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STRONG CONVERGENCE METHOD FOR A MONOTONE INCLUSION PROBLEM WITH ALTERNATING INERTIAL STEPS

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Abstract. This article proposes a strong convergence of the forward-backward splitting method for a monotone inclusion problem with alternated inertial extrapolation steps in a real Hilbert space. The proposed method converges strongly under some suitable and easy to verify assumptions. The advantage of our iterative scheme is that the single-valued operator is Lipschitz continuous monotone rather than cocoercive and Lipschitz constant does not require to be known. Finally, we give some numerical experiments of the proposed algorithm to demonstrate the advantages of our algorithm over the existing related ones.

1. Introduction

In this paper, our interest is to devise an alternating inertial algorithm to solve the monotone inclusion problem (MIP) in real Hilbert spaces. Our problem is described as follows:

$$\text{find} \quad x^* \in \mathcal{H} \qquad \text{such that } 0 \in (\mathcal{A}+\mathcal{B})x^*, \tag{1.1}$$

where $\mathcal{H}$ is a real Hilbert space with an inner product $\langle \cdot,\cdot \rangle$ and induced norm $\|\cdot\|$, $\mathcal{A}\colon \mathcal{H} \to \mathcal{H}$ is a monotone mapping, and $\mathcal{B}\colon \mathcal{H} \to 2^{\mathcal{H}}$ is a maximal monotone mapping. The solution set of (MIP) is denoted by Ω. It is known that many problems can be converted into the model of (MIP), such as image processing problems, convex minimization problems, split feasibility problems, equilibrium problems, variational inequalities and DC programming problems, see, e.g., [**1, 9, 10, 23, 24, 26, 29, 34**]. Therefore, a large number of researchers have been very interested in this problem and have developed many methods to solve such problems. One of the most famous of these approaches is the forward-backward algorithm (FBA), which generates an iterative sequence $\{x_n\}$ in the following way:

$$x_{n+1} = (I+\lambda_n\mathcal{B})^{-1}(I-\lambda_n\mathcal{A})x_n, \tag{1.2}$$

where stepsize $\lambda_n > 0$, I stands for identity mapping on $\mathcal{H}$, the operator $(I-\lambda_n\mathcal{A})$ is referred to as forward operator, and the operator $(I+\lambda_n\mathcal{B})^{-1}$ is

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the so-called backward operator (also referred to as resolvent operator). The FBA for monotone inclusion problems was first introduced by Lions and Mercier [**16**] (also by Passty [**22**], independently). In the past few decades, the convergence properties and the modified versions of this method have been extensively studied in the literature, see, e.g., [**2, 22, 20**] and references therein. It should be mentioned that the FBA defined by (1.2) requires mapping $\mathcal{A}$ to be inverse strongly monotone. This assumption is very strict and it is difficult to meet the practical problems. In order to avoid this restriction, many scholars have made a lot of efforts and achieved some important results.

1.1. Some efficient methods

In [**32**], Tseng proposed the splitting algorithm (also known as forward-backward-forward method), which is a two-step iterative scheme. More precisely, the form of the algorithm is as follows:

$$\begin{cases} y_n = (I + \lambda_n \mathcal{B})^{-1}(I - \lambda_n \mathcal{A})x_n, \\ x_{n+1} = y_n - \lambda_n(\mathcal{A}y_n - \mathcal{A}x_n), \end{cases} \tag{1.3}$$

where the step size $\{\lambda_n\}$ can be automatically updated by Armijo-type search methods. Whereas the mapping $\mathcal{A}$ is Lipschitz continuous monotone and the mapping $\mathcal{B}$ is maximal monotone, the sequence $\{x_n\}$ formed by iterative process (1.3) converges weakly to a solution of (1.1) in real Hilbert spaces. In 2018, Zhang and Wang [**35**] combined the projection and contraction method, and (1.2), and proposed another iterative scheme to overcome the strong assumption on mapping $\mathcal{A}$. To be more precise, the method is described as follows:

$$\begin{cases} y_n = (I + \lambda_n \mathcal{B})^{-1}(I - \lambda_n \mathcal{A})x_n, \\ x_{n+1} = x_n - \gamma \eta_n d_n, \end{cases} \tag{1.4}$$

where $d_n = x_n - y_n - \lambda_n(\mathcal{A}x_n - \mathcal{A}y_n)$, $\eta_n = \frac{\langle x_n - y_n, d_n \rangle}{\|d_n\|^2}$, $\gamma \in (0, 2)$, $\{\lambda_n\}$ is a control sequence, operator $\mathcal{A}$ is assumed to be Lipschitz continuous monotone, and operator $\mathcal{B}$ is assumed to be maximal monotone. They established the weak convergence of the iterative method (1.4) under some suitable conditions.

It is worth noting that the Tseng splitting method (1.3) and the Algorithm 1.4 are only weakly convergent in infinite-dimensional spaces. Examples in CT reconstruction and machine learning tell us that strong convergence is preferable to weak convergence in an infinite-dimensional space. Therefore, a natural question is how to modify method (1.2) such that it can achieve strong convergence in infinite-dimensional spaces. In fact, in the past few decades, researchers have proposed many modified forward-backward methods to achieve strong convergence in real Hilbert spaces, see, e.g., [**33, 11, 7, 27**] and the references therein. It should be pointed out that the algorithms mentioned in the above literatures also require operator $\mathcal{A}$ to be inverse strongly monotone. In 2018, Gibali and Thong [**12**] proposed two modifications of (1.2) based on Mann and viscosity ideas. They established two strong convergence theorems of the suggested algorithms in an infinite-dimensional Hilbert space. Moreover, Thong and Cholamjiak [**31**], and Gibali et al. [**14**] presented several new algorithms by means of the viscosity-type method and iterative method (1.4), and established the strong convergence theorems of the proposed algorithms in Hilbert spaces.

1.2. Some efficient methods with inertial steps

In recent years, the development of fast iterative algorithms has attracted enormous interest, especially for the inertial method, which is based on discrete version of a second-order dissipative dynamic system. Many researchers have constructed various fast iterative algorithms by using inertial technology, see, e.g., [**17, 25, 13, 8, 28, 30**] and references therein. One of the common features of these algorithms is that the next iteration depends on the combination of the previous two iterations. Note that these minor changes greatly improve the performance of the algorithms. In 2015, Lorenz and Pock [**17**] introduced the following intertial forward-backward algorithm (iFBA) for monotone inclusions:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (I + \lambda_n \mathcal{B})^{-1}(I - \lambda_n \mathcal{A}) w_n. \end{cases} \tag{1.5}$$

Note that the iFBA (1.5) still achieves weak convergence in real Hilbert spaces. Their numerical experiments on image restoration show that iFBA converges faster than some existing algorithms.

Recently, Tan Bing and Sun Young Cho [**3**] proposed the following inertial projection and contraction method for solving the MIP (1.1):

Algorithm 1.1.
The inertial Mann-type projection algorithm for solving (MIP)

Initialization:
Set $\delta > 0$, $\theta > 0$, $l \in (0,1)$, $\mu \in (0,1)$, $\gamma \in (0,2)$, and let $x_0, x_1 \in \mathcal{H}$ be arbitrary.
Iterative Steps:
Calculate x_{n+1} as follows:
Step 1.
Given the iterates x_{n-1} and x_n $(n \geq 1)$. Set $w_n = x_n + \theta_n(x_n - x_{n-1})$, where

$$\theta_n = \begin{cases} \min\left\{\dfrac{\varepsilon_n}{\|x_n - x_{n-1}\|}, \theta\right\} & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{1.6}$$

Step 2.
Compute $y_n = (I + \lambda_n \mathcal{B})^{-1}(I - \lambda_n \mathcal{A}) w_n$, where λ_n is chosen to be the largest $\lambda \in \{\delta, \delta l, \delta l^2, \dots\}$ satisfying the following

$$\lambda\langle \mathcal{A}w_n - \mathcal{A}y_n, w_n - y_n\rangle \leq \mu\|w_n - y_n\|^2. \tag{1.7}$$

If $w_n = y_n$, then stop and y_n is a solution of (MIP). Otherwise, go to *Step 3.*
Step 3.
Compute $z_n = w_n - \gamma\eta_n d_n$, where

$$d_n := w_n - y_n - \lambda_n(\mathcal{A}w_n - \mathcal{A}y_n), \qquad \eta_n := (1-\mu)\frac{\|w_n - y_n\|}{\|d_n\|^2}. \tag{1.8}$$

Step 4.
Compute $x_{n+1} = (1 - \alpha_n - \beta_n)w_n + \beta_n z_n$. Set $n := n+1$ and go back to *Step 1.*

Under some suitable assumptions, it has been shown that the sequence generated by their algorithm converges strongly to the solution of (1.1), with a linesearch for the choice of the single valued operator. Naturally, the following question should come to mind.
Can we design a strong convergence iterative method with an alternated inertial technique with a choice of step size that is independent of the Lipschitz constant and does not involve any linesearch procedure?

Motivated by the monotonic property of the alternated inertial step and the importance of strong convergence property, this article proposes a strong convergence forward-backward splitting method with an alternated inertial technique for solving a monotone inclusion problem in a real Hilbert space. The proposed method converges strongly under simple and easily verifiable assumptions. Moreover, this method can be implemented easily since the singled-valued operator does not need the knowlegde of the Lipschitz constant and does not involve any linesearch procedure. Our method to the best of our knowledge is the only strong convergence forward-backward splitting method with alternated inertial procedure. Additionally, numerical experiments to illustrate the computational performance of the proposed method is given with an application to an image processing problem to test the potential applicability of the method in comparision with some existing methods in the literature.

2. Preliminaries

In this section, we recall some basic notions and useful results in a real Hilbert space $\mathcal{H}$, which are needed for our convergence analysis. For any sequence $\{x_n\} \subset \mathcal{H}$, $\omega_\omega(x_n) := \{z \in \mathcal{H} : \exists\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup z\}$ denotes the weak ω-limit set of $\{x_n\}$. A point $x \in \mathcal{H}$ is called a fixed point of $\mathcal{A}$ if $\mathcal{A}x = x$. The operator $\mathcal{A}$ is said to be:

(i) α-inverse strongly monotone (ism) if there exists $\alpha > 0$ such that
$$\langle \mathcal{A}x - \mathcal{A}y, x - y\rangle \geq \alpha\|\mathcal{A}x - \mathcal{A}y\|^2 \qquad \text{for all } x, y, \in \mathcal{H},$$
(ii) monotone if
$$\langle \mathcal{A}x - \mathcal{A}y, x - y\rangle \geq 0 \qquad \text{for all } x, y \in \mathcal{H},$$
(iii) L-Lipschitz continuous if there exists a constant $L > 0$ such that
$$\|\mathcal{A}x - \mathcal{A}y\| \leq L\|x - y\| \qquad \text{for all } x, y \in \mathcal{H},$$
if $L = 1$, then $\mathcal{A}$ is called nonexpansive.

If $\mathcal{A}$ is a multivalued operator, i.e. $\mathcal{A}\colon \mathcal{H} \to 2^{\mathcal{H}}$, then $\mathcal{A}$ is called monotone if
$$\langle x - y, u - v\rangle \geq 0 \qquad \text{for all } x, y \in \mathcal{H},\ u \in \mathcal{A}(x),\ v \in \mathcal{A}(y),$$
and A is maximal monotone if the graph $G(\mathcal{A})$ of $\mathcal{A}$ defined by
$$G(\mathcal{A}) := \{(x, y) \in \mathcal{H} \times \mathcal{H} : y \in \mathcal{A}(x)\}$$

is not properly contained in the graph of any other monotone operator. It is generally known that $\mathcal{A}$ is maximal monotone if and only if for $(x,u) \in \mathcal{H} \times \mathcal{H}$, $\langle x-y, u-v\rangle \geq 0$ for all $(y,v) \in G(\mathcal{A})$, implies $u \in \mathcal{A}(x)$. The resolvent operator $J_\lambda^{\mathcal{A}}$ is associated with a multivalued operator $\mathcal{A}$ and λ is the mapping $J_\lambda^{\mathcal{A}} \colon \mathcal{H} \to 2^{\mathcal{H}}$ defined by $J_\lambda^{\mathcal{A}}(x) = (I+\lambda\mathcal{A})^{-1}(x),\ x \in \mathcal{H}$, $\lambda > 0$, where I is the identity operator on $\mathcal{H}$. It is well known that if the operator $\mathcal{A}$ is monotone, then $J_\lambda^{\mathcal{A}}$ is single-valued and nonexpansive.

Recall that for a nonempty closed and convex subset C of $\mathcal{H}$, the metric projection denoted as P_C, is a map defined on $\mathcal{H}$ onto C which assigns to each $x \in \mathcal{H}$, the unique point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| = \inf\{\|x-y\| : y \in C\}.$$

Lemma 2.1 ([**21**])**.** *Let C be a closed convex subset of $\mathcal{H}$. Given $\bar{z} \in \mathcal{H}$ and a point $z \in C$, then $z = P_C(\bar{z})$ if and only if*

$$\langle \bar{z} - z, y - z\rangle \leq 0 \qquad \text{for all } y \in C.$$

It is well known, for any $y, z, \bar{z}$ in a real Hilbert space $\mathcal{H}$ and for all $\sigma, \theta, \beta \in [0,1]$ with $\sigma + \theta + \beta = 1$, the following are satisfied

$$\begin{aligned} &\|y+\bar{z}\|^2 \leq \|y\|^2 + 2\langle \bar{z}, y+\bar{z}\rangle, \\ &\|\sigma y + \theta z + \beta \bar{z}\|^2 = \sigma\|y\|^2 + \theta\|z\|^2 + \beta\|\bar{z}\|^2 \\ &\qquad\qquad - \sigma\theta\|y-z\|^2 - \theta\beta\|z-\bar{z}\|^2 - \sigma\beta\|y-\bar{z}\|^2. \end{aligned} \tag{2.1}$$

Lemma 2.2 ([**4**])**.** *Let $\mathcal{H}$ be a real Hilbert space, $\mathcal{A} \colon \mathcal{H} \to \mathcal{H}$ be monotone and Lipschitz continuous operator, and $\mathcal{B} \colon \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator. Then, the operator $(\mathcal{A}+\mathcal{B}) \colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximal monotone.*

Lemma 2.3 ([**18**])**.** *Let $\{\gamma_n\}$, $\{\varepsilon_n\} \subset \mathbb{R}_+$, $\{\eta_n\} \subset (0,1)$, and $\{\kappa_n\}$ is a real sequence such that*

$$\gamma_{n+1} \leq (1-\eta_n)\gamma_n + \kappa_n + \varepsilon_n, \qquad n \geq 1.$$

Assume that $\sum_{n=1}^\infty \varepsilon_n < \infty$. Then the following results hold:

(i) *If $\kappa_n \leq \eta_n L$ for $L > 0$, then $\{\gamma_n\}$ is a bounded sequence.*
(ii) *If we have*

$$\sum_{n=1}^\infty \eta_n = \infty \qquad \text{and} \qquad \limsup_{n\to\infty} \frac{\kappa_n}{\eta_n} \leq 0,$$

then $\gamma_n \to 0$ as $n \to \infty$.

Lemma 2.4 ([**19**])**.** *Let $\{S_n\}$ be a sequence of real numbers that does not decrease at infinity. Also consider the sequence of integers $\{\tau(n)\}_{n\geq n_0}$ defined by*

$$\tau(n) = \max\{m \in \mathbb{N} : m \leq n, S_m \leq S_{m+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ is a decreasing sequence verifying $\lim_{n\to\infty} \tau(n) = \infty$, and for all $n \geq n_0$, the following two estimates hold

$$S_{\tau(n)} \leq S_{\tau(n)+1} \qquad \text{and} \qquad S_n \leq S_{\tau(n)+1}.$$

3. Main result

In this section, we present our method and discuss some of its features. We begin with the following assumptions under which our strong convergence is obtained.

Assumptions 3.1. *Let $\mathcal{H}$ be a real Hilbert space, we assume that the following hold:*

(a) $\mathcal{B}\colon \mathcal{H} \to 2^{\mathcal{H}}$ *is a maximal monotone operator and* $\mathcal{A}\colon \mathcal{H} \to \mathcal{H}$ *is a monotone and Lipschitz continuous operator but the Lipschitz constant need not to be known.*
(b) *The solution set* $\Omega := (\mathcal{A} + \mathcal{B})^{-1}(0)$ *is nonempty.*

Assumptions 3.2. *Suppose that* $\{\alpha_n\}$, $\{\theta_n\}$, *and* $\{\beta_n\}$ *are sequences in* $(0,1)$, *and* $\gamma \in (0,2)$ *satisfy the following conditions:*

(a) $\inf_{n\to\infty} \theta_n(1-\theta_n-\beta_n) > 0$,
(b) $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0$,
(c) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$.

Algorithm 3.3.

Initialization:
Choose the sequences $\{\alpha_n\}$, $\{\theta_n\}$, and $\{\beta_n\}$ such that the conditions from Assumptions 3.2 hold, and let $\lambda_1 > 0$, $\mu \in (0,1)$ and $x_1, x_0 \in \mathcal{H}$.
Iterative Steps:
For x_{n-1} and $x_n \in \mathcal{H}$, choose $\alpha \in [0,1)$ and α_n such that $0 \le \alpha_n \le \bar{\alpha_n}$, where

$$\bar{\alpha_n} := \begin{cases} \min\left\{\dfrac{1}{n^2\|x_n - x_{n-1}\|}, \alpha\right\} & \text{if } x_n \neq x_{n-1}, \\ \alpha, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 1. Compute

$$w_n = \begin{cases} x_n, & n = \text{even}, \\ x_n + \alpha_n(x_n - x_{n-1}), & n = \text{odd}, \end{cases} \tag{3.2}$$

and

$$y_n = J^{\mathcal{B}}_{\lambda_n}(I - \lambda_n\mathcal{A})w_n = (I + \lambda_n\mathcal{B})^{-1}(I - \lambda_n\mathcal{A})w_n, \tag{3.3}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|w_n - y_n\|}{\|\mathcal{A}w_n - \mathcal{A}y_n\|},\ \lambda_n\right\}, & \mathcal{A}w_n \neq \mathcal{A}y_n \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{3.4}$$

Step 2. Compute

$$d_n = w_n - y_n - \lambda_n(\mathcal{A}w_n - \mathcal{B}y_n) \qquad \text{for all } n \ge 1.$$

Step 3. Compute

$$x_{n+1} = (1 - \theta_n - \beta_n)x_n + \theta_n v_n, \tag{3.5}$$

where $v_n = w_n - \gamma\eta_n d_n$ and

$$\eta_n = \begin{cases} \dfrac{\langle w_n - y_n, d_n\rangle}{\|d_n\|^2}, & d_n \neq 0, \\ 0, & d_n = 0. \end{cases} \tag{3.6}$$

Step 4.
Set $n := n + 1$, and go back to *Step 1.*

Remark 3.4. (a) The stepsize given by (3.4) is generated at each iteration by some simple computations, which allow it to be easily implemented without prior knowlegde of the Lipschitz constant of the operator $\mathcal{A}$.

(b) Note that by (3.4), $\lambda_{n+1} \leq \lambda_n$ for all $n \geq 1$. Also, observe in Algorithm 3.3 that if $\mathcal{A}w_n \neq \mathcal{A}y_n$, then

$$\frac{\mu\|w_n - y_n\|}{\|\mathcal{A}w_n - \mathcal{A}y_n\|} \geq \frac{\mu\|w_n - y_n\|}{L\|w_n - y_n\|} = \frac{\mu}{L},$$

which implies that $0 < \min\{\lambda_1, \frac{\mu}{L}\} \leq \lambda_n$ for all $n \geq 1$. This means that $\lim_{n\to\infty} \lambda_n$ exists. Thus, there exists $\lambda > 0$ such that $\lim_{n\to\infty} \lambda_n = \lambda$.

(c) Obviously, from (3.1), we have

$$\alpha_n\|x_n - x_{n-1}\|^2 \leq \bar{\alpha_n}\|x_n - x_{n-1}\|^2 \leq \frac{1}{n^2}.$$

Therefore, it follows that

$$\sum_{n=0}^{\infty} \alpha_n\|x_n - x_{n-1}\|^2 < \infty.$$

(d) The iterates generated by some existing studies in the literature for the case when w_n in (3.2) is computed as $w_n = x_n + \alpha_n(x_n - x_{n-1})$, do not have a monotonic property with respect to a point in the solution. Consequently, it can swing back and forth around the solution set. This could be avoided using the new definition of w_n, which is one of the interesting properties of the alternating inertial method.

4. Convergence analysis

Lemma 4.1. *Let* $\{x_n\}$ *be a sequence generated by* Algorithm 3.3*. Then for each* $p^* \in \Omega$*, the following inequality holds*

$$\|v_{2n+1} - p^*\|^2 \leq \|w_{2n+1} - p^*\|^2 - \frac{2-\gamma}{\gamma}\|v_{2n+1} - w_{2n+1}\|^2.$$

Proof. Set $n = 2n + 1$ and choose $p^* \in \Omega := (\mathcal{A} + \mathcal{B})^{-1}(0)$, then we have

$$\begin{aligned} y_{2n+1} &= (I + \lambda_{2n+1}\mathcal{B})^{-1}(I - \lambda_{2n+1}\mathcal{A})w_{2n+1}, \\ v_{2n+1} &= w_{2n+1} - \gamma\eta_{2n+1}d_{2n+1}, \\ d_{2n+1} &= w_{2n+1} - y_{2n+1} - \lambda_{2n+1}(\mathcal{A}w_{2n+1} - \mathcal{A}y_{2n+1}), \end{aligned}$$

and

$$\eta_{2n+1} = \begin{cases} \dfrac{\langle w_{2n+1} - y_{2n+1}, d_{2n+1}\rangle}{\|d_{2n+1}\|^2}, & d_{2n+1} \neq 0, \\ 0, & d_{2n+1} = 0. \end{cases}$$

So,

$$\begin{aligned} \|v_{2n+1} - p^*\|^2 &= \|w_{2n+1} - p^* - \gamma\eta_{2n+1} d_{2n+1}\|^2 \\ &= \|w_{2n+1} - p^*\|^2 - 2\gamma\eta_{2n+1}\langle w_{2n+1} - p^*, d_{2n+1}\rangle \\ &\quad + \gamma^2\eta_{2n+1}^2\|d_{2n+1}\|^2. \end{aligned} \tag{4.1}$$

Note that

$$\begin{aligned} &\langle w_{2n+1} - p^*, d_{2n+1}\rangle \\ &= \langle w_{2n+1} - y_{2n+1}, d_{2n+1}\rangle + \langle y_{2n+1} - p^*, d_{2n+1}\rangle \\ &= \langle w_{2n+1} - y_{2n+1}, d_{2n+1}\rangle \\ &\quad + \langle y_{2n+1} - p^*, w_{2n+1} - y_{2n+1} - \lambda_{2n+1}(\mathcal{A}w_{2n+1} - \mathcal{A}y_{2n+1})\rangle. \end{aligned} \tag{4.2}$$

Since $y_{2n+1} = (I + \lambda_{2n+1}\mathcal{B})^{-1}(I - \lambda_{2n+1}\mathcal{A})w_{2n+1}$ then $(I - \lambda_{2n+1}\mathcal{A})w_{2n+1} \in (I + \lambda_{2n+1}\mathcal{B})y_{2n+1}$.

Using that $\mathcal{B}$ is maximal monotone, there exists $b_n \in \mathcal{B}y_{2n+1}$ such that

$$(I - \lambda_{2n+1}\mathcal{A})w_{2n+1} = y_{2n+1} + \lambda_{2n+1}b_n,$$

which implies

$$b_n = \frac{1}{\lambda_{2n+1}}(w_{2n+1} - y_{2n+1} - \lambda_{2n+1}\mathcal{A}w_{2n+1}).$$

$p^* \in \Omega$ implies that $0 \in (\mathcal{A} + \mathcal{B})p^*$. Hence, we have

$$\mathcal{A}y_{2n+1} + b_n \in (\mathcal{A} + \mathcal{B})y_{2n+1}.$$

Since $\mathcal{A} + \mathcal{B}$ is maximal monotone, we obtain

$$\langle \mathcal{A}y_{2n+1} + b_n, y_{2n+1} - p^*\rangle \geq 0. \tag{4.3}$$

Replacing $b_n = \frac{1}{\lambda_{2n+1}}(w_{2n+1} - y_{2n+1} - \lambda_{2n+1}\mathcal{A}w_{2n+1})$ in (4.3), we obtain

$$\Big\langle \frac{\lambda_{2n+1}}{\lambda_{2n+1}}\mathcal{A}y_{2n+1} + \frac{1}{\lambda_{2n+1}}(w_{2n+1} - y_{2n+1} - \lambda_{2n+1}\mathcal{A}w_{2n+1}), y_{2n+1} - p^*\Big\rangle \geq 0,$$

$$\frac{1}{\lambda_{2n+1}}\langle w_{2n+1} - y_{2n+1} - \lambda_{2n+1}\mathcal{A}w_{2n+1} + \lambda_{2n+1}\mathcal{A}y_{2n+1}, y_{2n+1} - p^*\rangle \geq 0.$$

This implies that

$$\langle w_{2n+1} - y_{2n+1} - \lambda_{2n+1}\mathcal{A}w_{2n+1} + \lambda_{2n+1}\mathcal{A}y_{2n+1}, y_{2n+1} - p^*\rangle \geq 0.$$

From (4.2), we obtain

$$\langle w_{2n+1} - p^*, d_{2n+1}\rangle \geq \langle w_{2n+1} - y_{2n+1}, d_{2n+1}\rangle. \tag{4.4}$$

Substituting (4.4) in (4.1) we have

$$\begin{aligned} &\|v_{2n+1} - p^*\| \\ &\leq \ \|w_{2n+1} - p^*\|^2 - 2\gamma\eta_{2n+1}\langle w_{2n+1} - y_{2n+1}, d_{2n+1}\rangle \\ &\quad + \gamma^2\eta_{2n+1}^2\|d_{2n+1}\|^2 \end{aligned}$$

$$
\begin{aligned}
&= \|w_{2n+1} - p^*\|^2 - 2\gamma\eta_{2n+1}\langle w_{2n+1} - y_{2n+1}, d_{2n+1}\rangle \\
&\quad + \gamma^2\eta_{2n+1}^2 \frac{\langle w_{2n+1} - y_{2n+1}, d_{2n+1}\rangle}{\eta_{2n+1}} \\
&= \|w_{2n+1} - p^*\|^2 - 2\gamma\eta_{2n+1}\langle w_{2n+1} - y_{2n+1}, d_{2n+1}\rangle \\
&\quad + \gamma^2\eta_{2n+1}\langle w_{2n+1} - y_{2n+1}, d_{2n+1}\rangle \\
&\leq \|w_{2n+1} - p^*\|^2 - \gamma(2-\gamma)\eta_{2n+1}\langle w_{2n+1} - y_{2n+1}, d_{2n+1}\rangle.
\end{aligned} \tag{4.5}
$$

Observe that

$$
\begin{aligned}
\eta_{2n+1}\langle w_{2n+1} - y_{2n+1}, d_{2n+1}\rangle &= \eta_{2n+1}^2\|d_{2n+1}\|^2 \\
&= \|\eta_{2n+1}d_{2n+1}\|^2 = \frac{1}{\gamma^2}\|v_{2n+1} - w_{2n+1}\|^2.
\end{aligned} \tag{4.6}
$$

Hence, from (4.5) and (4.6), we obtain

$$
\|v_{2n+1} - p^*\|^2 \leq \|w_{2n+1} - p^*\|^2 - \Big(\frac{2-\gamma}{\gamma}\Big)\|v_{2n+1} - w_{2n+1}\|^2. \tag{4.7}
$$

□

Theorem 4.2. *Let* $\{x_n\}$ *be a sequence generated by* Algorithm 3.3 *under the* Assumption 3.1 *and* Assumption 3.2. *Then,* $\{x_n\}$ *converges strongly to* $p^* \in \Omega$, *where*

$$
\|p^*\| = \min\{\|z\| : z \in \Omega\}.
$$

For simplicity, we divide the rest of the proof into claims.

Claim 1. We show that

$$
\begin{aligned}
&\|x_{2n+2} - p^*\|^2 \\
&\leq (1-\beta_{2n+1})\|x_{2n+1} - p^*\|^2 + \beta_{2n+1}\Big[\frac{\alpha_{2n+1}}{\beta_{2n+1}}\|x_{2n+1} - x_{2n}\|^2 3D_1(1-\beta_{2n+1}) \\
&\quad + 2\theta_{2n+1}\|x_{2n+1} - v_{2n+1}\|\|x_{2n+2} - p^*\| + 2\langle p^*, p^* - x_{2n+2}\rangle\Big].
\end{aligned} \tag{4.8}
$$

Proof. Notice that from (3.5), we have

$$
\begin{aligned}
x_{n+1} &= (1-\theta_n-\beta_n)x_n + \theta_n v_n \\
x_{2n+2} &= (1-\theta_{2n+1}-\beta_{2n+1})x_{2n+1} + \theta_{2n+1}v_{2n+1} \\
&= (1-\theta_{2n+1})x_{2n+1} + \theta_{2n+1}v_{2n+1} - \beta_{2n+1}x_{2n+1}.
\end{aligned}
$$

Let $\varphi_{2n+1} = (1-\theta_{2n+1})x_{2n+1} + \theta_{2n+1}v_{2n+1}$. Thus,

$$
\begin{aligned}
&\|\varphi_{2n+1} - p^*\|^2 \\
&= \|(1-\theta_{2n+1})x_{2n+1} + \theta_{2n+1}v_{2n+1} - p^*\|^2 \\
&= (1-\theta_{2n+1})^2\|x_{2n+1} - p^*\|^2 + \theta_{2n+1}^2\|v_{2n+1} - p^*\|^2 \\
&\quad + 2(1-\theta_{2n+1})\theta_{2n+1}\langle x_{2n+1} - p^*, v_{2n+1} - p^*\rangle \\
&\leq (1-\theta_{2n+1})^2\|x_{2n+1} - p^*\|^2 + \theta_{2n+1}^2\|v_{2n+1} - p^*\|^2 \\
&\quad + 2(1-\theta_{2n+1})\theta_{2n+1}\|x_{2n+1} - p^*\|\|v_{2n+1} - p^*\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\theta_{2n+1})^2\|x_{2n+1}-p^*\|^2+\theta_{2n+1}^2\|v_{2n+1}-p^*\|^2\\
&\quad +2(1-\theta_{2n+1})\theta_{2n+1}\|x_{2n+1}-p^*\|^2\\
&\quad +(1-\theta_{2n+1})\theta_{2n+1}\|v_{2n+1}-p^*\|^2\\
&=(1-\theta_{2n+1})\|x_{2n+1}-p^*\|^2+\theta_{2n+1}\|v_{2n+1}-p^*\|^2.
\end{aligned}\tag{4.9}
$$

From (4.7) and the last inequality, we get

$$
\|\varphi_{2n+1}-p^*\|^2\leq (1-\theta_{2n+1})\|x_{2n+1}-p^*\|^2+\theta_{2n+1}\|w_{2n+1}-p^*\|^2.\tag{4.10}
$$

On the other hand, we have

$$
\begin{aligned}
&\|w_{2n+1}-p^*\|^2\\
&=\|x_{2n+1}+\alpha_{2n+1}(x_{2n+1}-x_{2n})-p^*\|^2\\
&=\|x_{2n+1}-p^*\|^2+\alpha_{2n+1}^2\|x_{2n+1}-x_{2n}\|^2\\
&\quad +2\alpha_{2n+1}\langle x_{2n+1}-p^*,x_{2n+1}-x_{2n}\rangle\\
&\leq \|x_{2n+1}-p^*\|^2+\alpha_{2n+1}^2\|x_{2n+1}-x_{2n}\|^2\\
&\quad +2\alpha_{2n+1}\|x_{2n+1}-p^*\|\|x_{2n+1}-x_{2n}\|\\
&\leq \|x_{2n+1}-p^*\|^2+\alpha_{2n+1}\|x_{2n+1}-x_{2n}\|^2\\
&\quad +2\alpha_{2n+1}\|x_{2n+1}-p^*\|\|x_{2n+1}-x_{2n}\|\\
&\leq \|x_{2n+1}-p^*\|^2+3D_1\alpha_{2n+1}\|x_{2n+1}-x_{2n}\|^2,
\end{aligned}\tag{4.11}
$$

where $D_1=\sup_{n\geq 1}\{\|x_{2n+1}-p^*\|,\|x_{2n+1}-x_{2n}\|\}$. Substitute (4.11) in (4.10), we get

$$
\begin{aligned}
\|\varphi_{2n+1}-p^*\|^2&\leq (1-\theta_{2n+1})\|x_{2n+1}-p^*\|^2+\theta_{2n+1}\|x_{2n+1}-p^*\|^2\\
&\quad +3D_1\alpha_{2n+1}\theta_{2n+1}\|x_{2n+1}-x_{2n}\|^2\\
&=\|x_{2n+1}-p^*\|^2+3D_1\alpha_{2n+1}\theta_{2n+1}\|x_{2n+1}-x_{2n}\|^2.
\end{aligned}\tag{4.12}
$$

Since $\varphi_{2n+1}=(1-\theta_{2n+1})x_{2n+1}+\theta_{2n+1}v_{2n+1}$, we have

$$
x_{2n+1}-\varphi_{2n+1}=\theta_{2n+1}(x_{2n+1}-v_{2n+1}).
$$

Therefore, it follows that

$$
\begin{aligned}
x_{2n+2}&=\varphi_{2n+1}-\beta_{2n+1}x_{2n+1}\\
&=(1-\beta_{2n+1})\varphi_{2n+1}-\beta_{2n+1}(x_{2n+1}-\varphi_{2n+1})\\
&=(1-\beta_{2n+1})\varphi_{2n+1}-\beta_{2n+1}\theta_{2n+1}(x_{2n+1}-v_{2n+1}).
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&\|x_{2n+2}-p^*\|^2\\
&=\|(1-\beta_{2n+1})\varphi_{2n+1}-\beta_{2n+1}\theta_{2n+1}(x_{2n+1}-v_{2n+1})-p^*\|^2\\
&=\|(1-\beta_{2n+1})(\varphi_{2n+1}-p^*)\\
&\quad -(\beta_{2n+1}\theta_{2n+1}(x_{2n+1}-v_{2n+1})+\beta_{2n+1}p^*)\|^2\\
&\leq (1-\beta_{2n+1})^2\|\varphi_{2n+1}-p^*\|^2\\
&\quad -2\langle\beta_{2n+1}\theta_{2n+1}(x_{2n+1}-v_{2n+1})+\beta_{2n+1}p^*,x_{2n+2}-p^*\rangle
\end{aligned}
$$

$$\leq (1-\beta_{2n+1})\|\varphi_{2n+1}-p^*\|^2 + 2\langle \beta_{2n+1}\theta_{2n+1}(x_{2n+1}-v_{2n+1}), p^*-x_{2n+2}\rangle + 2\beta_{2n+1}\langle p^*, p^*-x_{2n+2}\rangle. \tag{4.13}$$

Now, substituting (4.12) in (4.13), we obtain

$$\begin{aligned}
&\|x_{2n+2}-p^*\|^2\\
&= \|(1-\beta_{2n+1})\|x_{2n+1}-p^*\|^2 + 3D_1\alpha_{2n+1}\theta_{2n+1}(1-\beta_{2n+1})\|x_{2n+1}-x_{2n}\|^2\\
&\quad + 2\langle \beta_{2n+1}\theta_{2n+1}(x_{2n+1}-v_{2n+1}), p^*-x_{2n+2}\rangle + 2\beta_{2n+1}\langle p^*, p^*-x_{2n+2}\rangle\\
&\leq (1-\beta_{2n+1})\|x_{2n+1}-p^*\|^2 + 3D_1\alpha_{2n+1}\theta_{2n+1}(1-\beta_{2n+1})\|x_{2n+1}-x_{2n}\|^2\\
&\quad + 2\beta_{2n+1}\theta_{2n+1}\|x_{2n+1}-v_{2n+1}\|\|x_{2n+1}-p^*\| + 2\beta_{2n+1}\langle p^*, p^*-x_{2n+2}\rangle\\
&\leq (1-\beta_{2n+1})\|x_{2n+1}-p^*\|^2 + \beta_{2n+1}\Big[\frac{\alpha_{2n+1}}{\beta_{2n+1}}\|x_{2n+1}-x_{2n}\|^2 3D_1(1-\beta_{2n+1})\\
&\quad + 2\theta_{2n+1}\|x_{2n+1}-v_{2n+1}\|\|x_{2n+2}-p^*\| + 2\langle p^*, p^*-x_{2n+2}\rangle\Big].
\end{aligned}$$

<u>*Claim 2:*</u> We show that the sequence $\{x_n\}$ is bounded. From the definition of x_{n+1}, we obtain

$$\begin{aligned}
&\|x_{2n+2}-p^*\|\\
&= \|(1-\theta_{2n+1}-\beta_{2n+1})x_{2n+1} + \theta_{2n+1}v_{2n+1} - p^*\|\\
&= \|(1-\theta_{2n+1}-\beta_{2n+1})(x_{2n+1}-p^*) + \theta_{2n+1}(v_{2n+1}-p^*) + \beta_{2n+1}(-p^*)\|\\
&\leq (1-\theta_{2n+1}-\beta_{2n+1})\|x_{2n+p}-p^*\| + \theta_{2n+1}\|v_{2n+1}-p^*\| + \beta_{2n+1}\|p^*\|.
\end{aligned} \tag{4.14}$$

Now, observe that

$$\begin{aligned}
\|w_{2n+1}-p^*\| &= \|x_{2n+1} + \alpha_{2n+1}(x_{2n+1}-x_{2n}) - p^*\|\\
&\leq \|x_{2n+1}-p^*\| + \alpha_{2n+1}\|x_{2n+1}-x_{2n}\|\\
&= \|x_{2n+1}-p^*\| + \beta_{2n+1}\Big(\frac{\alpha_{2n+1}}{\beta_{2n+1}}\|x_{2n+1}-x_{2n}\|\Big).
\end{aligned} \tag{4.15}$$

Since $\frac{\alpha_n}{\beta_n}\|x_{2n+1}-x_{2n}\| \to 0, n\to\infty$, this implies that for all $n \geq 1$, there exists $M_1 > 0$ such that $\frac{\alpha_{2n+1}}{\beta_{2n+1}}\|x_{2n+1}-x_{2n}\| \leq M_1$. Therefore, from equation (4.15), we get

$$\|w_{2n+1}-p^*\| \leq \|x_{2n+1}-p^*\| + \beta_{2n+1}M_1. \tag{4.16}$$

Now, putting (4.16) together with (4.7), we have

$$\|v_{2n+1}-p^*\| \leq \|w_{2n+1}-p^*\| \leq \|x_{2n+1}-p^*\| + \beta_{2n+1}M_1. \tag{4.17}$$

Substituting (4.17) in (4.14), we have

$$\|x_{2n+2}-p^*\| \leq (1-\beta_{2n+1})\|x_{2n+1}-p^*\| + \beta_{2n+1}(\theta_{2n+1}M_1 + \|p^*\|). \tag{4.18}$$

Since $\lim\limits_{n\to\infty}\beta_n = 0$, this implies that the sequence $\{\beta_{2n+1}\}_{n=0}^{\infty}$ is bounded. Setting $M := \max\{\|p\|, \theta_{2n+1}M_1\}$ and using Lemma 2.3(i), we conclude that the sequence $\{\|x_{2n+1}-p^*\|\}_{n=0}^{\infty}$ is bounded. Using similar argument in obtaining (4.18), it can easily be seen that the corresponding sequence $\{\|x_{2n}-p^*\|\}_{n=0}^{\infty}$ of the even terms is bounded as well. Consequently, the sequence $\{x_n\}_{n=0}^{\infty}$ is bounded.

Claim 3: We show that

$$\begin{aligned}&\|x_{2n+2}-p^*\|^2\\&\le(1-\beta_{2n+1})\|x_{2n}-p^*\|^2+[(1-\beta_{2n+1}+\theta_{2n+1}\alpha_{2n+1})\beta_{2n}+\beta_{2n+1}]\,\|p^*\|^2\\&\quad+2\theta_{2n+1}\alpha_{2n+1}\|x_{2n+1}-x_{2n}\|^2-(1-\theta_{2n}-\beta_{2n})\theta_{2n}\|v_{2n}-x_{2n}\|^2\\&\quad-\theta_{2n}\frac{2-\gamma}{\gamma}\|v_{2n}-w_{2n}\|^2.\end{aligned}$$

From the definition of w_{2n+1} and (2.2), we have

$$\begin{aligned}&\|w_{2n+1}-p^*\|^2\\&=\|x_{2n+1}+\alpha_{2n+1}(x_{2n+1}-x_{2n})-p^*\|^2\\&=\|(1+\alpha_{2n+1})(x_{2n+1}-p^*)-\alpha_{2n+1}(x_{2n}-p)\|^2\\&=(1+\alpha_{2n+1})\|x_{2n+1}-p^*\|^2\\&\quad-\alpha_{2n+1}\|x_{2n}-p^*\|^2+\alpha_{2n+1}(1+\alpha_{2n+1})\|x_{2n+1}-x_{2n}\|^2\\&\le(1+\alpha_{2n+1})\|x_{2n+1}-p^*\|^2-\alpha_{2n+1}\|x_{2n}-p^*\|^2\\&\quad+2\alpha_{2n+1}\|x_{2n+1}-x_{2n}\|^2.\end{aligned}\tag{4.19}$$

Using again (2.1) and the definition of x_{2n+2}, we have

(4.20)

$$\begin{aligned}&\|x_{2n+2}-p^*\|^2\\&=\|(1-\theta_{2n+1}-\beta_{2n+1})x_{2n+1}+\theta_{2n+1}v_{2n+1}-p^*\|^2\\&=\|(1-\theta_{2n+1}-\beta_{2n+1})(x_{2n+1}-p^*)+\theta_{2n+1}(v_{2n+1}-p^*)-\beta_{2n+1}(-p^*)\|^2\\&\le(1-\theta_{2n+1}-\beta_{2n+1})\|x_{2n+1}-p^*\|^2+\theta_{2n+1}\|v_{2n+1}-p^*\|^2+\beta_{2n+1}\|p^*\|^2\\&\quad-(1-\theta_{2n+1}-\beta_{2n+1})\theta_{2n+1}\|v_{2n+1}-x_{2n+1}\|^2.\end{aligned}$$

Substituting (4.7) in (4.20), we get

(4.21)

$$\begin{aligned}\|x_{2n+2}-p^*\|^2\le&(1-\theta_{2n+1}-\beta_{2n+1})\|x_{2n+1}-p^*\|^2+\theta_{2n+1}\|w_{2n+1}-p^*\|^2\\&-\theta_{2n+1}\frac{2-\gamma}{\gamma}\|v_{2n+1}-w_{2n+1}\|^2+\beta_{2n+1}\|p^*\|^2\\&-(1-\theta_{2n+1}-\beta_{2n+1})\theta_{2n+1}\|v_{2n+1}-x_{2n+1}\|^2.\end{aligned}$$

Substituting (4.19) in (4.21), we obtain

$$\begin{aligned}&\|x_{2n+2}-p^*\|^2\\&\le(1-\theta_{2n+1}-\beta_{2n+1})\|x_{2n+1}\\&\quad-p^*\|^2+\theta_{2n+1}[(1+\alpha_{2n+1})\|x_{2n+1}-p^*\|^2-\alpha_{2n+1}\|x_{2n}-p^*\|\\&\quad+2\alpha_{2n+1}\|x_{2n+1}-x_{2n}\|^2]\\&\quad-\theta_{2n+1}\frac{2-\gamma}{\gamma}\|v_{2n+1}-w_{2n+1}\|^2+\beta_{2n+1}\|p^*\|^2\\&\quad-(1-\theta_{2n+1}-\beta_{2n+1})\theta_{2n+1}\|v_{2n+1}-x_{2n+1}\|^2\end{aligned}$$

$$
\begin{aligned}
&\leq [(1-\theta_{2n+1}-\beta_{2n+1})+\theta_{2n+1}(1+\alpha_{2n+1})]\,\|x_{2n+1}-p^*\|^2 \\
&\quad -\theta_{2n+1}\alpha_{2n+1}\|x_{2n}-p^*\|^2+\beta_{2n+1}\|p^*\|^2 \\
&\quad +2\theta_{2n+1}\alpha_{2n+1}\|x_{2n+1}-x_{2n}\|^2 \\
&\quad -\theta_{2n+1}\frac{2-\gamma}{\gamma}\|v_{2n+1}-w_{2n+1}\|^2 \\
&\quad -(1-\theta_{2n+1}-\beta_{2n+1})\theta_{2n+1}\|v_{2n+1}-x_{2n+1}\|^2.
\end{aligned}
\tag{4.22}
$$

Using similar arguments in showing (4.22), one obtains

$$
\begin{aligned}
&\|x_{2n+1}-p^*\|^2 \\
&\leq (1-\theta_{2n}-\beta_{2n})\|x_{2n}-p^*\|^2+\theta_{2n}\|w_{2n}-p^*\|^2+\beta_{2n}\|p^*\|^2 \\
&\quad -\theta_{2n}\frac{(2-\gamma)}{\gamma}\|v_{2n}-w_{2n}\|^2-(1-\theta_{2n}-\beta_{2n})\theta_{2n}\|v_{2n}-x_{2n}\|^2 \\
&= (1-\theta_{2n}-\beta_{2n})\|x_{2n}-p^*\|^2+\theta_{2n}\|x_{2n}-p^*\|^2+\beta_{2n}\|p^*\|^2 \\
&\quad -\theta_{2n}\frac{(2-\gamma)}{\gamma}\|v_{2n}-w_{2n}\|^2-(1-\theta_{2n}-\beta_{2n})\theta_{2n}\|v_{2n}-x_{2n}\|^2 \\
&\leq (1-\beta_{2n})\|x_{2n}-p^*\|^2+\beta_{2n}\|p^*\|^2+(1-\theta_{2n}-\beta_{2n})\|v_{2n}-x_{2n}\|^2 \\
&\quad -\theta_{2n}\frac{(2-\gamma)}{\gamma}\|v_{2n}-w_{2n}\|^2.
\end{aligned}
\tag{4.23}
$$

Substituting (4.23) in (4.22), we get

$$
\begin{aligned}
&\|x_{2n+2}-p^*\|^2 \\
&\leq [(1-\theta_{2n+1}-\theta_{2n+1})+\theta_{2n+1}(1+\theta_{2n+1})]\Big[(1-\beta_{2n})\|x_{2n}-p^*\|^2+\beta_{2n}\|p^*\|^2 \\
&\quad -(1-\theta_{2n}-\beta_{2n})\theta_{2n}\|v_{2n}-x_{2n}\|^2-\theta_{2n}\frac{(2-\gamma)}{\gamma}\|v_{2n}-w_{2n}\|^2\Big] \\
&\quad -\theta_{2n+1}\alpha_{2n+1}\|x_{2n}-p^*\|^2+\beta_{2n+1}\|p^*\|^2+2\theta_{2n+1}\alpha_{2n+1}\|x_{2n+1}-x_{2n}\|^2 \\
&\quad -\theta_{2n+1}\frac{(2-\gamma)}{\gamma}\|v_{2n+1}-w_{2n+1}\|^2 \\
&\quad -(1-\theta_{2n+1}-\beta_{2n+1})\theta_{2n+1}\|v_{2n+1}-x_{2n+1}\|^2 \\
&= [((1-\theta_{2n+1}-\beta_{2n+1})+\theta_{2n+1}(1+\alpha_{2n+1}))(1-\beta_{2n})-\theta_{2n+1}\alpha_{2n+1}]\,\|x_{2n}-p^*\|^2 \\
&\quad +[((1-\theta_{2n+1}-\beta_{2n+1})+\theta_{2n+1}(1+\alpha_{2n+1}))\beta_{2n}+\beta_{2n+1}]\,\|p^*\|^2 \\
&\quad +2\theta_{2n+1}\alpha_{2n+1}\|x_{2n+1}-x_{2n}\|^2 \\
&\quad -(1-\theta_{2n}-\beta_{2n})\theta_{2n}((1-\theta_{2n+1}-\beta_{2n+1})+\theta_{2n+1}(1+\alpha_{2n+1}))\|v_{2n}-x_{2n}\|^2 \\
&\quad -2((1-\theta_{2n+1}-\beta_{2n+1})+\theta_{2n+1}(1+\alpha_{2n+1}))\theta_{2n}\frac{(2-\gamma)}{\gamma}\|v_{2n}-w_{2n}\|^2 \\
&\quad -\theta_{2n+1}\frac{(2-\gamma)}{\gamma}\|v_{2n+1}-w_{2n+1}\|^2 \\
&\quad -(1-\theta_{2n+1}-\beta_{2n+1})\theta_{2n+1}\|v_{2n+1}-x_{2n+1}\|^2.
\end{aligned}
\tag{4.24}
$$

Now observe that considering the given conditions of the sequences $\{\beta_n\}$, $\{\theta_n\}$, and $\{\alpha_n\}$, one can obtain

$$\begin{aligned}
&((1-\theta_{2n+1}-\beta_{2n+1})+\theta_{2n+1}(1+\alpha_{2n+1}))(1-\beta_{2n})-\theta_{2n+1}\alpha_{2n+1}\\
&=(1-\beta_{2n+1}+\theta_{2n+1}\alpha_{2n+1})(1-\beta_{2n})-\theta_{2n+1}\alpha_{2n+1}\\
&\leq(1-\beta_{2n+1}+\theta_{2n+1}\alpha_{2n+1})-\theta_{2n+1}\alpha_{2n+1}\\
&=(1-\beta_{2n+1}),
\end{aligned} \tag{4.25}$$

and

$$\begin{aligned}
&[(1-\theta_{2n+1}-\beta_{2n+1})+\theta_{2n+1}(1+\beta_{2n+1})]\,\beta_{2n}+\beta_{2n+1}\\
&=(1-\beta_{2n+1}+\theta_{2n+1}\alpha_{2n+1})\beta_{2n}+\beta_{2n+1}.
\end{aligned} \tag{4.26}$$

From (4.25), (4.26), and dropping some of the non neqative terms in (4.24) now it follows

$$\begin{aligned}
&\|x_{2n+2}-p^*\|^2\\
&\leq(1-\beta_{2n+1})\|x_{2n}-p^*\|^2+[(1-\beta_{2n+1}+\theta_{2n+1}\alpha_{2n+1})\beta_{2n}+\beta_{2n+1}]\,\|p^*\|^2\\
&\quad+2\theta_{2n+1}\alpha_{2n+1}\|x_{2n+1}-x_{2n}\|^2-(1-\theta_{2n}-\beta_{2n})\theta_{2n}\|v_{2n}-x_{2n}\|^2\\
&\quad-\theta_{2n}\frac{2-\gamma}{\gamma}\|v_{2n}-w_{2n}\|^2.
\end{aligned} \tag{4.27}$$

Claim 4. We show that $\{x_n\}$ converges strongly. We consider two cases to show the convergence. Observe that from (4.27), it follows

$$\begin{aligned}
&(1-\theta_{2n}-\beta_{2n})\theta_{2n}\|v_{2n}-x_{2n}\|^2+\theta_{2n}\frac{(2-\gamma)}{\gamma}\|v_{2n}-w_{2n}\|^2\\
&\leq(1-\beta_{2n+1})\|x_{2n}-p^*\|^2-\|x_{2n+2}-p^*\|^2\\
&\quad+[(1-\beta_{2n+1}+\theta_{2n+1}\alpha_{2n+1})\beta_{2n}+\beta_{2n+1}]\,\|p^*\|^2+2\theta_{2n+1}\alpha_{2n+1}\|x_{2n+1}-x_{2n}\|^2\\
&\leq\|x_{2n}-p^*\|^2-\|x_{2n+2}-p^*\|^2\\
&\quad+[(1-\beta_{2n+1}+\theta_{2n+1}\alpha_{2n+1})\beta_{2n}+\beta_{2n+1}]\,\|p^*\|^2+2\theta_{2n+1}\alpha_{2n+1}\|x_{2n+1}-x_{2n}\|^2.
\end{aligned} \tag{4.28}$$

Case 1. Suppose that there exists $\mathcal{N}\geq 0$ such that $\|x_{2n+1}-p^*\|\leq\|x_{2n}-p^*\|$ for all $n\geq 0$. In this case, $\lim_{n\to\infty}\|x_{2n}-p^*\|$ exists. Since $\lim_{n\to\infty}\beta_n=0$ and $\lim_{n\to\infty}\alpha_n\|x_n-x_{n-1}\|=0$, it now follows form (4.28) that

$$\lim_{n\to\infty}(1-\theta_{2n}-\beta_{2n})\theta_{2n}\|v_{2n}-x_{2n}\|^2=0$$

and

$$\lim_{n\to\infty}\|v_{2n}-w_{2n}\|=0. \tag{4.29}$$

From the assumption that $\inf_{n\geq 1}(1-\theta_{2n}-\theta_{2n})\theta_{2n}>0$, we obtain

$$\lim_{n\to\infty}\|v_{2n}-x_{2n}\|=0. \tag{4.30}$$

Also from (4.29) and (4.30), we get

$$\lim_{n\to\infty}\|w_{2n}-x_{2n}\|=0. \tag{4.31}$$

From Algorithm 3.3 and the definition of λ_{n+1}, we get

$$\begin{aligned}\|d_{2n}\| &= \|w_{2n} - y_{2n} - \lambda_{2n}(\mathcal{A}w_{2n} - \mathcal{B}y_{2n})\|\\ &\leq \|w_{2n} - y_{2n}\| + \lambda_{2n}\|\mathcal{A}w_{2n} - \mathcal{B}y_{2n}\| \leq \Big(1 + \frac{\mu\lambda_{2n}}{\lambda_{2n+1}}\Big)\|w_{2n} - y_{2n}\|.\end{aligned}$$

So,

$$\frac{1}{\|d_{2n}\|} \geq \frac{1}{\Big(1 + \frac{\mu\lambda_{2n}}{\lambda_{2n+1}}\Big)\|w_{2n} - y_{2n}\|}.$$

Now

$$\begin{aligned}\langle w_{2n} - y_{2n}, d_{2n}\rangle &= \langle w_{2n} - y_{2n}, w_{2n} - y_{2n} - \lambda_{2n}(\mathcal{A}w_{2n} - \mathcal{A}y_{2n})\rangle\\ &= \|w_{2n} - y_{2n}\|^2 - \langle w_{2n} - y_{2n}, \lambda_{2n}(\mathcal{A}w_{2n} - \mathcal{A}y_{2n})\rangle\\ &\geq \|w_{2n} - y_{2n}\|^2 - \lambda_{2n}\|\mathcal{A}w_{2n} - \mathcal{A}y_{2n}\|\|w_{2n} - y_{2n}\|\\ &\geq \|w_{2n} - y_{2n}\| - \frac{\mu\lambda_{2n}}{\lambda_{2n+1}}\|w_{2n} - y_{2n}\|^2\\ &= \Big(1 - \frac{\mu\lambda_{2n}}{\lambda_{2n+1}}\Big)\|w_{2n} - y_{2n}\|.\end{aligned}$$

From the definition of v_n, we get

(4.32)

$$\|v_{2n} - w_{2n}\| = \gamma\eta_{2n}\|d_{2n}\| = \gamma\frac{\langle w_{2n} - y_{2n}, d_{2n}\rangle}{\|d_{2n}\|} \geq \gamma\left[\frac{1 - \frac{\mu\lambda_{2n}}{\lambda_{2n+1}}}{1 + \frac{\mu\lambda_{2n}}{\lambda_{2n+1}}}\right]\|w_{2n} - y_{2n}\|.$$

Hence by (4.29), we get from (4.32) (noting that $\lim_{n\to\infty}\lambda_n = \lambda$) that

$$\lim_{n\to\infty}\|w_{2n} - y_{2n}\| = 0. \tag{4.33}$$

Also,

$$\lim_{n\to\infty}\|x_{2n} - y_{2n}\| \leq \lim_{n\to\infty}[\|x_{2n} - w_{2n}\| + \|w_{2n} - y_{2n}\|] = 0. \tag{4.34}$$

From (4.27), (4.29), (4.30), and the fact that $\lim_{n\to\infty}\beta_n = 0$, it follows that

$$\lim_{n\to\infty}\|x_{2n+1} - x_{2n}\| = 0. \tag{4.35}$$

Now, we show that $\omega_\omega(x_n) \subset \Omega$. Suppose that $z \in \omega_\omega(x_n)$ is an arbitrary element. Let the subsequence $\{x_{2n_k}\}$ of $\{x_{2n}\}$ be weakly convergent to a point z. Then, it follows that the subsequences $\{w_{2n_k}\}$ and $\{y_{2n_k}\}$ are also weakly convergent to $z \in \mathcal{H}$. Now, let $(v, u) \in G(\mathcal{A} + \mathcal{B})$, this implies that $u - \mathcal{A}v \in \mathcal{B}v$. Also, from (3.3), we obtain

$$\frac{1}{\lambda_{2n_k}}(w_{2n_k} - y_{2n_k} - \lambda_{2n_k}\mathcal{A}w_{2n_k}) \in \mathcal{B}y_{2n_k}.$$

Using maximal monotonicity of $\mathcal{B}$, we have

$$\Big\langle v - y_{2n_k}, u - \mathcal{A}v - \frac{1}{\lambda_{2n_k}}(w_{2n_k} - y_{2n_k} - \lambda_{2n_k}\mathcal{A}w_{2n_k})\Big\rangle \geq 0. \tag{4.36}$$

Using (4.36) and the monotonicity of $\mathcal{A}$, we obtain
(4.37)

$$\begin{aligned}\langle v - y_{2n_k}, \rangle &\geq \left\langle v - y_{2n_k}, \mathcal{A}v + \frac{1}{\lambda_{2n_k}}(w_{2n_k} - y_{2n_k} - \lambda_{2n_k} A w_{2n_k})\right\rangle \\ &= \langle v - y_{2n_k}, \mathcal{A}v - \mathcal{A}w_{2n_k}\rangle + \left\langle v - y_{2n_k}, \frac{1}{\lambda_{2n_k}}(w_{2n_k} - y_{2n_k})\right\rangle \\ &= \langle v - y_{2n_k}, \mathcal{A}v - \mathcal{A}y_{2n_k}\rangle + \langle v - y_{2n_k}, \mathcal{A}y_{2n_k} - \mathcal{A}w_{2n_k}\rangle \\ &\quad + \left\langle v - y_{2n_k}, \frac{1}{\lambda_{2n_k}}(w_{2n_k} - y_{2n_k})\right\rangle \\ &\geq \langle v - y_{2n_k}, \mathcal{A}y_{2n_k} - \mathcal{A}w_{2n_k}\rangle + \left\langle v - y_{2n_k}, \frac{1}{\lambda_{2n_k}}(w_{2n_k} - y_{2n_k})\right\rangle.\end{aligned}$$

Recall that $\lim_{k\to\infty} \lambda_{2n_k} > 0$, $\lim_{k\to\infty} \|w_{2n_k} - y_{2n_k}\| = 0$, and by the Lipschitz continuity $\mathcal{A}$, we obtain

$$\lim_{k\to\infty} \|\mathcal{A}y_{2n_k} - \mathcal{A}w_{2n_k}\| = 0.$$

Using $\|w_{2n_k} - y_{2n_k}\| \to 0$ as $k \to \infty$, which implies that $y_{2n_k} \to z$ as $k \to \infty$, we obtain

$$\langle v - z, u\rangle = \lim_{k\to\infty} \langle v - y_{2n_k}, u\rangle \geq 0.$$

Hence,

$$\langle v - v, u - 0\rangle \geq 0.$$

Also, by Lemma 2.2, $\mathcal{A}+\mathcal{B}$ is maximal monotone, thus we obtain that $0 \in (\mathcal{A}+\mathcal{B})z$ which implies that $z \in \Omega$. Since z is arbitrary, we conclude that $\omega_\omega(x_n) \subset \Omega$.

From *Claim 1*, we have
(4.38)

$$\begin{aligned}&\|x_{2n+2} - p^*\|^2 \\ &\leq (1 - \beta_{2n+1})\|x_{2n+1} - p^*\|^2 + \beta_{2n+1}\Big[\frac{\alpha_{2n+1}}{\beta_{2n+1}}\|x_{2n+1} - x_{2n}\|^2 3D_1(1 - \beta_{2n+1}) \\ &\quad + 2\theta_{2n+1}\|x_{2n+1} - v_{2n+1}\|\|x_{2n+2} - p^*\| + 2\langle p^*, p^* - x_{2n+2}\rangle\Big].\end{aligned}$$

Using similar argument to obtain (4.9), one gets

$$\begin{aligned}\|\varphi_{2n} - p^*\|^2 &\leq (1 - \theta_{2n})\|x_{2n} - p^*\|^2 + \theta_{2n}\|w_{2n} - p^*\|^2 \\ &= (1 - \theta_{2n})\|x_{2n} - p^*\|^2 + \theta_{2n}\|x_{2n} - p^*\|^2 \quad = \|x_{2n} - p^*\|^2.\end{aligned} \tag{4.39}$$

Again, using a similar argument as in the proof of *Claim 1* to obtain (4.13), one obtains
(4.40)

$$\begin{aligned}\|x_{2n+1} - p^*\|^2 \leq{}& (1 - \beta_{2n})\|\varphi_{2n} - p^*\|^2 + 2\beta_{2n}\theta_{2n}\|x_{2n} - v_{2n}\|\|x_{2n+1} - p^*\| \\ &+ 2\beta_{2n}\langle p^*, p^* - x_{2n+1}\rangle.\end{aligned}$$

Substituting (4.39) in (4.40), we have

$$\begin{aligned}\|x_{2n+1} - p^*\|^2 \leq (1 - \beta_{2n})\|\varphi_{2n} - p^*\|^2 + \beta_{2n}[2\|x_{2n} - v_{2n}\|\|x_{2n+1} - p^*\| \\ + 2\langle p^*, p^* - x_{2n+1}\rangle].\end{aligned} \tag{4.41}$$

It remains only to show that

$$\limsup_{n\to\infty}\langle p^*, p^* - x_{2n+1}\rangle \le 0.$$

In fact, since $p^* = P_\Omega(0)$, by the characterization of the metric space projection in Lemma 2.1, we get

$$\limsup_{n\to\infty}\langle p^*, p^* - x_{2n+1}\rangle = \max_{z\in\omega_\omega(x_n)}\langle p^*, p^* - z\rangle \le 0. \tag{4.42}$$

Now from, (4.30), (4.41), (4.42), and Lemma 2.3, it follows that the sequence $\{x_{2n}\}$ converges strongly to $p^* = P_\Omega(0)$.

Case 2: Suppose that there exists a subsequence $\{\|x_{2n_m} - p^*\|\}_{m=0}^{\infty} \subset \{\|x_{2n} - p^*\|\}_{n=0}^{\infty}$ such that $\|x_{2n_m} - p^*\| \le \|x_{2n_m+1} - p^*\|$ for all $m \ge 1$. For this case, we define $\tau\colon \mathbb{N}\to\mathbb{N}$ by

$$\tau(n) := \max\{k \le n : \|x_{2k} - p^*\| \le \|x_{2k+1} - p^*\|\}.$$

Then, we have from Lemma 2.4 that $\tau(n)\to\infty$ as $n\to\infty$ and $\|x_{\tau(n)} - p^*\| \le \|x_{\tau(n)+1} - p^*\|$, so that we have from (4.27),

(4.43)

$$\begin{aligned}
&(1-\theta_{2\tau(n)} - \beta_{2\tau(n)})\theta_{2\tau(n)}\|v_{2\tau(n)} - x_{2\tau(n)}\|^2 + \theta_{2\tau(n)}\frac{2-\gamma}{\gamma}\|v_{2\tau(n)} - w_{2\tau(n)}\|^2\\
&\le \left(1-\beta_{2\tau(n)+1}\right)\|x_{2\tau(n)} - p^*\|^2 - \|x_{2\tau(n)+2} - p^*\|^2\\
&\quad + \left[\left(1-\beta_{2\tau(n)+1} + \theta_{2\tau(n)+1}\alpha_{2\tau(n)+1}\right)\beta_{2\tau(n)} + \beta_{2\tau(n)+1}\right]\|p^*\|^2\\
&\quad + 2\theta_{2\tau(n)+1}\alpha_{2\tau(n)+1}\|x_{2\tau(n)+1} - x_{2\tau(n)}\|^2.
\end{aligned}$$

Following the same lines of proof as in *Case 1*, we infer from (4.43) that

$$\lim_{n\to\infty}\|v_{2\tau(n)} - x_{2\tau(n)}\|^2 = 0, \tag{4.44}$$

$$\lim_{n\to\infty}\|v_{2\tau(n)} - w_{2\tau(n)}\|^2 = 0, \tag{4.45}$$

$$\limsup_{n\to\infty}\langle p^*, p^* - x_{2\tau(n)+1}\rangle = \max_{z\in\omega_\omega(x_{\tau(n)})}\langle p^*, p^* - z\rangle \le 0, \tag{4.46}$$

and

$$\begin{aligned}
\|x_{2\tau(n)+1} - p^*\|^2 &\le (1-\beta_{2\tau(n)})\|x_{2\tau(n)} - p^*\|^2\\
&\quad + \beta_{2\tau(n)}\Big[2\|x_{2\tau(n)} - v_{2\tau(n)}\|\|x_{2\tau(n)+1} - p^*\|\\
&\quad + \langle p^*, p^* - x_{2\tau(n)+1}\rangle\Big].
\end{aligned} \tag{4.47}$$

Since $\|x_{\tau(n)} - p^*\| \le \|x_{\tau(n)+1} - p^*\|$, we have from (4.47) that

$$\|x_{2\tau(n)} - p^*\|^{\le}2\|x_{2\tau(n)} - v_{2\tau(n)}\|\|x_{2\tau(n)+1} - p^*\| + 2\langle p^*, p^* - x_{2\tau(n)+1}\rangle. \tag{4.48}$$

Combininig (4.44), (4.46), and (4.48) yields

$$\limsup_{n\to\infty}\|x_{2\tau(n)} - p^*\|^2 \le 0,$$

and hence,

$$\lim_{n\to\infty}\|x_{2\tau(n)} - p^*\|^2 = 0.$$

From (4.47), we have

$$\limsup_{n\to\infty} \|x_{2\tau(n)+1} - p^*\|^2 \le \limsup_{n\to\infty} \|x_{2\tau(n)} - p^*\|^2.$$

Thus

$$\lim_{n\to\infty} \|x_{2\tau(n)+1} - p^*\|^2 = 0.$$

Therefore, by Lemma 2.4, we obtain

$$0 \le \|x_{2\tau(n)} - p^*\| \le \max\left\{\|x_{2\tau(n)} - p^*\|, \|x_{2n} - p^*\|\right\} \le \|x_{2\tau(n)+1} - p^*\| \to 0.$$

Consequently, the sequence $\{x_{2n}\}$ converges strongly to $p^* = P_\Omega(0)$.

Now, we show the convergence of the sequence of odd terms $\{x_{2n+1}\}$. Note that by *Case 1,* since $\lim_{n\to\infty} \|x_{2n} - p^*\|$ exists and $\lim_{m\to\infty} \|x_{2n_m} - p^*\| = 0$, we get that $\lim_{n\to\infty} \|x_{2n} - p^*\| = 0$. Therefore, p^* is unique.

It follows from (4.35), (4.39), the condition on θ_{2n+1}, and α_{2n+1} that

$$\lim_{n\to\infty} (1 - \beta_{2n+1}) \frac{\alpha_{2n+1}}{\beta_{2n+1}} \|x_{2n+1} - x_{2n}\| = 0.$$

Using similar arguments in obtaining (4.35), we have

$$\lim_{n\to\infty} \|x_{2n+1} - v_{2n+1}\| = 0.$$

To apply Lemma 2.3(ii), we use similar arguments as in proof of *Case 1* to show that

$$\limsup_{n\to\infty} \langle p^*, p^* - x_{2n+1}\rangle = \max_{z\in\omega_\omega(x_n)} \langle p^*, p^* - z\rangle \le 0.$$

Hence, it follows from Lemma 2.3 that the sequence $\{x_{2n+1}\}$ converges strongly to $p^* = P_\Omega(0)$, which is the minimum norm solution of the Inclusion Problem.

For second case, we suppose that there exists a subsequence $\{\|x_{2n_m} - p^*\|\}_{m=0}^{\infty} \subset \{\|x_{2n} - p^*\|\}_{n=0}^{\infty}$ such that $\|x_{2n_m} - p^*\| \le \|x_{2n_m+1} - p^*\|$ for all $m \ge 1$. Following similar argument as in the proof of *Case 2* above, it can be shown that the sequence of odd terms $\{x_{2n+1}\}$ converges strongly to $p^* = P_\Omega(0)$. Thus, the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to $p^* = P_\Omega(0)$, and hence, the proof is complete. □

5. Applications

5.1. Application to image processing problems

Using known information from the contaminated signal/image to estimate the original and clean signal/image is called the signal processing/image restoration problem. This kind of problem can usually be expressed as the following linear inverse problem:

$$\mathbf{b} = \mathbf{C}\mathbf{x} + \mathbf{w},$$

where $\mathbf{C}$, $\mathbf{x}$, $\mathbf{b}$, and $\mathbf{w}$ represent degradation operator, unknown real image, contaminated image, and noise function, respectively. Regularization methods have aroused considerable interest in many researchers for dealing with such problems. In particular, the l_1 regularization method considers finding the solution to the following problem:

$$\min_x \left\{ \frac{1}{2}\|\mathbf{Cx}-\mathbf{b}\|^2 + \gamma\|\mathbf{x}\|_{\mathbf{1}} \right\},$$

where γ stands for the regularization parameter, and $\|\mathbf{x}\|_1$ represents the sum of the absolute values of the components of $\mathbf{x}$. Set $h(\mathbf{x}) = \frac{1}{2}\|\mathbf{Cx}-\mathbf{b}\|^2$ and $g(\mathbf{x}) = \|\mathbf{x}\|_1$, then $\nabla h(\mathbf{x}) = \mathbf{C}^*(\mathbf{Cx}-\mathbf{b})$ and thus it is a Lipschitz continuous with constant $L(h) = \|\mathbf{C}^*\mathbf{C}\|$. The proximal map of $g(\mathbf{x}) = \gamma\|\mathbf{x}\|_1$ is expressed as $\mathrm{prox}_{\lambda g}(\mathbf{x}) = (I+\lambda\partial g)^{-1}$ and it can be calculated by the following:

$$\begin{aligned}\mathrm{prox}_{\lambda g}(\mathbf{x}) &= \mathrm{prox}_{\lambda\gamma\|\cdot\|_1}(\mathbf{x}) = \Big(\mathrm{prox}_{\lambda\gamma|\cdot|_1}(x_1), \dots, \mathrm{prox}_{\lambda\gamma|\cdot|_1}(x_n)\Big)\\ &= (p_1, \dots, p_n),\end{aligned}$$

where $p_k = \mathrm{sgn}(x_k)\max\{|x_k| - \lambda\gamma, 0\}$ for $1, 2, \dots n$. Set $\mathcal{A} = \nabla h$ and $\mathcal{B} = \partial g$, then we immediately get the following result by Theorem 4.2.

Corollary 5.1. *Let the mappings $\mathcal{A}$, and $\mathcal{B}$ be defined above. Suppose that $\Omega \neq \emptyset$, and Assumption 3.2 holds. Let $\{x_n\}$ be a sequence generated by*

Algorithm 5.2.

Initialization:
Choose the sequences $\{\alpha_n\}$, $\{\theta_n\}$, and $\{\beta_n\}$ such that the conditions from Assumptions 3.2 hold and let $\lambda_1 > 0$, $\mu \in (0,1)$, and $x_1, x_0 \in \mathcal{H}$.
Iterative Steps:
For x_{n-1} and $x_n \in \mathcal{H}$, choose $\alpha \in [0,1)$ and α_n such that $0 \le \alpha_n \le \bar{\alpha_n}$, where

$$\bar{\alpha_n} := \begin{cases} \min\left\{\dfrac{1}{n^2\|x_n - x_{n-1}\|}, \alpha\right\} & \text{if } x_n \neq x_{n-1}, \\ \alpha, & \text{otherwise.} \end{cases} \tag{5.1}$$

Step 1. Compute

$$w_n = \begin{cases} x_n, & n = \text{even}, \\ x_n + \alpha_n(x_n - x_{n-1}), & n = \text{odd}, \end{cases} \tag{5.2}$$

and

$$y_n = \mathrm{prox}_{\lambda_n g}(I - \lambda_n \nabla h)w_n, \tag{5.3}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|w_n - y_n\|}{\|\nabla h(w_n) - \nabla h(y_n)\|},\ \lambda_n\right\}, & \nabla h(w_n) \neq \nabla h(y_n), \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{5.4}$$

Step 2. Compute

$$d_n = w_n - y_n - \lambda_n(\nabla h(w_n \nabla h(y_n)) \qquad \text{for all } n \ge 1.$$

Step 3. Compute

$$x_{n+1} = (1 - \theta_n - \beta_n)x_n + \theta_n v_n, \tag{5.5}$$

where $v_n = w_n - \gamma\eta_n d_n$.
Update

$$\eta_n = \begin{cases} \dfrac{\langle w_n - y_n, d_n\rangle}{\|d_n\|^2}, & d_n \neq 0, \\ 0, & d_n = 0. \end{cases} \tag{5.6}$$

Step 4.
Set $n := n+1$, and go back to *Step 1.*

Then, the iterative sequence $\{x_n\}$ presented above converges strongly to $p^* \in \Omega$, where

$$\|p^*\| = \min\{\|z\| : z \in \Omega\}.$$

5.2. Application to split feasibility problems

Suppose that $\mathcal{C}$ and $\mathcal{Q}$ are nonempty closed convex subsets of real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The split feasibility problem (SFP) is described as follows:

$$\text{find } x^* \in \mathcal{C} \ \text{ such that } \ Tx^* \in \mathcal{Q}, \tag{5.7}$$

where $T\colon \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator. We also use Υ to represent the solution set of (SFP) (5.7). Problem (5.7) appears in image reconstruction and signal processing. From an optimization point of view, $x^* \in \Upsilon$ if and only if x^* is a solution of the following minimization problem with zero optimal value:

$$\min_{x\in C} h(x) := \frac{1}{2}\|Tx - P_{\mathcal{Q}}Tx\|^2.$$

It should be noted that h is convex difference. Moreover, note that $\nabla h(x) = T^*(I - P_{\mathcal{Q}})Tx$ and it is $\|T\|^2$-Lipschitz contiunuous monotone. Thus, x^* solves SFP (5.7) if and only if x^* solves the following variational inclusion problem:

$$\text{find } x \in \mathcal{H}_1 \ \text{ such that } \ 0 \in \nabla h(x) + \partial\delta_{\mathcal{C}}(x),$$

where $\delta_{\mathcal{C}}$ is the indicator function of C. In Theorem (4.2), choosing $\mathcal{A} = \nabla h$ and $\mathcal{B} = \partial\delta_{\mathcal{C}}$, the we obtain following result.

Corollary 5.3. *Let the mappings $\mathcal{A}$ and $\mathcal{B}$ be defined above. Suppose that $\Upsilon \neq \emptyset$ and the Assumption 3.2 hold. Let $\{x_n\}$ be a sequence generated by*

Algorithm 5.4.

Initialization: Choose the sequences $\{\alpha_n\}$, $\{\theta_n\}$, and $\{\beta_n\}$ such that the conditions from Assumptions 3.2 hold and let $\lambda_1 > 0$, $\mu \in (0,1)$, and $x_1, x_0 \in \mathcal{H}$.
Iterative Steps:
For x_{n-1} and $x_n \in \mathcal{H}$, choose $\alpha \in [0,1)$ and α_n such that $0 \le \alpha_n \le \bar{\alpha_n}$ where

$$\bar{\alpha_n} := \begin{cases} \min\left\{\dfrac{1}{n^2\|x_n - x_{n-1}\|}, \alpha\right\} & \text{if } x_n \neq x_{n-1}, \\ \alpha, & \text{otherwise.} \end{cases} \tag{5.8}$$

Step 1. Compute

$$w_n = \begin{cases} x_n, & n = \text{even}, \\ x_n + \alpha_n(x_n - x_{n-1}), & n = \text{odd}, \end{cases} \tag{5.9}$$

and

$$y_n = P_{\mathcal{C}}(I - \lambda_n \nabla h)w_n, \tag{5.10}$$

where

$$\lambda_{n+1} = \begin{cases} \min\Big\{\dfrac{\mu\|w_n - y_n\|}{\|\nabla h(w_n) - \nabla h(y_n)\|}, \lambda_n\Big\}, & \nabla h(w_n) \neq \nabla h(y_n), \\ \lambda_n, & \text{otherwise}. \end{cases} \tag{5.11}$$

Step 2. Compute

$$d_n = w_n - y_n - \lambda_n(\nabla h(w_n) - \nabla h(y_n)) \qquad \text{for all } n \geq 1.$$

Step 3. Compute

$$x_{n+1} = (1 - \theta_n - \beta_n)x_n + \theta_n v_n, \tag{5.12}$$

where $v_n = w_n - \gamma\eta_n d_n$.
Update

$$\eta_n = \begin{cases} \dfrac{\langle w_n - y_n, d_n\rangle}{\|d_n\|^2}, & d_n \neq 0, \\ 0, & d_n = 0. \end{cases} \tag{5.13}$$

Step 4.
Set $n := n + 1$, and go back to *Step 1.*

Then the iterative sequence $\{x_n\}$ presented above converges strongly to $p^* \in \Upsilon$, where

$$\|p^*\| = \min\{\|z\| : z \in \Upsilon\}.$$

6. Numerical example

In this section, we provide some numerical examples occurring in infinite dimensional spaces to show the advantages of our algorithm and compare them with some known strongly convergent algorithms, including Tan Bing and Sun Young Cho Algorithm 5.4 and Duong Viet Thong and Prasit Cholamjiak of [**31**] (Algorithm 3.1). For easy referencing, we term Algorithm 1.1, and Algorithm 3.1 of [**31**] as SUN and THONG, respectively. Numerical experiments were carried out on MATLAB R2015a version. All programs ran on a 64-bit OS PC with Intel(R) Core(TM) i7-3540M CPU @ 1.00GHz 1.19 GHz and 3GB RAM. All figures were plotted using the $\log\log$ plot command.

Example 6.1. Let $\mathcal{H} = L^2([0,1])$ be endowed with inner product

$$\langle x, y\rangle = \int_0^1 x(t)y(t)\mathrm{d}t \quad \text{and} \quad \|x\| := \Big(\int_0^1 |x(t)|^2\mathrm{d}t\Big)^{1/2} \quad \text{for all } x, y \in L_2([0,1]).$$

Now, define the operator $A, B\colon L_2([0,1]) \to L_2([0,1])$ by

$$\mathcal{A}x(t) = \int_0^1 \Big(x(t) - \Big(\frac{2tse^{t+s}}{e\sqrt{e^2-1}}\Big)\cos x(s)\Big)ds + \frac{2te^t}{e\sqrt{e^2-1}}, \qquad x \in L_2([0,1]),$$
$$\mathcal{B}x(t) = \max\{0, x(t)\}, \qquad t \in [0,1].$$

Then $\mathcal{A}$ is Lipschitz continuous and monotone, and $\mathcal{B}$ is maximal monotone on $L_2([0,1])$ (see [**15**]). The integrals were approximated using the *trapz* and *int* command on MATLAB over the interval $[0,1]$. The results of the experiment are displayed in Table 1 and Figures 1, 2, and 3.

Table 1. Computational Results for Example 6.1.

Algorithms	No. of Iterations	γ	$\|x_{101} - x_{100}\|_2$	Time(secs)
Algorithm 3.1	$1.0e+02$	$2.5e-01$	$1.1671e-04$	71.4678
Thong			$4.5432e-04$	74.2397
Sun			$1.1566e-04$	75.6076
Algorithm 3.1	$1.0e+02$	$1.0e-01$	$1.3548e-04$	92.1468
Thong			$4.7427e-04$	96.7780
Sun			$2.2921e-04$	101.1242
Algorithm 3.1	$1.0e+02$	$1.0e-02$	$1.6913e-04$	109.0433
Thong			$4.8065e-04$	119.2423
Sun			$1.6927e-04$	124.0408

Table 2. Computational Results for Example 6.2.

Algorithms	No. of Iterations	γ	$\|x_{101} - x_{100}\|_2$	Time(secs)
Algorithm 3.1	$1.0e+02$	$2.5e-01$	$3.2715e-06$	$9.2404e-02$
Thong			$5.3333e-04$	$6.5388e-02$
Sun			$6.1917e-04$	$7.0700e-02$
Algorithm 3.1	$1.0e+02$	$1.0e-01$	$5.5460e-05$	$8.5640e-02$
Thong			$5.0468e-04$	$6.8222e-02$
Sun			$1.8265e-04$	$7.3701e-02$
Algorithm 3.1	$1.0e+02$	$1.0e-02$	$1.7413e-04$	$9.5250e-02$
Thong			$4.8400e-04$	$6.7798e-02$
Sun			$2.1891e-04$	$7.2760e-02$

Example 6.2. In this example, we explore the proposed methods to solve the split feasibility problem (SFP) in infinite-dimensional Hilbert spaces. For any $x, y \in L^2([0,1])$, we consider $\mathcal{H}_1 = \mathcal{H}_2 = L^2([0,1])$ embedded with the inner product $\langle x, y\rangle : \int_0^1 x(t)y(t)dt$ and the induced norm $\|x\| := \Big(\int_0^1 |x(t)|^2 dt\Big)^{\frac{1}{2}}$. Consider the following nonempty closed and convex subsets $\mathcal{C}$ and $\mathcal{Q}$ in $L^2([0,1])$,

$$\mathcal{C} = \Big\{x \in L^2([0,1]) : \int_0^1 x(t)dt \le 1\Big\},$$
$$\mathcal{Q} = \Big\{x \in L^2([0,1]) : \int_0^1 |x(t) - \sin(t)|^2 dt \le 16\Big\}.$$

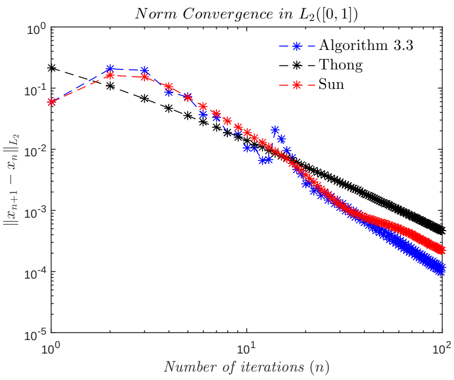

Figure 1. Example 6.1 with $\gamma = 0.25$.

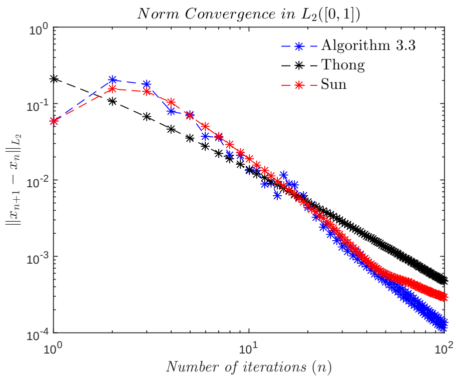

Figure 2. Example 6.1 with $\gamma = 0.1$.

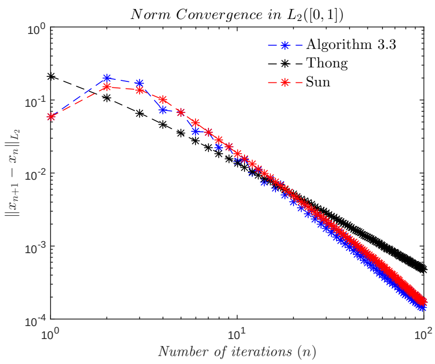

Figure 3. Example 6.1 with $\gamma = 0.01$.

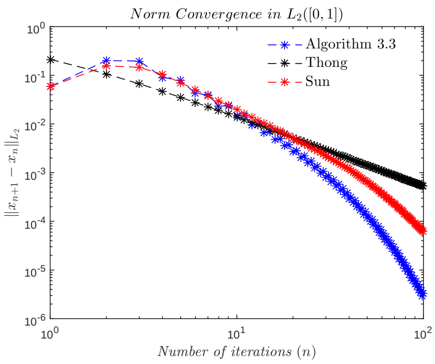

Figure 4. Example 6.2 with $\gamma = 0.25$.

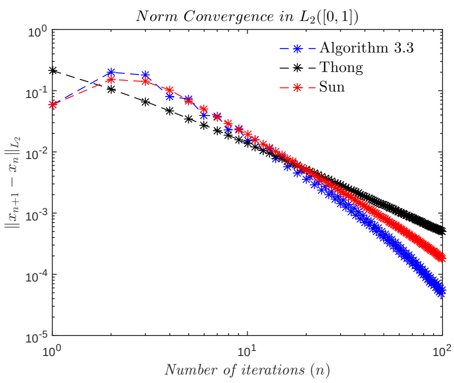

Figure 5. Example 6.2 with $\gamma = 0.1$.

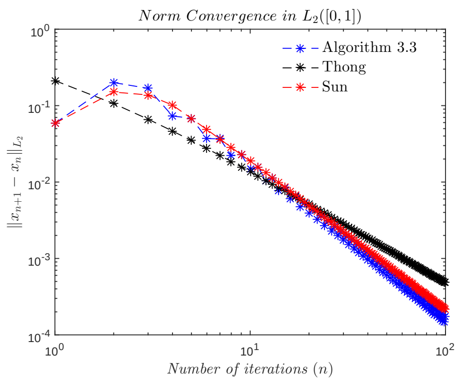

Figure 6. Example 6.2 with $\gamma = 0.01$.

Let $T\colon L^2([0,1]) \to L^2([0,1])$ be the Volterra integration operator, which is given by $(Tx)(t) = \int_0^t x(s)\mathrm{d}s$ for all $t \in [0,1]$, $x \in \mathcal{H}$. Then T is a bounded linear operator (see [**5**, Exercise 20.16]) and its operator norm is $\|T\| = \frac{2}{\pi}$. Moreover, the adjoint T^* of T is defined by $(T^*x)(t) = \int_t^1 x(s)\mathrm{d}s$. Note that $x(t) = 0$ is a solution of SFP (5.7), and thus the solution set of the problems is nonempty. On the other hand, it is known that projections on set $\mathcal{C}$ and $\mathcal{Q}$ have display formulas, that is,

$$P_{\mathcal{C}}(x) = \begin{cases} 1 - a + x, & a > 1, \\ x, & a \le 1, \end{cases} \quad \text{and} \quad P_{\mathcal{Q}}(x) = \begin{cases} \sin(\cdot) + \frac{4(x - \sin(\cdot))}{\sqrt{b}}, & b > 16, \\ x, & b \le 16, \end{cases}$$

where $a := \int_0^1 x(t)\mathrm{d}t$ and $b := \int_0^1 |x(t) - \sin(t)|\mathrm{d}t$.

Remark 6.3. From the results displayed in Tables 1 and 2, it is clear that the fastness of the convergence of all the three algorithms heavily depends on the value of γ. For instance, as the value of γ gets smaller, the time taken to reach the specified tolerences or the number of iteration increases, likewise, the slower it converges. In all cases, Algorithm 3.1 seems to perform better than its counterparts. Finally, μ is taken to be 0.5 throughout the experiment.

7. Conclusion

A strong convergence alternated inertial iterative method for solving monotone inclusion problems has been studied in this research. Incorporation of an alternated inertial extrapolation step in the method has shown a remarkable performance in terms of speed and CPU time of the proposed method, in comparison with some existing iterative methods in the literature. The stepsize is chosen self adaptively in such a way that prior information of the Lipschitz constant of the operator is not needed during implementation. Numerical experiments presented have shown that the method is easy to implement and the results indicate that the method is efficient and robust.

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