_2(z) = U^{-1}V^{-1}\left[UP^T_{m_0-1}E_{m_0+2}(z) - VP^T_{m_0-1}E_{m_0+1}(z)\right], \tag{13}$$

respectively, for $z \in J_0$. The function $F_n(z)$ is called the Jost solution of BVP and by using the boundary condition (2), we obtain the Jost function of BVP by

$$F_0(z) := F(z) = D_2(z).$$

The Jost function F is analytic in $\mathbb{C}_-$ and continuous in $\overline{\mathbb{C}}_-$.

Theorem 3.1. *For all* $z \in \left[-\frac{\pi}{2}\mathrm{i}, \frac{3\pi}{2}\mathrm{i}\right] \smallsetminus \{0, \pi\mathrm{i}\}$, $\det F(z) \neq 0$, *where* $F(z)$ *is the Jost function of BVP given in the last equation.*

Proof. We think the following solution $G(z) = \{G_n(z)\}$ of (5) to get the proof of Theorem 3.1

$$G_n(z) = \begin{cases} P_n(z), & n \in \{0, 1, \ldots, m_0 - 1\} \\ E_n(z)D_3(z) + E_n(-z)D_4(z), & n \in \{m_0 + 1, m_0 + 2, \ldots\} \end{cases}$$

for $z \in \left[-\frac{\pi}{2}\mathrm{i}, \frac{3\pi}{2}\mathrm{i}\right] \smallsetminus \{0, \pi\mathrm{i}\}$. Using the jump conditions (3), we write

$$E_{m_0+1}(z)D_3(z) + E_{m_0+1}(-z)D_4(z) = UP_{m_0-1}(z), \tag{14}$$

$$E_{m_0+2}(z)D_3(z) + E_{m_0+2}(-z)D_4(z) = VP_{m_0-2}(z). \tag{15}$$

By (6), we get easily that

$$W\left[E(z), E^T(z)\right] = 0, \qquad W\left[E(-z), E^T(z)\right] = -2\sinh z.$$

Then using these wronskian equalities in (14) and (15), we obtain

$$D_3(z) = -\frac{1}{2\sinh z}\left[UE^T_{m_0+2}(-z)P_{m_0-1}(z) - VE^T_{m_0+1}(-z)P_{m_0-2}(z)\right], \tag{16}$$

$$D_4(z) = \frac{1}{2\sinh z}\left[UE^T_{m_0+2}(z)P_{m_0-1}(z) - VE^T_{m_0+1}(z)P_{m_0-2}(z)\right] \tag{17}$$

for $z \in \left[-\frac{\pi}{2}\mathrm{i}, \frac{3\pi}{2}\mathrm{i}\right] \setminus \{0, \pi\mathrm{i}\}$. By using (13), (16), and (17), it is clear to show the following relations between coefficients for all $z \in \left[-\frac{\pi}{2}\mathrm{i}, \frac{3\pi}{2}\mathrm{i}\right] \setminus \{0, \pi\mathrm{i}\}$

$$D^T_4(z) = D^T_3(-z) = \frac{1}{2\sinh z}UVD_2(z). \tag{18}$$

To complete the proof of Theorem 3.1, we assume that there exists a point z_0 in $\left[-\frac{\pi}{2}\mathrm{i}, \frac{3\pi}{2}\mathrm{i}\right] \setminus \{0, \pi\mathrm{i}\}$ such that $\det F(z_0) = 0$. From (18), it is evident that $\det D_4(z_0) = \det D_3(z_0) = 0$. Then the solution G is equal to zero identically. It gives a trivial solution of BVP, but it is a contradiction. Finally, we obtain $\det F(z) \neq 0$ for all $z \in \left[-\frac{\pi}{2}\mathrm{i}, \frac{3\pi}{2}\mathrm{i}\right] \setminus \{0, \pi\mathrm{i}\}$. ☐

Theorem 3.1 says that the inverse of the function F exists and the following definition is meaningful.

Definition 3.2. For $z \in \left[-\frac{\pi}{2}\mathrm{i}, \frac{3\pi}{2}\mathrm{i}\right] \setminus \{0, \pi\mathrm{i}\}$, the matrix function

$$S(z) = F^{-1}(z)F(-z)$$

exists and it is called the scattering matrix of BVP.

Theorem 3.3. *The matrix function $S(z)$ is an uniter matrix and for all $z \in \left[-\frac{\pi}{2}\mathrm{i}, \frac{3\pi}{2}\mathrm{i}\right] \setminus \{0, \pi\mathrm{i}\}$, it satisfies $S(-z) = S^{-1}(z) = S^*(z)$, where $*$ denotes the adjoint operator.*

Proof. Using Definition 3.2, we find

$$S(-z) = F^{-1}(-z)F(z), \qquad z \in \left[-\frac{\pi}{2}\mathrm{i}, \frac{3\pi}{2}\mathrm{i}\right] \setminus \{0, \pi\mathrm{i}\}.$$

Last equation helps to get

$$S(z)S(-z) = S(-z)S(z) = I, \qquad z \in \left[-\frac{\pi}{2}\mathrm{i}, \frac{3\pi}{2}\mathrm{i}\right] \setminus \{0, \pi\mathrm{i}\},$$

and it gives $S(-z) = S^{-1}(z)$. Let consider the solutions $F_n(z)$, $F_n(-z)$ and $G_n(z)$ in order to obtain $S^*(z) = S(-z)$. Therefore, we write

$$\begin{aligned} G_n(z) &= F_n(z)\gamma + F_n(-z)\zeta, \\ G_{n+1}(z) &= F_{n+1}(z)\gamma + F_{n+1}(-z)\zeta, \end{aligned} \tag{19}$$

where γ, ζ are matrices not depending on n. By using (19), we easily get

$$\begin{aligned} \gamma &= W^{-1}\left[F(z), F^*(z)\right]\left\{F^*_n(z)G_{n+1}(z) - F^*_{n+1}(z)G_n(z)\right\}, \\ \zeta &= W^{-1}\left[F^*(-z), F(-z)\right]\left\{F^*_n(-z)G_{n+1}(z) - F^*_{n+1}(-z)G_n(z)\right\}. \end{aligned}$$

Because of the characteristic features of equations with jump conditions, we find

$$W^{-1}\left[F(z), F^*(z)\right] = -W^{-1}\left[F(-z), F^*(-z)\right].$$

Letting $n = 0$ in the equation (19), we obtain

$$F(z)F^*(z) = F(-z), F^*(-z). \tag{20}$$

Using Definition 3.2 and (20), we complete the proof of $S^*(z) = S(-z)$. Finally, it is clear that $\|S\| = I$ and $SS^* = S^*S = I$, these equations prove that S is uniter. □

Let us give the following Lemma to investigate the relationship between the wronskians of the scattering solutions of BVP

Lemma 3.4. *For all* $z \in \left[-\frac{\pi}{2}\mathrm{i}, \frac{3\pi}{2}\mathrm{i}\right] \smallsetminus \{0, \pi\mathrm{i}\}$*, the following wronskian holds*

$$W[F(z), G^T(z)](n) = \begin{cases} -D_2(z), & n \in \{0, 1, \dots, m_0 - 1\} \\ UVD_2(z), & n \in \{m_0 + 1, m_0 + 2, \dots\}. \end{cases}$$

Proof. Using (6), we find

$$W\left[F(z), G^T(z)\right](n) = G_0^T(z)F_1(z) - G_1^T(z)F_0(z) \qquad \text{for } n = 0, 1, \dots, m_0 - 1.$$

Since $P_0(z) = 0$, $P_1(z) = I$, $Q_0(z) = I$ and $Q_1(z) = 0$, we obtain

$$W[F(z), G^T(z)](n) = -D_2(z), \qquad n = 0, 1, \dots, m_0 - 1.$$

Similarly, for $n = m_0 + 1, m_0 + 2, \dots$, we get $W[F(z), G^T(z)](n) = 2 \sinh z D_4^T(z)$. Then by using (18), we find

$$W[F(z), G^T(z)](n) = UVD_2(z), \quad n = m_0 + 1, m_0 + 2, \dots$$

It completes the proof. □

4. Resolvent operator, eigenvalues, spectral singularities and continuous spectrum of $\mathcal{L}$

Now, we define another solution of BVP

$$H_n(z) = \begin{cases} P_n(z), & n \in \{0, 1, \dots, m_0 - 1\} \\ E_n(z)D_5(z) + \hat{E}_n(z)D_6(z), & n \in \{m_0 + 1, m_0 + 2, \dots\} \end{cases}$$

for $z \in J_0$. Using the jump conditions (3), we get

$$E_{m_0+1}(z)D_5(z) + \hat{E}_{m_0+1}(z)D_6(z) = UP_{m_0-1}(z),$$
$$E_{m_0+2}(z)D_5(z) + \hat{E}_{m_0+2}(z)D_6(z) = UP_{m_0-2}(z).$$

It can be easily calculated from (6),

$$W\left[E(z), E^T(z)\right] 0, \qquad W\left[\hat{E}(z), \hat{E}^T(z)\right] = 0,$$
$$W\left[\hat{E}(z), E^T(z)\right] = -2\sinh z, \qquad W\left[E(z), \hat{E}^T(z)\right] = 2\sinh z.$$

For $z \in J_0$, the coefficients $D_5(z)$ and $D_6(z)$ are obtained as

$$D_5(z) = -\frac{1}{2\sinh z}\left[U\hat{E}_{m_0+2}(z)P_{m_0-1}(z) - V\hat{E}_{m_0+1}(z)P_{m_0-2}(z)\right]$$

and

$$D_6(z) = \frac{1}{2\sinh z}\left[UE_{m_0+2}^T(z)P_{m_0-1}(z) - VE_{m_0+1}^T(z)P_{m_0-2}(z)\right].$$

Note that

$$D_6^T(z) = \frac{1}{2\sinh z} UVD_2(z), \quad z \in J_0. \tag{21}$$

For $z \in J_0$, the wronskian of the solutions $F(z)$ and $H(z)$ is found as

$$W[F(z), H^T(z)](n) = \begin{cases} -D_2(z), & n \in \{0, 1, \ldots, m_0 - 1\} \\ UVD_2(z), & n \in \{m_0 + 1, m_0 + 2, \ldots\}. \end{cases} \tag{22}$$

Theorem 4.1. *For all $z \in J_0$ and $k, n \neq m_0$, the resolvent operator of $\mathcal{L}$ is defined by*

$$(\mathcal{R}_\mu(\mathcal{L})\psi)_n = \sum_{k=0}^{\infty} \mathcal{G}_{n,k}(z)\psi(k), \qquad \psi := \{\psi_k\} \in l_2\left(\mathbb{N}, \mathbb{C}^h\right),$$

where

$$\mathcal{G}_{n,k} = \begin{cases} F_n(z)\mathcal{A}^{-1}(z)H_k^T(z), & k < n \\ H_n(z)\left[\mathcal{A}^{-1}(z)\right]^T F_k^T(z), & k > n, \end{cases}$$

is the Green function of $\mathcal{L}$ for $k, n \neq m_0$, and $\mathcal{A} := W\left[F(z), H^T(z)\right]$.

Proof. We need to solve the following equation

$$\nabla(\triangle Y_n) + C_n Y_n - \mu Y_n = \psi_n \tag{23}$$

to obtain the Green function, where $C_n = 2I_n + B_n$. Since $F(z)$ and $G(z)$ are linearly independent fundamental solutions of the equation (5), we can write the general solution of (23)

$$Y_n(z) = L_n F_n(z) + T_n H_n(z),$$

where $\{L_n\}_{n\in\mathbb{N}} := L$ and $\{T_n\}_{n\in\mathbb{N}} := T$ are selfadjoint diagonal matrices in $\mathbb{C}^h$. To get the coefficients L and T, we use the method of variation of parameters and obtain them as

$$L_n = L_0 + \sum_{k=1}^{n} \frac{H_k^T(z)\psi_k(z)}{\mathcal{A}(z)}, \qquad T_n = \nu + \sum_{k=n+1}^{\infty} \frac{F_k^T(z)\psi_k(z)}{\mathcal{A}^T(z)},$$

where L_0 and ν are selfadjoint diagonal matrices in $\mathbb{C}^h$. Since $\psi(z) \in l_2\left(\mathbb{N}, \mathbb{C}^h\right)$, ν must be equal to zero. If we use the boundary condition and Theorem 3.1, we also find that L_0 is equal to zero. Finally, we obtain Green function and resolvent operator of $\mathcal{L}$. □

Theorem 4.1 and (22) show that in order to investigate the quantitative properties of $\mathcal{L}$, it is sufficient to find the quantitative properties of zeros of the function $\det F(z)$. Therefore, we define the sets of eigenvalues and spectral singularities of BVP by σ_d and σ_{ss} as

$$\sigma_d(\mathcal{L}) = \{\mu = 2\cosh z : z \in J, \det F(z) = 0\},$$

$$\sigma_{ss}(\mathcal{L}) = \left\{\mu = 2\cosh z : z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \setminus \{0, \pi\},\ \det F(z) = 0\right\},$$ respectively.

Theorem 4.2. *Under the condition* (4), *the function* $F(z)$ *satisfies the following asymptotic equation for* $z \in J_0$

$$F(z) = U^{-1}V^{-1}\left(U - V\right)\left[I + o(1)\right], \qquad |z| \to \infty. \tag{24}$$

Proof. Since the polynomial function $P_n(z)$ is of $(n-1)$. degree according to μ, we clearly find that

$$P_n^T(z)\mathrm{e}^{(n-1)z} = \left[I + o(1)\right], \qquad |z| \to \infty,\ z \in J_0. \tag{25}$$

Since $F(z) = D_2(z)$, the following equation is written by using (13)

$$\begin{aligned} J(z) = U^{-1}V^{-1}[&UP_{m_0-1}^T(z)\mathrm{e}^{(m_0-2)z}\mathrm{e}^{-(m_0-2)z}E_{m_0+2}(z)\mathrm{e}^{-(m_0+2)z}\mathrm{e}^{(m_0+2)z} \\ &- VP_{m_0-2}^T(z)\mathrm{e}^{(m_0-3)z}\mathrm{e}^{-(m_0-3)z}E_{m_0+1}(z)\mathrm{e}^{-(m_0+1)z}\mathrm{e}^{(m_0+1)z}]. \end{aligned} \tag{26}$$

By the help of (7), (25) and (26), we obtain the following asymptotic equation

$$F(z) = \mathrm{e}^{4z}U^{-1}V^{-1}\left(U - V\right)\left[I + o(1)\right], \qquad |z| \to \infty,\ z \in J_0. \qquad \square$$

Theorem 4.3. *Under the condition* (4), $\sigma_c(\mathcal{L}) = [-2, 2]$, *where* $\sigma_c(\mathcal{L})$ *denotes the continuous spectrum of* $\mathcal{L}$.

Proof. We first introduce the difference operators $\mathcal{L}_0$ and $\mathcal{L}_1$ generated by the following difference expressions in $l_2\left(\mathbb{N}, \mathbb{C}^h\right)$ together with (2) and (3)

$$\begin{aligned} \left(\mathcal{L}_0 y\right)_n &= Y_{n-1} + Y_{n+1}, && n \in \mathbb{N} \setminus \{m_0 - 1, m_0 + 1\}, \\ \left(\mathcal{L}_1 y\right)_n &= B_n Y_n, && n \in \mathbb{N} \setminus \{m_0\}, \end{aligned}$$

respectively. It is evident that $\mathcal{L}_0$ is a selfadjoint operator with $\sigma_c(\mathcal{L}_0) = [-2, 2]$ in $l_2\left(\mathbb{N}, \mathbb{C}^h\right)$ [**20**]. On the other hand, under the assumption (4), $\mathcal{L}_1$ is a compact operator [**20**]. By using the Weyl theorem of a compact perturbation [**16**], the continuous spectrum of the operator $\mathcal{L}$ and the continuous spectrum of the selfadjoint operator are the same. So, it completes the proof. $\square$

5. Unperturbated equation with jump conditions

In this section, we handle an unperturbated discrete impulsive Sturm-Liouville equation which is the special case of main problem. We find Jost function and investigate spectral singularities and eigenvalues of this example. It gives a new perspective to understand our main results.

Example 5.1. Let us consider the following unperturbated discrete Sturm-Liouville problem with jump conditions

$$\begin{cases} Y_{n-1} + Y_{n+1} = 2\cosh z Y_n, & n \in \mathbb{N} \setminus \{2, 3, 4\}, \\ Y_0 = 0, \\ Y_4 = UY_2, \quad Y_5 = VY_1, \end{cases} \tag{27}$$

where $m_0 = 3$, U and V are selfadjoint diagonal matrices defined as $U := \left[\alpha_{ij}\right]_{nxn}$ and $V := \left[\beta_{ij}\right]_{nxn}$ in $\mathbb{C}^h$. Differently from BVP, throughout the example, we assume that the matrix B is a zero matrix in $\mathbb{C}^h$.

Then the solution $E_n(z)$ turns into e^{nz} and the fundamental solutions $P_n(z)$ and $Q_n(z)$ of the problem (27) have the following values for $n = 0, 1, 2$:

$$\begin{aligned} P_0(z) &= 0, \qquad P_1(z) = I, \qquad P_2(z) = \mu I, \\ Q_0(z) &= I, \qquad Q_1(z) = 0, \qquad Q_2(z) = -I. \end{aligned}$$

From (13), we get the Jost solution of (27) as follows:

$$F(z) = U^{-1}V^{-1}e^{4z}\left[Ue^{2z} + U - V\right].$$

In order to investigate the eigenvalues and spectral singularities of the problem (27), it is sufficient to find the zeros of the function $\det F(z)$. Since all eigenvalues of U and V are different from zero, $\det F(z) = 0$ if and only if

$$\det \begin{bmatrix} \alpha_{11}e^{2z} + \alpha_{11} - \beta_{11} & 0 & 0 & \dots & 0 \\ 0 & \alpha_{22}e^{2z} + \alpha_{22} - \beta_{22} & 0 & \dots & 0 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & \alpha_{nn}e^{2z} + \alpha_{nn} - \beta_{nn} \end{bmatrix} = 0.$$

So it is clear that

$$\prod_{j=1}^{n} \left(\alpha_{jj}e^{2z} + \alpha_{jj} - \beta_{jj}\right) = 0. \tag{28}$$

The equation (28) shows that for any j integer in $\{1, 2, \dots, n\}$, $\det F(z) = 0$ whenever $\alpha_{jj}e^{2z} + \alpha_{jj} - \beta_{jj} = 0$. By using last equation, we write

$$e^{2z} = \frac{\beta_{jj} - \alpha_{jj}}{\alpha_{jj}} := R_j, \quad z = \ln\sqrt{R_j}.$$

Now, we analyze two special cases to give information about the eigenvalues and spectral singularities of (27).

Case 1.: Let $\alpha_{jj} > \beta_{jj}$ for all j integer in $\{1, 2, \dots, n\}$.

(i) For $\alpha_{jj} > 0$, we get $R_j < 0$. Since z is not defined, the problem (27) does not have any eigenvalue and spectral singularity.

(ii) Assume $\alpha_{jj} < 0$ and $1 < \frac{\beta_{jj}}{\alpha_{jj}} < 2$. Then, we find that $0 < R_j < 1$. Hence the spectral singularity of (27) does not exist but (27) has eigenvalues.

Case 2.: Let $\alpha_{jj} < \beta_{jj}$ for all j integers in $\{1, 2, \dots, n\}$.

(i) Let $\alpha_{jj} > 0$ and $1 < \dfrac{\beta_{jj}}{\alpha_{jj}} < 2$. Similarly to the Case 1 (ii), the problem (27) does not have any spectral singularity but it has eigenvalues.

(ii) For $\alpha_{jj} < 0$, we obtain that $R_j < 0$. Similar to Case 1 (i), there is no spectral singularity and eigenvalue of (27).

6. Conclusion

This paper is the first and important work that studies the scattering solutions of a matrix difference equation with jump conditions and hyperbolic eigenparameter. These solutions help to obtain the scattering matrix of BVP. These solutions and this matrix inform us about the scattering data of this study. Furthermore, we find the resolvent operator, continuous spectrum, and discrete spectrum of this BVP. Finally, we handle an unperturbated equation as an example and apply our main results on it. This study prepares a groundwork for many researchers working on scattering theory.

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