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ON THE ENESTRÖM-KAKEYA THEOREM

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Abstract. In this paper, we obtain some refinements of a well-known result of Eneström-Kakeya concerning the bounds for the moduli of the zeros of polynomials with complex coefficients which among other things are also improved upon some known results in this direction.

1. Introduction

If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n with real coefficients satisfying

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0,$$

then according to a well-known result of Eneström-Kakeya (see [**9, 10**]) all the zeros of $P(z)$ lie in $|z|\leq 1$.

We may apply this result to $P(tz)$ to obtain the following, more general result.

Theorem A. *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n with real coefficients satisfying*

$$t^n a_n \geq t^{n-1}a_{n-1} \geq \cdots \geq ta_1 \geq a_0 > 0$$

for some $t>0$, then all the zeros of $P(z)$ lie in $|z|\leq t$.

In literature [**1, 2, 3, 4, 5, 6, 7, 8, 9, 10**] there exit several extensions of Eneström-Kakeya theorem. A. Aziz and Q. G. Mohammad [**3**] used Schwarz's Lemma and proved among other things the following generalization of Theorem A.

Theorem B. *Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with real and positive coefficients. If $t_1>t_2\geq 0$ can be found such that*

$$t_1t_2a_r+(t_1-t_2)a_{r-1}-a_{r-2}\geq 0 \qquad \text{for} \quad r=1,2,\dots,n+1, \quad (a_{-1}=a_{n+1}=0),$$

then all the zeros of $P(z)$ lie in $|z|\leq t_1$.

For the polynomials with complex coefficients, A. Aziz and Q. G. Mohammad [**4**] also proved the following generalization of Theorem A.

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Theorem C. *Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with complex coefficients such that for $k=0,1,2,\dots,n$ and for some $t>0$,*

$$t^n|a_n|\le t^{n-1}|a_{n-1}|\le\cdots\le t^k|a_k|\ge t^{k-1}|a_{k-1}|\ge\cdots\ge t|a_1|\ge|a_0|,$$

then $P(z)$ has all its zeros in the circle

$$|z|\le t\left\{\frac{2t^k|a_k|}{t^n|a_n|}-1\right\}+2\sum_{j=0}^{n}\frac{\big|a_j-|a_j|\big|}{|a_n|t_1^{n-j-1}}. \tag{1}$$

Rather et. al. [**11**] extended Theorem B to the polynomials with complex coefficients by proving the following result which is also generalization of Theorem C.

Theorem D. *Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with complex coefficients. If $t_1>t_2\ge0$ can be found such that*

$$t_1t_2|a_r|+(t_1-t_2)|a_{r-1}|-|a_{r-2}|\ge0\qquad\text{for}\quad r=1,2,\dots,k+1,$$

and

$$t_1t_2|a_r|+(t_1-t_2)|a_{r-1}|-|a_{r-2}|\le0\qquad\text{for}\quad r=k+2,\dots,n+1,$$

$0\le k\le n$, $a_{-1}=a_{n+1}=0$, *then all the zeros of $P(z)$ lie in*

$$|z|\le t_1\left\{\frac{2|a_k|+2t_2|a_{k+1}|}{t_1^{n-k}|a_n|}-1\right\}+\frac{2}{|a_n|}\sum_{j=0}^{n}\frac{\big|a_j-|a_j|\big|}{t_1^{n-j-1}}. \tag{2}$$

The following generalization of Theorem A is due to A. Aziz and Q. G. Mohammad [**4**].

Theorem E. *Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with complex coefficients. Let $\operatorname{Re}a_j=\alpha_j$ and $\operatorname{Im}a_j=\beta_j$ for $j=0,1,2,\dots,n$. If for some $t>0$, and $0\le k\le n$,*

$$0<t^n\alpha_n\le t^{n-1}\alpha_{n-1}\le\cdots\le t^k\alpha_k\ge t^{k-1}\alpha_{k-1}\ge\cdots\ge t\alpha_1\ge\alpha_0\ge0,$$

then $P(z)$ has all its zeros in the circle

$$|z|\le t\left\{\frac{2t^k\alpha_k}{t^n\alpha_n}-1\right\}+\frac{2}{\alpha_n}\sum_{j=0}^{n}\frac{|\beta_j|}{t^{n-j-1}}. \tag{3}$$

Rather et. al [**11**] also obtained the following generalization of Theorem E.

Theorem F. *Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with complex coefficients. Let $\operatorname{Re}a_j=\alpha_j$ and $\operatorname{Im}a_j=\beta_j$ for $j=0,1,2,\dots,n$. If $t_1>t_2\ge0$ can be found such that*

$$t_1t_2\alpha_r+(t_1-t_2)\alpha_{r-1}-\alpha_{r-2}\ge0\qquad\text{for}\quad r=1,2,\dots,k+1,$$

and

$$t_1t_2\alpha_r+(t_1-t_2)\alpha_{r-1}-\alpha_{r-2}\le0\qquad\text{for}\quad r=k+2,\dots,n+1,$$

$0\le k\le n$, $\alpha_{-1}=\alpha_{n+1}=0$, $\alpha_n>0$, *then all the zeros of $P(z)$ lie in*

$$|z|\le t_1\left\{\frac{2\alpha_k+t_2\alpha_{k+1}}{t_1^{n-k}\alpha_n}-1\right\}+\frac{2}{\alpha_n}\sum_{j=0}^{n}\frac{|\beta_j|}{t_1^{n-j-1}}. \tag{4}$$

In this paper, we first present the following result which among the other things provides an interesting refinement of Theorem D for $0 \le k \le n-1$.

Theorem 1.1. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n \ge 2$ with complex coefficients. If $t_1 \neq 0$, $t_1 \ge t_2 \ge 0$ can be found such that*

$$t_1 t_2 |a_r| + (t_1 - t_2)|a_{r-1}| - |a_{r-2}| \ge 0 \qquad \text{for } \ r = 1, 2, \ldots, k+1,$$

and

$$t_1 t_2 |a_r| + (t_1 - t_2)|a_{r-1}| - |a_{r-2}| \le 0 \qquad \text{for } \ r = k+2, \ldots, n+1,$$

$0 \le k \le n-1$, $a_{-1} = a_{n+1} = 0$, then all the zeros of $P(z)$ lie in

$$\left| z + \frac{a_{n-1}}{a_n} - (t_1 - t_2) \right| \le 2\left(\frac{t_2|a_{k+1}| + |a_k|}{t_1^{n-k-1}|a_n|} \right) + 2\sum_{\nu=0}^{n} \frac{|a_\nu - |a_\nu||}{t_1^{n-\nu-1}|a_n|}$$

$$- \left(t_2 + \frac{|a_{n-1}| + \big|a_{n-1} - |a_{n-1}|\big| + t_1\big|a_n - |a_n|\big|}{|a_n|} \right). \tag{5}$$

Remark 1.2. In general, Theorem 1.1 gives much better result than Theorem D for $0 \le k \le n-1$. To see this, we show that the circle defined by (5) is contained in the circle defined by (2). Let $z = w$ be any point belonging to the circle defined by (5), then

$$\left| w + \frac{a_{n-1}}{a_n} - (t_1 - t_2) \right| \le 2\left(\frac{t_2|a_{k+1}| + |a_k|}{t_1^{n-k-1}|a_n|} \right) + 2\sum_{\nu=0}^{n} \frac{|a_\nu - |a_\nu||}{t_1^{n-\nu-1}|a_n|}$$

$$- \left(t_2 + \frac{|a_{n-1}| + \big|a_{n-1} - |a_{n-1}|\big| + t_1\big|a_n - |a_n|\big|}{|a_n|} \right).$$

This implies

$$\begin{aligned}
|w| &= \left| w + \frac{a_{n-1}}{a_n} - (t_1 - t_2) + (t_1 - t_2) - \frac{a_{n-1}}{a_n} \right| \\
&\le \left| w + \frac{a_{n-1}}{a_n} - (t_1 - t_2) \right| + \left| (t_1 - t_2) - \frac{a_{n-1}}{a_n} \right| \\
&\le 2\left(\frac{t_2|a_{k+1}| + |a_k|}{t_1^{n-k-1}|a_n|} \right) + 2\sum_{\nu=0}^{n} \frac{|a_\nu - |a_\nu||}{t_1^{n-\nu-1}|a_n|} \\
&\quad - \left(t_2 + \frac{|a_{n-1}| + \big|a_{n-1} - |a_{n-1}|\big| + t_1\big|a_n - |a_n|\big|}{|a_n|} \right) \\
&\quad + \frac{\big||a_n|(t_1 - t_2) - |a_{n-1}|\big|}{|a_n|} + \frac{\big|a_n - |a_n|\big|(t_1 - t_2) + \big|a_{n-1} - |a_{n-1}|\big|}{|a_n|} \\
&= 2\left(\frac{t_2|a_{k+1}| + |a_k|}{t_1^{n-k-1}|a_n|} \right) + 2\sum_{\nu=0}^{n} \frac{|a_\nu - |a_\nu||}{t_1^{n-\nu-1}|a_n|} - \left(t_2 + \frac{|a_{n-1}| + t_1\big|a_n - |a_n|\big|}{|a_n|} \right) \\
&\quad + \frac{|a_{n-1}| - |a_n|(t_1 - t_2)}{|a_n|} + \frac{\big|a_n - |a_n|\big|(t_1 - t_2)}{|a_n|}
\end{aligned}$$

$$\leq 2t_1\left(\frac{t_2|a_{k+1}|+|a_k|}{t_1^{n-k}|a_n|}\right)+\frac{2}{|a_n|}\sum_{\nu=0}^{n}\frac{|a_\nu-|a_\nu||}{t_1^{n-\nu-1}}-t_1$$
$$=t_1\left\{\frac{2|a_k|+2t_2|a_{k+1}|}{t_1^{n-k}|a_n|}-1\right\}+\frac{2}{|a_n|}\sum_{\nu=0}^{n}\frac{\big|a_\nu-|a_\nu|\big|}{t_1^{n-\nu-1}}.$$

This shows that the point $z=w$ belongs to the circle defined by (2). Hence the circle defined by (5) is contained in the circle defined by (2).

The following corollary follows immediately by taking $t_2=0$ in Theorem 1.1.

Corollary 1.3. *Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with complex coefficients. If $t>0$ can be found such that*

$$t^n|a_n|\leq t^{n-1}|a_{n-1}|\leq\cdots\leq t^{k+1}|a_{k+1}|\leq t^k|a_k|,$$
$$t^k|a_k|\geq t^{k-1}|a_{k-1}|\geq\cdots\geq t|a_1|\geq|a_0|,$$

$0\leq k\leq n-1$, *then all the zeros of $P(z)$ lie in*

$$\left|z+\frac{a_{n-1}}{a_n}-t\right|\leq\frac{2|a_k|}{t^{n-k-1}|a_n|}+\frac{2}{|a_n|}\sum_{\nu=0}^{n}\frac{|a_\nu-|a_\nu||}{t^{n-\nu-1}}-\frac{|a_{n-1}|+\big|a_{n-1}-|a_{n-1}|\big|+t\big|a_n-|a_n|\big|}{|a_n|}.$$

For polynomial $P(z)=\sum_{j=0}^{n}a_jz^j$ with real and positive coefficients, we obtain the following result.

Corollary 1.4. *Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with real and positive coefficients. If $t_1>t_2\geq0$ can be found such that*

$$t_1t_2a_r+(t_1-t_2)a_{r-1}-a_{r-2}\geq0\qquad\text{for}\quad r=1,2,\ldots,k+1,$$

and

$$t_1t_2a_r+(t_1-t_2)a_{r-1}-a_{r-2}\leq0\qquad\text{for}\quad r=k+2,\ldots,n+1,$$

$0\leq k\leq n-1$, $a_{-1}=a_{n+1}=0$, *then all the zeros of $P(z)$ lie in*

$$\left|z+\frac{a_{n-1}}{a_n}-(t_1-t_2)\right|\leq2\left(\frac{t_2a_{k+1}+a_k}{t_1^{n-k-1}a_n}\right)-\left(t_2+\frac{a_{n-1}}{a_n}\right).$$

Next we present the following result which considerably improves upon the bound of the Theorem F for $0\leq k\leq n-1$.

Theorem 1.5. *Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with complex coefficients. Let $\operatorname{Re}a_j=\alpha_j$ and $\operatorname{Im}a_j=\beta_j$ for $j=0,1,2,\ldots,n$. If $t_1\neq0$, $t_1\geq t_2\geq0$ can be found such that*

$$t_1t_2\alpha_r+(t_1-t_2)\alpha_{r-1}-\alpha_{r-2}\geq0\qquad\text{for}\quad r=1,2,\ldots,k+1,$$

and

$$t_1t_2\alpha_r+(t_1-t_2)\alpha_{r-1}-\alpha_{r-2}\leq0\qquad\text{for}\quad r=k+2,\ldots,n+1,$$

$0 \leq k \leq n-1$, $a_{-1} = a_{n+1} = 0$, $\alpha_n > 0$, *then all the zeros of* $P(z)$ *lie in*

$$
\begin{aligned}
&\left|z+\frac{\alpha_{n-1}-(t_1-t_2)\alpha_n}{a_n}\right| \\
&\qquad \leq 2\frac{(\alpha_{k+1}t_2+\alpha_k)}{|a_n|t_1^{n-k-1}}+\frac{2}{|a_n|}\sum_{\nu=0}^{n}\frac{|\beta_\nu|}{t_1^{n-\nu-1}}-\frac{t_1|\beta_n|+\alpha_n t_2+\alpha_{n-1}}{|a_n|}.
\end{aligned} \tag{6}
$$

Remark 1.6. In general Theorem 1.5 also gives much better result than Theorem F for $0 \leq k \leq n-1$. We show that the circle defined by (6) is contained in the circle defined by (4). Let $z = w$ be any point belonging to the circle defined by (6), then

$$
\begin{aligned}
&\left|w+\frac{\alpha_{n-1}-(t_1-t_2)\alpha_n}{a_n}\right| \\
&\qquad \leq 2\frac{(\alpha_{k+1}t_2+\alpha_k)}{|a_n|t_1^{n-k-1}}+\frac{2}{|a_n|}\sum_{\nu=0}^{n}\frac{|\beta_\nu|}{t_1^{n-\nu-1}}-\frac{t_1|\beta_n|+\alpha_n t_2+\alpha_{n-1}}{|a_n|}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
|w| &= \left|w+\frac{\alpha_{n-1}-(t_1-t_2)\alpha_n}{a_n}-\frac{\alpha_{n-1}-(t_1-t_2)\alpha_n}{a_n}\right| \\
&\leq \left|w+\frac{\alpha_{n-1}-(t_1-t_2)\alpha_n}{a_n}\right|+\left|\frac{\alpha_{n-1}-(t_1-t_2)\alpha_n}{a_n}\right| \\
&\leq 2\frac{(\alpha_{k+1}t_2+\alpha_k)}{|a_n|t_1^{n-k-1}}+\frac{2}{|a_n|}\sum_{\nu=0}^{n}\frac{|\beta_\nu|}{t_1^{n-\nu-1}}-\frac{t_1|\beta_n|+\alpha_n t_2+\alpha_{n-1}}{|a_n|} \\
&\qquad +\frac{\alpha_{n-1}-(t_1-t_2)\alpha_n}{|a_n|} \\
&= \frac{1}{|a_n|}\left\{2\frac{(\alpha_{k+1}t_2+\alpha_k)}{t_1^{n-k-1}}+2\sum_{\nu=0}^{n}\frac{|\beta_\nu|}{t_1^{n-\nu-1}}+-t_1\alpha_n-t_1|\beta_n|\right\} \\
&\leq \frac{1}{\alpha_n}\left\{2\frac{(\alpha_{k+1}t_2+\alpha_k)}{t_1^{n-k-1}}+2\sum_{\nu=0}^{n}\frac{|\beta_\nu|}{t_1^{n-\nu-1}}-t_1\alpha_n\right\} \\
&= t_1\left\{\frac{2\alpha_k+2t_2a_{k+1}}{t_1^{n-k}\alpha_n}-1\right\}+\frac{2}{\alpha_n}\sum_{\nu=0}^{n}\frac{|\beta_\nu|}{t_1^{n-\nu-1}}.
\end{aligned}
$$

Hence the point $z = w$ belongs to the circle defined by (4) and therefore, the circle defined by (13) is contained in the circle defined (4).

The following result also follows from Theorem 1.5 by taking $t_2 = 0$.

Corollary 1.7. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with complex coefficients. Let* $\operatorname{Re} a_j = \alpha_j$ *and* $\operatorname{Im} a_j = \beta_j$ *for* $j = 0, 1, 2, \ldots, n$. *If* $t > 0$ *can be found such that*

$$t^n\alpha_n \leq t^{n-1}\alpha_{n-1} \leq \cdots \leq t^{k+1}\alpha_{k+1} \leq t^k\alpha_k,$$

$$t^k\alpha_k \geq t^{k-1}\alpha_{k-1} \geq \cdots \geq t\alpha_1 \geq \alpha_0$$

$0 \leq k \leq n-1$, $a_{-1} = a_{n+1} = 0$, $\alpha_n > 0$, *then all the zeros of* $P(z)$ *lie in*

$$\left|z + \frac{\alpha_{n-1} - t\alpha_n}{a_n}\right| \leq \frac{2\alpha_k}{|a_n|t^{n-k-1}} + \frac{2}{|a_n|}\sum_{\nu=0}^{n}\frac{|\beta_\nu|}{t^{n-\nu-1}} - \frac{t|\beta_n| + \alpha_{n-1}}{|a_n|}. \tag{7}$$

2. Proof of theorems

Proof of Theorem 1.1. Consider the polynomial

$$\begin{aligned} F(z) &= (t_1 - z)(t_2 + z)P(z) \\ &= -a_n z^{n+2} + (a_n(t_1 - t_2) - a_{n-1})z^{n+1} \\ &\quad + \sum_{\nu=2}^{n}(a_\nu t_1 t_2 + a_{\nu-1}(t_1 - t_2) - a_{\nu-2})z^\nu \\ &\quad + (a_1 t_1 t_2 + a_0(t_1 - t_2))z + a_0 t_1 t_2 \\ &= -a_n z^{n+2} + (a_n(t_1 - t_2) - a_{n-1})z^{n+1} \\ &\quad + \sum_{\nu=0}^{n}(a_\nu t_1 t_2 + a_{\nu-1}(t_1 - t_2) - a_{\nu-2})z^\nu \qquad (a_{-2} = a_{-1} = 0). \end{aligned}$$

Let $|z| > t_1$, then

$$\begin{aligned} |F(z)| &\geq |a_n||z|^{n+1}\left[\left|z + \frac{a_{n-1}}{a_n} - (t_1 - t_2)\right| \right. \\ &\quad \left. - \sum_{\nu=0}^{n}\left|\frac{a_\nu t_1 t_2 + a_{\nu-1}(t_1 - t_2) - a_{\nu-2}}{a_n}\right|\frac{1}{|z|^{n-\nu+1}}\right] \\ &> |a_n||z|^{n+1}\left[\left|z + \frac{a_{n-1}}{a_n} - (t_1 - t_2)\right| \right. \\ &\quad \left. - \frac{1}{|a_n|}\sum_{\nu=0}^{n}\frac{|a_\nu t_1 t_2 + a_{\nu-1}(t_1 - t_2) - a_{\nu-2}|}{t_1^{n-\nu+1}}\right]. \end{aligned} \tag{8}$$

Now,

$$\begin{aligned} &\sum_{\nu=0}^{n}\frac{|a_\nu t_1 t_2 + a_{\nu-1}(t_1 - t_2) - a_{\nu-2}|}{t_1^{n-\nu+1}} \\ &\leq \sum_{\nu=0}^{n}\frac{\big||a_\nu|t_1 t_2 + |a_{\nu-1}|(t_1 - t_2) - |a_{\nu-2}|\big|}{t_1^{n-\nu+1}} \\ &\quad + \sum_{\nu=0}^{n}\frac{\big|a_\nu - |a_\nu|\big|t_1 t_2 + \big|a_{\nu-1} - |a_{\nu-1}|\big|(t_1 - t_2) + \big|a_{\nu-2} - |a_{\nu-2}|\big|}{t_1^{n-\nu+1}} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\nu=0}^{k+1} \frac{|a_\nu| t_1 t_2 + |a_{\nu-1}|(t_1 - t_2) - |a_{\nu-2}|}{t_1^{n-\nu+1}} \\
&\quad + \sum_{\nu=k+2}^{n} \frac{|a_{\nu-2}| - |a_{\nu-1}|(t_1 - t_2) - |a_\nu| t_1 t_2}{t_1^{n-\nu+1}} \\
&\quad + \sum_{\nu=1}^{n} \frac{t_1 \big|a_{\nu-1} - |a_{\nu-1}|\big| + \big|a_{\nu-2} - |a_{\nu-2}|\big|}{t_1^{n-\nu+1}} \\
&\le 2\left(\frac{t_2 |a_{k+1}| + |a_k|}{t_1^{n-k-1}}\right) + 2\sum_{\nu=0}^{n} \frac{\big|a_\nu - |a_\nu|\big|}{t_1^{n-\nu-1}} \\
&\quad - \left(t_2 |a_n| + |a_{n-1}| + \big|a_{n-1} - |a_{n-1}|\big| + t_1 \big|a_n - |a_n|\big|\right).
\end{aligned}
$$

Using this in (8), we obtain for $|z| > t_1$,

$$
\begin{aligned}
|F(z)| \ge |a_n||z|^{n+1} \Bigg[& \left| z + \frac{a_{n-1}}{a_n} - (t_1 - t_2) \right| \\
& - 2\left(\frac{t_2 |a_{k+1}| + |a_k|}{t_1^{n-k-1} |a_n|}\right) - 2\sum_{\nu=0}^{n} \frac{\big|a_\nu - |a_\nu|\big|}{t_1^{n-\nu-1} |a_n|} \\
& + \left(t_2 + \frac{|a_{n-1}| + \big|a_{n-1} - |a_{n-1}|\big| + t_1 \big|a_n - |a_n|\big|}{|a_n|}\right) > 0,
\end{aligned}
$$

if

$$
\begin{aligned}
\left| z + \frac{a_{n-1}}{a_n} - (t_1 - t_2) \right| > {} & 2\left(\frac{t_2 |a_{k+1}| + |a_k|}{t_1^{n-k-1} |a_n|}\right) + 2\sum_{\nu=0}^{n} \frac{\big|a_\nu - |a_\nu|\big|}{t_1^{n-\nu-1} |a_n|} \\
& - \left(t_2 + \frac{|a_{n-1}| + \big|a_{n-1} - |a_{n-1}|\big| + t_1 \big|a_n - |a_n|\big|}{|a_n|}\right).
\end{aligned}
$$

Therefore, it follows that all the zeros of $F(z)$ whose modulus is greater than t_1 lie in the circle

$$
\begin{aligned}
\left| z + \frac{a_{n-1}}{a_n} - (t_1 - t_2) \right| \le {} & 2\left(\frac{t_2 |a_{k+1}| + |a_k|}{t_1^{n-k-1} |a_n|}\right) + 2\sum_{\nu=0}^{n} \frac{\big|a_\nu - |a_\nu|\big|}{t_1^{n-\nu-1} |a_n|} \\
& - \left(t_2 + \frac{|a_{n-1}| + \big|a_{n-1} - |a_{n-1}|\big| + t_1 \big|a_n - |a_n|\big|}{|a_n|}\right). \qquad (9)
\end{aligned}
$$

We now show that all those zeros of $F(z)$ whose modulus is less than or equal to t_1 also satisfy (9).

Let $|z| \leq t_1$,

$$
\begin{aligned}
\left|z+\frac{a_{n-1}}{a_n}-(t_1-t_2)\right| &\leq t_1+\left|\frac{a_{n-1}}{a_n}-(t_1-t_2)\right| \\
&\leq t_1+\frac{\big||a_n|(t_1-t_2)-|a_{n-1}|\big|}{|a_n|} \\
&\quad +\frac{\big|a_n-|a_n|\big|(t_1-t_2)+\big|a_{n-1}-|a_{n-1}|\big|}{|a_n|} \\
&= t_1+\frac{|a_{n-1}|-|a_n|(t_1-t_2)}{|a_n|} \\
&\quad +\frac{\big|a_n-|a_n|\big|(t_1-t_2)+\big|a_{n-1}-|a_{n-1}|\big|}{|a_n|} \\
&= \left(t_2+\frac{|a_{n-1}|}{|a_n|}\right)+\frac{\big|a_{n-1}-|a_{n-1}|\big|}{|a_n|} \\
&\quad +(t_1-t_2)\frac{\big|a_n-|a_n|\big|}{|a_n|} \\
&= 2\left(t_2+\frac{|a_{n-1}|}{|a_n|}\right)+2\frac{\big|a_{n-1}-|a_{n-1}|\big|}{|a_n|} \\
&\quad +(t_1-t_2)\frac{\big|a_n-|a_n|\big|}{|a_n|}-\left(t_2+\frac{|a_{n-1}|}{|a_n|}\right) \\
&\quad -\frac{\big|a_{n-1}-|a_{n-1}|\big|}{|a_n|}.
\end{aligned}
\tag{10}
$$

Now, by hypothesis,

$$\sum_{\nu=k+2}^{n}\frac{|a_\nu|t_1t_2+|a_{\nu-1}|(t_1-t_2)-|a_{\nu-2}|}{t_1^{n-\nu+1}}\leq 0.$$

This gives

$$2\left(t_2+\left|\frac{a_{n-1}}{a_n}\right|\right)\leq\frac{2t_2|a_{k+1}|+2|a_k|}{|a_n|t_1^{n-k-1}},\qquad 0\leq k\leq n-1. \tag{11}$$

Using (11) in (10) for $0\leq k\leq n-1$, we obtain

$$
\begin{aligned}
\left|z+\frac{a_{n-1}}{a_n}-(t_1-t_2)\right| &\leq 2\left(\frac{t_2|a_{k+1}|+|a_k|}{t_1^{n-k-1}|a_n|}\right)+2\frac{\big|a_{n-1}-|a_{n-1}|\big|}{|a_n|} \\
&\quad -\left(t_2+\frac{|a_{n-1}|+\big|a_{n-1}-|a_{n-1}|\big|}{|a_n|}\right) \\
&\quad +(t_1-t_2)\frac{\big|a_n-|a_n|\big|}{|a_n|}
\end{aligned}
$$

$$\leq 2\left(\frac{t_2|a_{k+1}|+|a_k|}{t_1^{n-k-1}|a_n|}\right)+2\sum_{\nu=0}^{n}\frac{|a_\nu-|a_\nu||}{t_1^{n-\nu-1}|a_n|} - \left(t_2+\frac{|a_{n-1}|+\big|a_{n-1}-|a_{n-1}|\big|+t_1\big|a_n-|a_n|\big|}{|a_n|}\right).$$

Thus, we have shown that if $|z|\leq t_1$, then for $0\leq k\leq n-1$,

$$\left|z+\frac{a_{n-1}}{a_n}-(t_1-t_2)\right|\leq 2\left(\frac{t_2|a_{k+1}|+|a_k|}{t_1^{n-k-1}|a_n|}\right)+2\sum_{\nu=0}^{n}\frac{|a_\nu-|a_\nu||}{t_1^{n-\nu-1}|a_n|} - \left(t_2+\frac{|a_{n-1}|+\big|a_{n-1}-|a_{n-1}|\big|+t_1\big|a_n-|a_n|\big|}{|a_n|}\right).$$

Hence all the zeros of $F(z)$ lie in the circle defined by (9). But all the zeros of $P(z)$ are also the zeros of $F(z)$, therefore, we conclude that all the zeros of $P(z)$ lie in the circle defined by (9). This proves the Theorem 1.1. □

Proof of Theorem 1.5. Consider the polynomial,

$$\begin{aligned}F(z)=&(t_1-z)(t_2+z)P(z)\\ =&-a_nz^{n+2}+(a_n(t_1-t_2)-a_{n-1})z^{n+1}\\ &+\sum_{\nu=2}^{n}(a_\nu t_1t_2+a_{\nu-1}(t_1-t_2)-a_{\nu-2})z^\nu\\ &+(a_1t_1t_2+a_0(t_1-t_2))z+a_0t_1t_2\\ =&-a_nz^{n+2}+(a_n(t_1-t_2)-a_{n-1})z^{n+1}\\ &+\sum_{\nu=0}^{n}(a_\nu t_1t_2+a_{\nu-1}(t_1-t_2)-a_{\nu-2})z^\nu \qquad (a_{-2}=a_{-1}=0).\end{aligned}$$

Let $|z|>t_1$, then

$$\begin{aligned}|F(z)|\geq |z|^{n+1}&\Bigg[|a_nz+a_{n-1}-(t_1-t_2)a_n|\\ &-\sum_{\nu=0}^{n}|a_\nu t_1t_2+a_{\nu-1}(t_1-t_2)-a_{\nu-2}|\,\frac{1}{|z|^{n-\nu+1}}\Bigg]\\ >|z|^{n+1}&\Bigg[|a_nz+\alpha_{n-1}-(t_1-t_2)\alpha_n|-|\beta_{n-1}|-(t_1-t_2)|\beta_n|\\ &-\sum_{\nu=0}^{n}|a_\nu t_1t_2+a_{\nu-1}(t_1-t_2)-a_{\nu-2}|\frac{1}{t_1^{n-\nu+1}}\Bigg].\end{aligned}\tag{12}$$

Now by hypothesis,

$$\begin{aligned}
&\sum_{\nu=0}^{n}|a_\nu t_1t_2+a_{\nu-1}(t_1-t_2)-a_{\nu-2}|t_1^\nu\\
&\qquad\le \sum_{\nu=0}^{n}|\alpha_\nu t_1t_2+\alpha_{\nu-1}(t_1-t_2)-\alpha_{\nu-2}|t_1^\nu\\
&\qquad\quad+\sum_{\nu=0}^{n}|\beta_\nu t_1t_2+\beta_{\nu-1}(t_1-t_2)-\beta_{\nu-2}|t_1^\nu\\
&\qquad\le \sum_{\nu=0}^{k+1}|\alpha_\nu t_1t_2+\alpha_{\nu-1}(t_1-t_2)-\alpha_{\nu-2}|t_1^\nu\\
&\qquad\quad+\sum_{\nu=k+2}^{n}|\alpha_\nu t_1t_2+\alpha_{\nu-1}(t_1-t_2)-\alpha_{\nu-2}|t_1^\nu\\
&\qquad\quad+\sum_{\nu=0}^{n}\left(|\beta_\nu|t_1t_2+|\beta_{\nu-1}|(t_1-t_2)+|\beta_{\nu-2}|\right)t_1^\nu\\
&\qquad=2(\alpha_{k+1}t_2+\alpha_k)t_1^{k+2}-(\alpha_nt_2+\alpha_{n-1})t_1^{n+1}\\
&\qquad\quad+t_1^{n+1}\left(|\beta_n|t_2+2\sum_{\nu=0}^{n-1}\frac{|\beta_\nu|}{t_1^{n-\nu-1}}-|\beta_{n-1}|\right).
\end{aligned}$$

Using this in (12), we obtain

$$\begin{aligned}
|F(z)|\ge |z|^{n+1}\Bigg\{&|a_nz+\alpha_{n-1}-(t_1-t_2)\alpha_n|-|\beta_{n-1}|-(t_1-t_2)|\beta_n|\\
&-2(\alpha_{k+1}t_2+\alpha_k)\frac{t_1^{k+1}}{t_1^n}+(\alpha_nt_2+\alpha_{n-1})\\
&-|\beta_n|t_2-2\sum_{\nu=0}^{n-1}\frac{|\beta_\nu|}{t_1^{n-\nu-1}}+|\beta_{n-1}|\Bigg\}\\
=\Bigg\{&|a_nz+\alpha_{n-1}-(t_1-t_2)\alpha_n|-t_1|\beta_n|\\
&-2(\alpha_{k+1}t_2+\alpha_k)\frac{t_1^{k+1}}{t_1^n}+(\alpha_nt_2+\alpha_{n-1})-2\sum_{\nu=0}^{n-1}\frac{|\beta_\nu|}{t_1^{n-\nu-1}}\Bigg\}>0
\end{aligned}$$

if

$$\begin{aligned}
&\left|z+\frac{\alpha_{n-1}-(t_1-t_2)\alpha_n}{a_n}\right|\\
&\quad>\frac{1}{|a_n|}\left\{t_1|\beta_n|+2\sum_{\nu=0}^{n-1}\frac{|\beta_\nu|}{t_1^{n-\nu-1}}\right\}+2\frac{(\alpha_{k+1}t_2+\alpha_k)}{|a_n|t_1^{n-k-1}}-\frac{(\alpha_nt_2+\alpha_{n-1})}{|a_n|}
\end{aligned}$$

$$= 2\frac{(\alpha_{k+1}t_2 + \alpha_k)}{|a_n|t_1^{n-k-1}} + \frac{2}{|a_n|}\sum_{\nu=0}^{n}\frac{|\beta_\nu|}{t_1^{n-\nu-1}} - \frac{t_1|\beta_n| + \alpha_n t_2 + \alpha_{n-1}}{|a_n|}.$$

Hence all the zeros of $F(z)$ whose modulus is greater than t_1 lie in the circle

$$\left|z+ \frac{\alpha_{n-1} - (t_1 - t_2)\alpha_n}{a_n}\right| \leq 2\frac{(\alpha_{k+1}t_2 + \alpha_k)}{|a_n|t_1^{n-k-1}} + \frac{2}{|a_n|}\sum_{\nu=0}^{n}\frac{|\beta_\nu|}{t_1^{n-\nu-1}} - \frac{t_1|\beta_n| + \alpha_n t_2 + \alpha_{n-1}}{|a_n|}. \tag{13}$$

We now show that all those zeros of $F(z)$ whose modulus is less than or equal to t_1 also satisfy (13). Let $|z| \leq t_1$, then we have

$$\begin{aligned} \left|a_n z + \alpha_{n-1} - (t_1 - t_2)\alpha_n\right| &\leq |a_n|t_1 + |\alpha_{n-1} - (t_1 - t_2)\alpha_n| \\ &\leq t_1|\alpha_n| + t_1|\beta_n| + \alpha_{n-1} - (t_1 - t_2)\alpha_n \\ &= t_1|\beta_n| + 2(t_2\alpha_n + \alpha_{n-1}) - (t_2\alpha_n + \alpha_{n-1}) \end{aligned} \tag{14}$$

Again, by hypothesis, we have

$$\begin{aligned} &\sum_{\nu=k+2}^{n}\frac{\alpha_\nu t_1 t_2 + \alpha_{\nu-1}(t_1 - t_2) - \alpha_{\nu-2}}{t_1^{n-\nu+1}} \leq 0, \\ \text{or} \quad &2\left(t_2\alpha_n + \alpha_{n-1}\right) \leq \frac{2t_2\alpha_{k+1} + 2\alpha_k}{t_1^{n-k-1}}, \qquad 0 \leq k \leq n-1. \end{aligned} \tag{15}$$

Using (15) in (14) for $0 \leq k \leq n-1$, we obtain

$$\begin{aligned} &\left|a_n z + \alpha_{n-1} - (t_1 - t_2)\alpha_n\right| \\ &\leq t_1|\beta_n| + \frac{2t_2\alpha_{k+1} + 2\alpha_k}{t_1^{n-k-1}} - (t_2\alpha_n + \alpha_{n-1}) \\ &\leq t_1|\beta_n| + \frac{2t_2\alpha_{k+1} + 2\alpha_k}{t_1^{n-k-1}} - (t_2\alpha_n + \alpha_{n-1}) + 2\sum_{\nu=0}^{n-1}\frac{|\beta_\nu|}{t_1^{n-\nu-1}} \\ &= 2\frac{(\alpha_{k+1}t_2 + \alpha_k)}{t_1^{n-k-1}} + 2\sum_{\nu=0}^{n}\frac{|\beta_\nu|}{t_1^{n-\nu-1}} - \left(t_1|\beta_n| + \alpha_n t_2 + \alpha_{n-1}\right). \end{aligned}$$

This shows that all the zeros of $F(z)$ whose modulus is less than or equal to t_1 also satisfy the inequality (13). Thus we conclude that all the zeros of $F(z)$ and hence that of $P(z)$ lie in the circle defined by (13). This completes the proof of Theorem 1.5. □

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