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MINIMAL OUTER CONNECTED MONOPHONIC SETS IN GRAPHS

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Abstract. For a connected graph G of order at least two, an outer connected monophonic set S in a graph G is called a *minimal outer connected monophonic set* if no proper subset of S is an outer connected monophonic set of G. The *upper outer connected monophonic number* $m^+_{oc}(G)$ of G is the maximum cardinality of a minimal outer connected monophonic set of G. We determine bounds for it and find the upper outer connected monophonic number of certain classes of graphs. It is shown that for any three positive integers a, b, c with $3 \leq a \leq b \leq c$, there is a connected graph G with $m(G) = a, m_{oc}(G) = b, m^+_{oc}(G) = c$, where $m(G)$ is the monophonic number of a graph and $m_{oc}(G)$ is the outer connected monophonic number of a graph. Also, it is shown that for any three positive integers a, b, and n with $3 \leq a \leq n \leq b$, there is a connected graph G with $m_{oc}(G) = a, m^+_{oc}(G) = b$, and a minimal outer connected monophonic set of cardinality n.

1. Introduction

By a graph $G = (V, E)$, we mean a simple undirected connected graph, where V is the set of vertices and E is the set of edges of G. The order and size of G are denoted by p and q, respectively. For basic graph theoretic terminology we refer to Harary [**1, 9**]. A vertex v of G is called an *extreme vertex* if the subgraph induced by its neighbors is complete. A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. A set S of vertices of G is a *monophonic set* of G if each vertex v of G lies on an $x - y$ monophonic path for some x and y in S. The minimum cardinality of a monophonic set of G is the *monophonic number* of G and is denoted by $m(G)$. The monophonic number of a graph, algorithmic aspects of monophonic and its related concepts were introduced and studied in [**2, 3, 4, 10, 11, 14, 15**]. A set S of vertices in a graph G is said to be an *outer connected monophonic set* if S is a monophonic set of G, and either $S = V$ or the subgraph induced by $V - S$ is connected. The minimum cardinality of an outer connected monophonic set of G is the *outer connected monophonic number* of G and is denoted by $m_{oc}(G)$. The

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outer connected monophonic number of a graph was introduced in [**5**], and further studied in [**6, 7, 8**].

For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest $u-v$ monophonic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = \max\{d_m(v, u) : u \in V(G)\}$. The *monophonic radius*, $\mathrm{rad}_m(G)$ of G is $\mathrm{rad}_m(G) = \min\ \{e_m(v) : v \in V(G)\}$ and the *monophonic diameter*, $\mathrm{diam}_m(G)$ of G is $\mathrm{diam}_m(G) = \max\ \{e_m(v) : v \in V(G)\}$. The monophonic distance was introduced in [**12**], and further studied in [**13**]. These concepts have interesting applications in Channel Assignment Problem in FM radio technologies. The monophonic matrix is used to discuss different aspects of certain molecular graphs associated to the molecules arising in special situations of molecular problems in theoretical chemistry.

The following theorems are further used in the sequel.

Theorem 1.1 ([**14**]). *Each extreme vertex of a connected graph G belongs to every monophonic set of G.*

Theorem 1.2 ([**14**]). *If T is a tree with k end-vertices, then $m(T) = k$.*

Theorem 1.3 ([**5**]). *If T is a tree with k end-vertices, then $m_{oc}(T) = k$.*

Throughout this paper, G denotes a connected graph with at least two vertices.

2. Main results

Definition 2.1. An outer connected monophonic set S in a graph G is called a *minimal outer connected monophonic set* if no proper subset of S is an outer connected monophonic set of G. The *upper outer connected monophonic number* $m^+_{oc}(G)$ of G is the maximum cardinality of a minimal outer connected monophonic set of G.

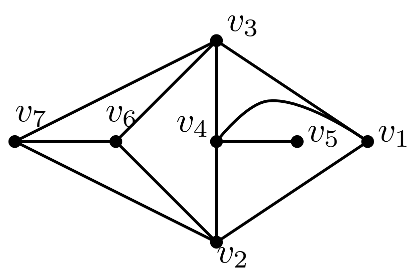

Figure 2.1. *G.*

Example 2.2. For the graph G given in Figure 2.1 of order 7, it is clear that no 2-element subset of $V(G)$ is a monophonic set of G. The set $S = \{v_2, v_3, v_5\}$ is a monophonic set of G, and so $m(G) = 3$. Since the subgraph induced by $V - S$ is not connected, S is not an outer connected monophonic set of G. It is clear that no 2-element or 3-element subset of $V(G)$ is an outer connected monophonic set of G. The minimal outer connected monophonic sets of G are $S_1 = \{v_1, v_5, v_6, v_7\}$, $S_2 = \{v_1, v_2, v_3, v_4, v_5\}$, and $S_3 = \{v_2, v_3, v_5, v_6, v_7\}$. By the definitions of the outer connected monophonic number, and the upper outer connected monophonic number of a graph, we have $m_{oc}(G) = 4$ and $m^+_{oc}(G) = 5$. Thus the monophonic number, the outer connected monophonic number and the upper outer connected monophonic number of a graph are different.

An application point of view, Wireless Sensor Networks (WSN's) is represented as a graph $G = (V, E)$, where each vertex is related to a sensor node and each edge is a wireless communication link between sensor nodes. Each sensor node sends and receives a message; and also performs some computation from its neighbors. In WSN's, we make a set of nodes S as an anchor node set such that any node on this network lies on a monophonic path joining a pair of nodes in S and if the anchor node set S fails, then all other nodes in the network are still able to communicate with each other. An anchor node set S is called minimal if no proper subset of S is an anchor node set in the given network. The problem is to identify the maximum cardinality of a minimal anchor node set in the given network. Then the model of this problem in WSN's to find the upper outer connected monophonic number of a network. Since no intervention is possible in monophonic paths, the maximum cardinality of a minimal anchor node set involves in WSN's is more secured.

Remark 2.3. Every minimum outer connected monophonic set of G is a minimal outer connected monophonic set of G and the converse need not be true. For example, the graph G given in Figure 2.1, $S_2 = \{v_1, v_2, v_3, v_4, v_5\}$ is a minimal outer connected monophonic set and it is not a minimum outer connected monophonic set of G.

The following results are clear from the fact that each minimal outer connected monophonic set is an outer connected monophonic set of G and every outer connected monophonic set of G is a monophonic set of G, and also every monophonic set of G contains its extreme vertices.

Remark 2.4. Each extreme vertex of a connected graph G belongs to every minimal outer connected monophonic set of G.

Remark 2.5. For the complete graph K_p, $m_{oc}(K_p) = m^+_{oc}(K_p) = p$.

Theorem 2.6. *Let G be a connected graph with cut-vertices and let S be a minimal outer connected monophonic set of G. If v is a cut-vertex of G, then every component of $G - v$ contains an element of S.*

Proof. Suppose that there is a component G_1 of $G - v$ such that G_1 contains no vertex of S. Let u be a vertex in G_1. Since S is a minimal outer connected monophonic set of G, there exist vertices $x, y \in S$ such that u lies on some $x - y$ monophonic path $P\colon x = u_0, u_1, \ldots u, \ldots, u_n = y$ in G. Let P_1 be the $x - u$ subpath of P and P_2 be the $u - y$ subpath of P. Since v is a cut-vertex of G, both

P_1 and P_2 contain v so that P is not a path, which is a contradiction. Thus every component of $G - v$ contains an element of S. □

Theorem 2.7. *For any connected graph G of order p, $2 \leq m(G) \leq m_{oc}(G) \leq m_{oc}^{+}(G) \leq p$. Moreover, $m_{oc}(G) = p$ if and only if $m_{oc}^{+}(G) = p$.*

Proof. Any monophonic set needs at least two vertices, and so $m(G) \geq 2$. Since every outer connected monophonic set of G is a monophonic set of G, $m(G) \leq m_{oc}(G)$. Since every minimal outer connected monophonic set of G is an outer connected monophonic set of G, $m_{oc}(G) \leq m_{oc}^{+}(G)$. Also, $V(G)$ is an outer connected monophonic set of G, it is clear that $m_{oc}^{+}(G) \leq p$. Thus $2 \leq m(G) \leq m_{oc}(G) \leq m_{oc}^{+}(G) \leq p$.

Let $m_{oc}^{+}(G) = p$. Then $S = V(G)$ is the unique minimal outer connected monophonic set of G. Since no proper subset of S is an outer connected monophonic set of G, it is clear that S is the unique minimum outer connected monophonic set of G, and so $m_{oc}(G) = p$. The converse is clear. □

Remark 2.8. The bounds in Theorem 2.7 are sharp. By Theorem 1.2, any non-trivial path P_n $(n \geq 2)$, $m(P_n) = 2$. By Remark 2.5, for the complete graph $K_p(p \geq 2)$, $m_{oc}^{+}(K_p) = p$. Also, all the inequalities in Theorem 2.7 can be strict. For the graph G given in Figure 2.1, $m(G) = 3$, $m_{oc}(G) = 4$, and $m_{oc}^{+}(G) = 5$. Thus, we have $2 < m(G) < m_{oc}(G) < m_{oc}^{+}(G) < p$.

Theorem 2.9. *If G is a connected graph of order p with $m_{oc}(G) = p - 1$, then $m_{oc}^{+}(G) = p - 1$.*

Proof. If $m_{oc}(G) = p - 1$, then by Theorem 2.7 we have $m_{oc}^{+}(G) = p - 1$ or p. If $m_{oc}^{+}(G) = p$, by Theorem 2.7, $m_{oc}(G) = p$, which is a contradiction. Hence $m_{oc}^{+}(G) = p - 1$. □

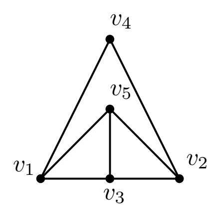

Figure 2.2. *G*.

Remark 2.10. The converse of Theorem 2.9 need not be true. For the graph G given in Figure 2.2 of order 5, no 2-element subset of $V(G)$ is an outer connected monophonic set of G. It is clear that $S = \{v_1, v_2, v_4\}$ is an outer connected monophonic set of G, and so $m_{oc}(G) = 3$. The minimal outer connected monophonic sets of G are $S = \{v_1, v_2, v_4\}$, $S_1 = \{v_1, v_3, v_4, v_5\}$, and $S_2 = \{v_2, v_3, v_4, v_5\}$. By the definition of the upper outer connected monophonic number of a graph, we have $m_{oc}^{+}(G) = 4 = p - 1$.

Observation 2.11. The upper outer connected monophonic number of some standard graphs:

- For any tree T with k end-vertices, $m_{oc}(T) = m_{oc}^{+}(T) = k$.
- For the wheel W_n $(n \geq 5)$, $m_{oc}(G) = m_{oc}^{+}(G) = 2$.
- For the star $G = K_{1,p-1}$, $m_{oc}(G) = m_{oc}^{+}(G) = p - 1$.
- For the complement of the cycle C_n $(n \geq 6)$, $m_{oc}(\bar{C_n}) = m_{oc}^{+}(\bar{C_n}) = 2$.

3. Some realization results

In view of Theorem 2.7, we have the following realization result.

Theorem 3.1. *For any three integers a, b, and c such that $3 \leq a \leq b \leq c$, there is a connected graph G with $m(G) = a$, $m_{oc}(G) = b$, and $m_{oc}^{+}(G) = c$.*

Proof. We prove this theorem by considering four cases.
Case 1. $3 \leq a = b = c$.
Let G be any tree with a end-vertices. Then by Theorems 1.2, 1.3, and Observation 2.11, G has the desired properties.

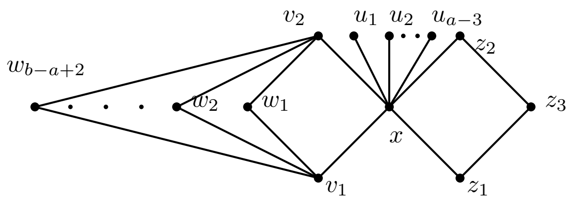

Figure 3.1. *G*.

Case 2. $3 \leq a < b = c$.
Let $K_{1,4}$ be a star having the vertex set x, v_1, v_2, z_1, z_2 with x as the cut-vertex. The required graph G is obtained from the star $K_{1,4}$ by adding b new vertices $w_1, w_2, \ldots, w_{b-a+2}, z_3, u_1, u_2, \ldots, u_{a-3}$ and joining each w_i $(1 \leq i \leq b - a + 2)$ to the vertices v_1 and v_2, and joining the vertex z_3 to the vertices z_1 and z_2, and also joining each u_i $(1 \leq i \leq a - 3)$ to the vertex x, thereby producing the graph G as shown in the Figure 3.1. Then $S = \{u_1, u_2, \ldots, u_{a-3}\}$ is the set of all extreme vertices of G. Since every outer connected monophonic set of G is a monophonic set of G and by Theorem 1.1, Remark 2.4, every monophonic set, every outer connected monophonic set, and every minimal outer connected monophonic set of G contain S. It is clear that S is not a monophonic set of G. Also, for any two vertices $u, v \in V(G) - S$, $S \cup \{u, v\}$ is not a monophonic set of G. Clearly, $S_1 = S \cup \{v_1, v_2, z_3\}$ is a minimum monophonic set of G, and so $m(G) = a$. Since the subgraph induced by $V - S_1$ is not connected, S_1 is not an outer connected monophonic set of G. It is easy to observe that every minimum outer connected monophonic set and every minimal outer connected monophonic set of G contains $\{w_1, w_2, \ldots, w_{b-a+2}\}$. With this, it is clear that $S_2 = S \cup \{w_1, w_2, \ldots, w_{b-a+2}, z_3\}$

is the unique minimal outer connected monophonic set of G, and hence $m_{oc}(G) = b$ and $m^+_{oc}(G) = b$.

Case 3. $3 \leq a = b < c$.

Let G be the graph obtained from the path $P_3\colon x, y, z$ of order 3 by adding c new vertices $v_1, v_2, \ldots, v_{c-a+1}, u_1, u_2, \ldots, u_{a-1}$ and joining each $u_i (1 \leq i \leq a-1)$ with the vertices y and z of P_3; and joining each v_i $(1 \leq i \leq c-a)$ to v_j $(i+1 \leq j \leq c-a+1)$; and also joining each v_i $(1 \leq i \leq c-a+1)$ with the vertices x and z of P_3, thereby producing the graph G as shown in Figure 3.2. Then $S = \{u_1, u_2, \ldots, u_{a-1}\}$ is the set of all extreme vertices of G. Since every outer connected monophonic set of G is a monophonic set of G and by Theorem 1.1, every monophonic set and every outer connected monophonic set of G contain S. It is clear that S is not a monophonic set of G. However, $S_1 = S \cup \{x\}$ is a monophonic set of G, and so $m(G) = a$. Since the subgraph induced by $V - S_1$ is connected, S_1 is an outer connected monophonic set of G, and so $m_{oc}(G) = a$.

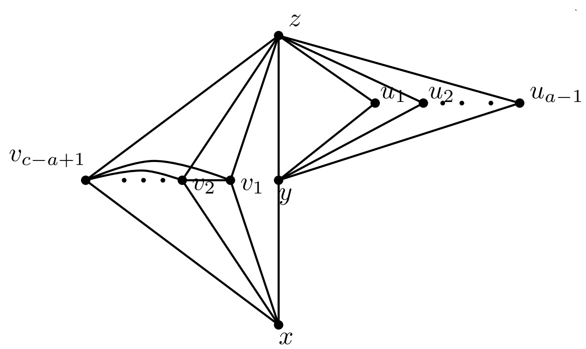

Figure 3.2. G.

Next, we prove that $m^+_{oc}(G) = c$. Clearly $S_2 = S \cup \{v_1, v_2, \ldots, v_{c-a+1}\}$ is an outer connected monophonic set of G. Now, we claim that S_2 is a minimal outer connected monophonic set of G. Assume to the contrary, that S_2 is not a minimal outer connected monophonic set of G. Then there is a proper subset T of S_2 such that T is an outer connected monophonic set of G. Hence there exists a vertex $v \in S_2$ such that $v \notin T$. By Remark 2.4, $v \neq u_i$ $(1 \leq i \leq a-1)$. Therefore, $v = v_j$ for some j $(1 \leq j \leq c-a+1)$. It is easily verified that the vertex v_j does not lie on any monophonic path joining a pair of vertices of T, and so T is not a monophonic set of G, which is a contradiction to T is an outer connected monophonic set of G. Thus S_2 is a minimal outer connected monophonic set of G, and so $m^+_{oc}(G) \geq c$. Next, we prove that $m^+_{oc}(G) = c$. Let S_3 be an outer connected monophonic set of G with $|S_3| \geq c+1$. Necessarily, $S \subseteq S_3$. Observe that for any $v \in \{v_1, v_2, \ldots, v_{c-a+1}\}$, $(S_2 - \{v\}) \cup \{z, y\}$ is not a monophonic set. Thus, S_3 contains S_1 or S_2 as a proper subset, and so S_3 is not a minimal outer connected monophonic set of G. Therefore, $m^+_{oc}(G) = c$.

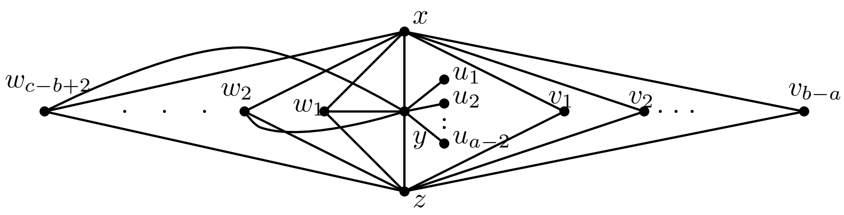

Figure 3.3. G.

<u>*Case 4.*</u> $3 \leq a < b < c$. Let $P_3\colon x, y, z$ be a path of order 3. Let G be the graph obtained from P_3 by adding c new vertices $u_1, u_2, \ldots, u_{a-2}, w_1, w_2, \ldots, w_{c-b+2}, v_1, v_2, \ldots, v_{b-a}$, and joining each u_i $(1 \leq i \leq a-2)$ with the vertex y, and joining each w_i $(1 \leq i \leq c-b+2)$ with the vertices x, y and z; and joining each v_i $(1 \leq i \leq b-a)$ with the vertices x and z, thereby producing the graph G as shown in Figure 3.3. Then $S = \{u_1, u_2, \ldots, u_{a-2}\}$ is the set of all extreme vertices of G. Since every outer connected monophonic set of G is a monophonic set of G and by Theorem 1.1, every monophonic set and every outer connected monophonic set of G contain S. It is clear that S is not a monophonic set of G. Also, for any vertex $u \in V(G) - S$, $S \cup \{u\}$ is not a monophonic set of G. Clearly, $S_1 = S \cup \{x, z\}$ is a minimum monophonic set of G, and so $m(G) = a$. Since the subgraph induced by $V - S_1$ is not connected, S_1 is not an outer connected monophonic set of G. It is easy to observe that every outer connected monophonic set of G contains $\{v_1, v_2, \ldots, v_{b-a}\}$ and since $S_2 = S_1 \cup \{v_1, v_2, \ldots, v_{b-a}\}$ is a minimum outer connected monophonic set of G, it follows that $m_{oc}(G) = b$.

Clearly, the set $T = S \cup \{w_1, w_2, \ldots, w_{c-b+2}, v_1, v_2, \ldots, v_{b-a}\}$ is an outer connected monophonic set of G. We show that T is a minimal outer connected monophonic set of G. Assume to the contrary, that T is not a minimal outer connected monophonic set of G, then there is a proper subset W of T such that W is an outer connected monophonic set of G. Hence there exists a vertex say $v \in T$ such that $v \notin W$. By Remark 2.4, $v \neq u_i$ for all $i = 1, 2, \ldots, a-2$. Then either $v = v_i$ $(1 \leq i \leq b-a)$ for some i, or $v = w_j$ $(1 \leq j \leq c-b+2)$ for some j. Clearly, the vertex v does not lie on any monophonic path joining a pair of vertices of W, and so W is not a monophonic set of G, which is a contradiction to W is an outer connected monophonic set of G. Thus T is a minimal outer connected monophonic set of G and so $m^+_{oc}(G) \geq |T| = c$. Next, we prove that $m^+_{oc}(G) = c$. Suppose that $m^+_{oc}(G) \geq c+1$. Let T' be a minimal outer connected monophonic set of G with $|T'| \geq c+1$. Then the set T' is of the form $T' = T \cup \{x, y\}$ or $T' = T \cup \{y, z\}$ or $T' = V(G)$. Since T is a proper subset of T', T' is not a minimal outer connected monophonic set of G. Therefore, $m^+_{oc}(G) = c$. $\square$

Corollary 3.2. *Let n be a positive integer. Then there exist connected graphs G and H such that $m_{oc}(G) - m(G) = n$ and $m^+_{oc}(H) - m_{oc}(H) = n$.*

Theorem 3.3. *For any three integers a, b, and n with $3 \leq a \leq n \leq b$, there is a connected graph G with $m_{oc}(G) = a$, $m^+_{oc}(G) = b$, and a minimal outer connected monophonic set of cardinality n.*

Proof. We prove this theorem by considering four cases.
Case 1. $a = n = b$.
Let G be any tree with a end-vertices. Then by Theorem 1.3 and Observation 2.11, $m_{oc}(G) = m^+_{oc}(G) = a$.
Case 2. $a = n < b$.
For the graph G given in Figure 3.2 of Theorem 3.1 (put $c = b$), it is proved that $m_{oc}(G) = a$, $m^+_{oc}(G) = b$, and $S = \{u_1, u_2, \ldots, u_{a-1}, x\}$ is a minimal outer connected monophonic set of cardinality n.
Case 3. $a < n = b$.
For the graph G given in Figure 3.2 of Theorem 3.1 (put $c = b$), it is proved that $m_{oc}(G) = a$, $m^+_{oc}(G) = b$ and $S = \{u_1, u_2, \ldots, u_{a-1}, v_1, v_2, \ldots, v_{b-a+1}\}$ is a minimal outer connected monophonic set of cardinality n.
Case 4. $a < n < b$.
Let $C : z_1, z_2, z_3, z_4$ and $C' : v_1, v_2, v_3, v_4$ be two cycles, each of order 4. Let H be the graph obtained from C and C' by identifying the vertex z_1 in C and the vertex v_1 in C'. Let G be the graph obtained from H by adding b new vertices $u_1, u_2, \ldots, u_{a-2}, y_1, y_2, \ldots, y_{b-n+1}, x_1, x_2, \ldots, x_{n-a+1}$ and joining each u_i $(1 \leq i \leq a-2)$ with the vertex v_1; and joining each y_i $(1 \leq i \leq b-n+1)$ with the vertices z_2 and v_2; and also joining each x_i $(1 \leq i \leq n-a+1)$ with the vertices z_4 and v_4, thereby producing the graph G as shown in Figure 3.4. Then $S = \{u_1, u_2, \ldots, u_{a-2}\}$ is the set of all extreme vertices of G. Since every outer connected monophonic set of G is a monophonic set of G and by Theorem 1.1, every outer connected monophonic set of G contains S. It is clear that S is not an outer connected monophonic set of G. Also for any vertex $u \in V(G) - S$, $S \cup \{u\}$ is not an outer connected monophonic set of G. It is easily verified that $S_1 = S \cup \{z_3, v_3\}$ is an outer connected monophonic set of G, and so $m_{oc}(G) = a$.

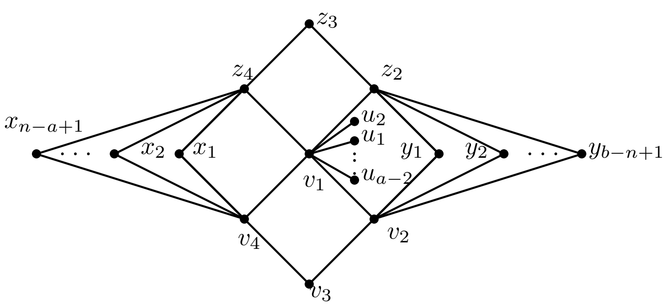

Figure 3.4. G.

Next, we show that $m^+_{oc}(G) = b$. Let $M = S \cup \{y_1, y_2, \ldots, y_{b-n+1}, x_1, x_2, \ldots, x_{n-a+1}\}$. It is clear that M is an outer connected monophonic set of G.

We claim that M is a minimal outer connected monophonic set of G. Assume that M is not a minimal outer connected monophonic set of G. Then there is a proper subset M_1 of M such that M_1 is an outer connected monophonic set of G. Hence there exists a vertex $w \in M$ such that $w \notin M_1$. By Remark 2.4, $w \neq u_i$ $(1 \leq i \leq a-2)$. Then either $w = y_i$ $(1 \leq i \leq b-n+1)$ or $w = x_j$ $(1 \leq j \leq n-a+1)$. If $w = y_i$ $(1 \leq i \leq b-n+1)$ or $w = x_j$ $(1 \leq j \leq n-a+1)$, then the vertex w does not lie on any $x-z$ monophonic path for some $x, z \in M_1$, which is a contradiction to M_1 is an outer connected monophonic set of G. Hence M is a minimal outer connected monophonic set of G, and so $m^+_{oc}(G) \geq b$. Now, we prove that $m^+_{oc}(G) = b$. Suppose that $m^+_{oc}(G) \geq b+1$. Let X be a minimal outer connected monophonic set of G with cardinality $|X| \geq b+1$. Then there exists at least one vertex, say, $v \in X$ such that $v \notin M$. Thus $v \in \{v_1, z_2, z_3, z_4, v_2, v_3, v_4\}$. If $v \in \{z_2, z_3, v_2, v_3\}$, then $X_1 = (M - \{y_1, y_2, \ldots y_{b-n+1}\}) \cup \{v\}$ is an outer connected monophonic set of G and also it is a proper subset of X, which is a contradiction to X is a minimal outer connected monophonic set of G. If $v \in \{z_4, v_4\}$, then $X_2 = (M - \{x_1, x_2, \ldots x_{n-a+1}\}) \cup \{v\}$ is an outer connected monophonic set of G and also it is a proper subset of X, which is a contradiction to X is a minimal outer connected monophonic set of G. If $v = v_1$, then the subgraph induced by $V - X$ is not connected, which is a contradiction to X is a minimal outer connected monophonic set of G. Thus $m^+_{oc}(G) = b$.

Finally, we show that there is a minimal outer connected monophonic set of cardinality n. Let $T = S \cup \{z_3, x_1, x_2, \ldots, x_{n-a+1}\}$. It is clear that T is an outer connected monophonic set of G. We claim that T is a minimal outer connected monophonic set of G. Assume that T is not a minimal outer connected monophonic set of G. Then there is a proper subset T_1 of T such that T_1 is an outer connected monophonic set of G. Hence there exist a vertex $t \in T$ such that $t \notin T_1$. By Remark 2.4 shows that $t \neq u_i$ $(i = 1, 2, \ldots, a-2)$. Then either $t = z_3$ or $t = x_j$ $(1 \leq j \leq n-a+1)$. If $t = z_3$ or $t = x_j$ $(1 \leq j \leq n-a+1)$, then the vertex t is not an internal vertex of any $x-y$ monophonic path for some $x, y \in T_1$, which is a contradiction to T_1 is an outer connected monophonic set of G. Thus T is a minimal outer connected monophonic set of G with cardinality n. Hence the theorem. □

For any connected graph G, $\mathrm{rad}_m(G) \leq \mathrm{diam}_m(G)$. It is shown in [**12**] that every two positive integers a and b with $a \leq b$ are realizable as the monophonic radius and monophonic diameter, respectively, of some connected graph. This theorem can also be extended so that the upper outer connected monophonic number can be prescribed when $\mathrm{rad}_m(G) < \mathrm{diam}_m(G)$.

Theorem 3.4. *For any three positive integers r, d, and $k \geq 2$ with $r < d$, there is a connected graph G such that $\mathrm{rad}_m(G) = r$, $\mathrm{diam}_m(G) = d$, and $m^+_{oc}(G) = k$.*

Proof. Now, let $r = 1$ and $d \geq 2$. Let G be the graph obtained from the cycle $C_{d+2}\colon v_1, v_2, \ldots, v_{d+2}, v_1$ of order $d+2$ by adding $k-2$ new vertices $u_1, u_2, \ldots, u_{k-2}$ to C_{d+2} and joining each vertex $x \in \{u_1, u_2, \ldots, u_{k-2}, v_3, v_4, \ldots, v_{d+1}\}$ to the vertex v_1 of C_{d+2}. The graph G is shown in Figure 3.5. It is easily verified that

$1 \leq e_m(u) \leq d$ for any vertex u in G, $e_m(v_1) = 1$ and $e_m(v_2) = e_m(v_{d+2}) = d$. Thus $\mathrm{rad}_m(G) = r$ and $\mathrm{diam}_m(G) = d$. Let $S = \{u_1, u_2, \ldots, u_{k-2}, v_2, v_{d+2}\}$ be the set of all extreme vertices of G. By Remark 2.4, every minimal outer connected monophonic set of G contains S. It is clear that S is the unique minimal outer connected monophonic set of G, and so $m^+_{oc}(G) = |S| = k$.

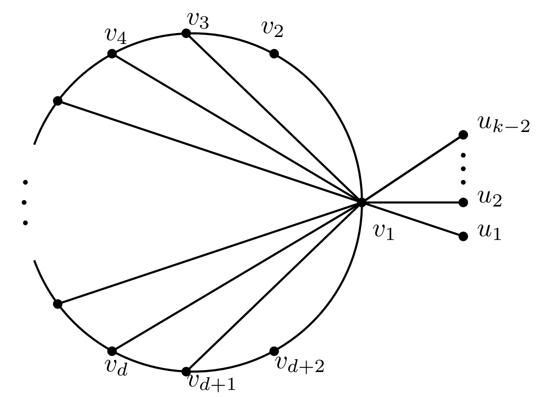

Figure 3.5. G.

Now, let $r \geq 2$ and $r < d$. Let H be the graph obtained from the cycle $C_{r+2}\colon v_1, v_2, \ldots, v_{r+2}, v_1$ of order $r + 2$ and the path $P_{d-r+1}\colon w_0, w_1, \ldots, w_{d-r}$ of order $d-r+1$ by identifying the vertex v_{r+1} in C_{r+2} and the vertex w_0 in P_{d-r+1}, and also joining each vertex $w_i (1 \leq i \leq d-r)$ in P_{d-r+1} with the vertex v_{r+2} in C_{r+2}. Now, let G be the graph obtained from H by adding $k-2$ new vertices $u_1, u_2, \ldots, u_{k-2}$ and joining each vertex u_i $(1 \leq i \leq k-2)$ to the vertex v_{r+2} in H. The graph G is shown in Figure 3.6. It is easily verified that $r \leq e_m(x) \leq d$ for any vertex x in G. Also $e_m(v_{r+2}) = r$ and $e_m(v_1) = e_m(w_{d-r}) = d$. Thus $\mathrm{rad}_m(G) = r$ and $\mathrm{diam}_m(G) = d$. Let $S = \{u_1, u_2, \ldots, u_{k-2}, w_{d-r}\}$ be the set of all extreme vertices of G. By Remark 2.4, every minimal outer connected monophonic set of G contains S. Clearly, S is not an outer connected monophonic set of G. It is easily verified that $S' = S \cup \{v_1\}$ is a minimal outer connected monophonic set of G, $m^+_{oc}(G) \geq k$. The minimal outer connected monophonic sets of G are $S \cup \{x\}$ where $x \in V(C_{r+2}) - \{v_{r+1}, v_{r+2}\}$. By the definition of the upper outer connected monophonic number of a graph G, we have $m^+_{oc}(G) = k$. □

We leave the following problem as an open question.

Problem 3.5. For any three positive integers r, d, and $k \geq 2$ with $r = d$, does there exist a connected graph G with $\mathrm{rad}_m(G) = r$, $\mathrm{diam}_m(G) = d$, and $m^+_{oc}(G) = k$?

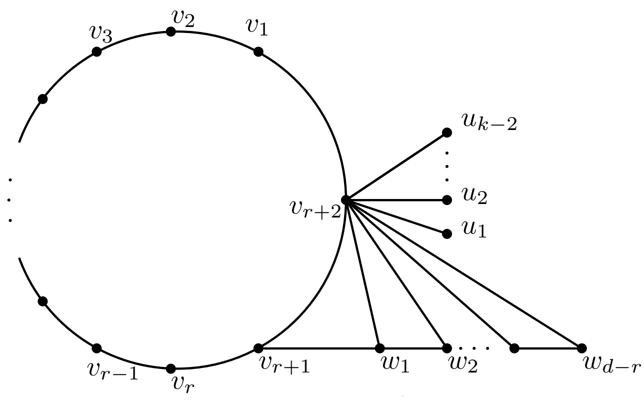

Figure 3.6. G.

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