Acta Math. Univ. Comenianae
Vol. XCII, 4 (2023), pp. 313–320

CONE NORMALITY NOTIONS AND CHARACTERIZATIONS OF BANACH ALGEBRAS

M. A. NEJJARI and J. HLAL

Abstract. In the frame of involutive Banach algebras, the notions of spectral normality, normality, and supernormality are used to characterize hermitian Banach algebras, C^*-algebras, and finite dimensional C^*-algebras, respectively.

1. Introduction

The notion of normal cone is one of the most important notions of convex cone. In [**6**], G. Isac introduced the notion of supernormal cone for optimization reasons. In [**5**], we defined the notion of spectrally normal cone to give some characterizations of strictly real Banach algebras. Here we exploit those notions to give some characterizations of involutive Banach algebras. We start with a comparison of spectral normality and normality in the context of normed algebras. We show that the second notion is not always stronger than the first one, and that these two notions are in general different (Examples 3.4 and 3.5).

Let A be a Banach algebra with involution * and $A_+ = \{x \in A : x = x^*$ and $\mathrm{Sp}(x) \subset \mathbb{R}^+\}$. It is well known that the subset A_+ is a convex cone whenever A is hermitian. We show that it is, moreover, automatically spectrally normal (Proposition 3.6). However, Example 3.7 shows that the spectral normality of the cone A_+ fails to characterize hermitian Banach algebras. Then we consider the cone K_0 of finite sums of squares of self-adjoint elements. In general, this cone is different from A_+ and seems to be more compatible with the normality notions to give such characterizations. Indeed, a Banach algebra with involution is hermitian if and only if the cone K_0 is spectrally normal. Moreover, it becomes a C^*-algebra if K_0 is normal, and it is finite dimensional if normality is replaced by supernormality (Theorem 3.12). In the commutative case, we characterize C^*-algebras (resp. finite dimensional C^*-algebras) among involutive Banach algebras, in general, by the normality (resp. the supernormality) of their cone K_0. Finally, we exploit the normality and the spectral normality of the cone P_+ of positive elements relatively to numerical range to give some Vidav-Palmer type results.

Received January 31, 2022; revised October 25, 2023.
2020 *Mathematics Subject Classification.* Primary 46H05; Secondary 46K05.
Key words and phrases. Hermitian algebras; C^*-algebras; normality notions; positive elements.

2. Preliminaries

Let $(A, \|.\|)$ be a unitary complex Banach algebra with an involution *. We denote by $\mathrm{Sym}\,(A)$, A_+, K_0, $\mathrm{Her}(A)$, and P_+, respectively, the subsets of self-adjoint elements, positive elements, finite sums of squares of self-adjoint elements, hermitian elements, and positive elements relatively to numerical range:

$$\mathrm{Sym}\,(A) = \{x \in A : x = x^*\},$$

$$A_+ = \{x \in A : x = x^* \text{ and } \mathrm{Sp}(x) \subset \mathbb{R}^+\},$$

$$K_0 = \left\{ \sum_{\text{finite}} h^2 : h \in \mathrm{Sym}\,(A) \right\},$$

$$\mathrm{Her}(A) = \{x \in A : V(x) \subset \mathbb{R}\},$$

and

$$P_+ = \left\{x \in A : V(x) \subset \mathbb{R}^+\right\},$$

where $V(x) = \{f(x) : f \in A' \text{ and } \|f\| = f(1) = 1\}$ denotes the numerical range of x with A' the topological dual space of A.

For $a \in A$, by $\mathrm{Sp}\,(a)$, $\rho\,(a)$, $\nu\,(a)$, and $p_A\,(a)$, we denote the spectrum, the spectral radius, the numerical radius, and the Pták function of a, respectively. An involutive algebra A is said to be hermitian if $\mathrm{Sp}(a) \subset \mathbb{R}$ for every $a \in \mathrm{Sym}(A)$.

A subset K of A is called a convex cone if it satisfies $K + K \subset K$, $\lambda K \subset K$ for every $\lambda > 0$. Given a convex cone $K \subset A$, the relation $x \leq y$ iff $y - x \in K$ defines a partial order on A. A convex cone K is said to be: *normal* if there exists $\alpha > 0$ such that $\|x\| \leq \alpha\,\|y\|$ whenever $0 \leq x \leq y$, *spectrally normal* if there exists $\alpha > 0$ such that $\rho\,(x) \leq \alpha\rho\,(y)$ whenever $0 \leq x \leq y$, and *supernormal* if there exists $f \in A'$ such that $\|x\| \leq f\,(x)$ for every $x \in K$.

3. Some characterizations of involutive Banach algebras via cone normality notions

In the context of Banach algebras, the normality seems to be stronger than the spectral normality. This is particularly true when the cone K is stable by product ($K.K \subset K$). We will see later that this is not always the case in general. In the following proposition, we give a property which is equivalent to normality and which we will use in the sequel.

Proposition 3.1. *Let K be a convex cone in A. The following two statements are equivalent:*

(i) *K is normal.*
(ii) *There exists $\mu > 0$ such that $\|x + y\| \geq \mu\,\|x\|$ for every x, y in K.*

Proof. (i) $\Longrightarrow$ (ii). Suppose that K is normal. There exists $\alpha > 0$ such that $\|x\| \leq \alpha\,\|y\|$ whenever $0 \leq x \leq y$. Since $x \leq x + y$ for every x, y in K, one gets $\|x\| \leq \alpha\,\|x + y\|$. Hence $\|x + y\| \geq \mu\,\|x\|$ for every x, y in K with $\mu = \frac{1}{\alpha}$.

(ii) $\Longrightarrow$ (i). Suppose that there exists $\mu > 0$ such that $\|x+y\| \geq \mu \|x\|$ for every x, y in K. Let x, y in K such that $0 \leq x \leq y$. Then $y - x \in K$ and $\|x\| \leq \frac{1}{\mu} \|x + y - x\| = \frac{1}{\mu} \|y\|$. □

Remark 3.2. By the same arguments used in the previous proposition, we can establish that the following two properties are equivalent:

(i) K is spectrally normal.
(ii) There exists $\alpha > 0$ such that $\rho(x+y) \geq \alpha\rho(x)$ for every $x, y \in K$.

Proposition 3.3.

(i) *A normal cone that is stable by product is spectrally normal.*
(ii) *The closure* $\overline{K}$ *of a normal cone* K *is normal.*

Proof. (i) Let K be a normal cone, there exists $\alpha > 0$ such that for every x, y in K, $\|x\| \leq \alpha \|y\|$ whenever $0 \leq x \leq y$. Let x, y in A such that $0 \leq x \leq y$. Since K is assumed to be stable by product, we have $y^2 - x^2 = y(y-x) - (y-x)x \in K$, and therefore, $0 \leq x^2 \leq y^2$. Now by iteration, one gets $\left\|x^{2^n}\right\|^{\frac{1}{2^n}} \leq \alpha^{\frac{1}{2^n}} \left\|y^{2^n}\right\|^{\frac{1}{2^n}}$ for every $n \in \mathbb{N}$. We conclude by going to limit.

(ii) Let x, y in $\overline{K}$. There exist $(x_n)_n$ and $(y_n)_n$ in K such that $x = \lim_n x_n$ and $y = \lim_n y_n$. Since K is assumed to be normal, there exists $\alpha > 0$ such that $\|x_n + y_n\| \geq \alpha \|x_n\|$ for every n. Hence $\|x+y\| \geq \alpha \|x\|$ for all x, y in $\overline{K}$. We conclude by Proposition 3.1. □

Now, by examples, we show that the two notions of normality and spectral normality are in general different and that the stability by product assumed in (i) of the previous proposition is not superfluous.

Example 3.4. Consider the algebra $A = C^1[0,1]$ endowed with the usual operations and involution and with the norm defined by $\|f\| = \operatorname{Sup}_{x\in[0,1]} |f(x)| + \operatorname{Sup}_{x\in[0,1]} |f'(x)|$. The cone of positive elements $A_+ = \{f \in A : f(x) \in \mathbb{R}_+\}$ is spectrally normal, but not normal. To see that A_+ is not normal, suppose that there exists $\alpha > 0$ such that for every $f, g \in A$, $\|f\| \leq \alpha \|g\|$ whenever $0 \leq f \leq g$. Let $n \in \mathbb{N}$ and $f_n, g_n \in A$ be defined by $f_n(t) = \exp(nt)$ and $g_n(t) = \exp(t) + \exp(n)$ for all $t \in [0,1]$. It is clear that $0 \leq f_n \leq g_n$ for every $n \in \mathbb{N}$. By the above assumption, it follows that $\|f_n\| - \alpha \|g_n\| \leq 0$ for every $n \in \mathbb{N}$. But since $\|f_n\| - \alpha \|g_n\| = (n+1)\,\mathrm{e}^n - \alpha(2\mathrm{e} + \mathrm{e}^n) = (n+1-\alpha)\,\mathrm{e}^n - 2\alpha\mathrm{e}$ and $\lim_n (n+1-\alpha)\,\mathrm{e}^n = +\infty$, there exists $n \in \mathbb{N}$ such that $\|f_n\| - \alpha \|g_n\| > 0$, that is a contradiction.

Example 3.5. Let $A = M_2(\mathbb{C})$ be the algebra of a 2×2 square matrices endowed with the usual operations and the matricial norm $\|(a_{ij})\|_1 = \operatorname{Max}_{1\leq j\leq 2} \sum_{i=1}^2 |a_{ij}|$. We consider the subset K of A defined by

$$K = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b, c \in \mathbb{R} \text{ and } b \geq \max(|a|, |c|) \right\}.$$

K is a normal convex cone which is not spectrally normal. Indeed, let $\Phi = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$, $\Psi = \begin{pmatrix} a' & 0 \\ b' & c' \end{pmatrix} \in K$ and $\lambda \geq 0$. Since $\lambda b \geq \max(\lambda |a|, \lambda |c|)$ and

$$\begin{aligned} b + b' &\geq \max\left(|a| + |a'|, |c| + |c'|\right) \\ &\geq \max\left(|a + a'|, |c + c'|\right), \end{aligned}$$

we have $\lambda\Phi$, $\Phi + \Psi \in K$, and so K is a convex cone. If $\Phi \leq \Psi$, then $b' - b \geq 0$, and hence $\|\Phi\| \leq 2\|\Psi\|$ since $\|\Phi\| = |a| + b$ and $\|\Psi\| = |a'| + b'$. Now, suppose that there exists $\alpha > 0$ such that $\rho(\Phi) \leq \alpha\rho(\Psi)$ whenever $0 \leq \Phi \leq \Psi$. For $\Phi = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\Psi = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}$, we have $0 \leq \Phi \leq \Psi$ and $\rho(\Phi) = 1$, $\rho(\Psi) = 0$, that is a contradiction.

Proposition 3.6.

(i) *If A_+ is a convex cone, then it is automatically spectrally normal.*
(ii) *The cone P_+ is normal and spectrally normal.*

Proof. (i) Suppose that A_+ is a convex cone. For every x, y in A_+, one has

$$\rho(x + y) - x = \rho(x + y) - (x + y) + y \in A_+.$$

It follows that $\operatorname{Sp}(\rho(x + y) - x) \subset \mathbb{R}_+$, and hence $\rho(x + y) \geq \rho(x)$.

(ii) Let $x, y \in P_+$ be such that $x \leq y$. For $f \in A'$ such that $\|f\| = f(1) = 1$, we have $f(x) \leq f(y)$. Then $\nu(x) \leq \nu(y)$, where $\nu(x) = \max\{\lambda : \lambda \in V(x)\}$ is the numerical radius of x. Now, by [**3**, p. 56, Theorem 14], we have

$$\frac{1}{\mathrm{e}}\|x\| \leq \nu(x) \leq \nu(y) \leq \|y\|.$$

This proves that P_+ is normal. To see that P_+ is also spectrally normal, it suffices to use the fact that $\rho(h) = \|h\|$ for every $h \in \operatorname{Her}(A)$ [**3**, p. 57, Theorem 17]. □

In any hermitian Banach algebra, the subset A_+ is a convex cone which is spectrally normal (Proposition 3.6). The converse is not true as the following example shows.

Example 3.7. Consider the algebra $A = \mathbb{C}^2$ endowed with the involution, the multiplication, and the norm defined by

$$\begin{aligned} (\alpha, \beta)^* &= \left(\overline{\beta}, \overline{\alpha}\right), \\ (\alpha, \beta)(\alpha', \beta') &= (\alpha\alpha', \beta\beta'), \\ \|(\alpha, \beta)\| &= \max(|\alpha|, |\beta|) \quad \text{for every } (\alpha, \beta) \in A. \end{aligned}$$

Then $\operatorname{Sym}(A) = \{(z, \overline{z}) : z \in \mathbb{C}\}$ and $A_+ = \{(x, x) : x \in \mathbb{R}_+\}$ since for every $(z_1, z_2) \in A$, $\operatorname{Sp}((z_1, z_2)) = \{z_1, z_2\}$. The subset A_+ is a convex cone which is normal and spectrally normal, but the algebra A is not hermitian. To see that A_+ is normal, let $a = (x, x)$ and $b = (y, y)$ in A_+ be such that $a \leq b$. Then $0 \leq x \leq y$, and so $\|a\| \leq \|b\|$ since $\|a\| = x$ and $\|b\| = y$.

Now, consider the cone K_0 of finite sums of squares of self-adjoint elements $K_0 = \{ \sum_{\text{finite}} h^2 : h \in \text{Sym}(A) \}$. This cone is generally different from A_+. In the previous example, $K_0 = \text{Sym}(A) = \{(\alpha, \overline{\alpha}) : \alpha \in \mathbb{C}\} \neq A_+$. Using the cone K_0 and the notions of normality, we give a classification of involutive Banach algebras. We start by giving some lemmas that we need in the sequel.

Lemma 3.8 (Ford's Lemma)**.** *Let A be a Banach algebra with involution and let $h \in \text{Sym}(A)$ with $\rho(1-h) < 1$. Then there exists $u \in \text{Sym}(A)$ such that $u^2 = h$.*

Lemma 3.9 ([**8**, Theorem 8.4])**.** *Let A be a Banach algebra with involution. The following are equivalent:*

1. *A has an equivalent C^*-norm.*
2. *The involution is hermitian and there exists $\beta > 0$ such that $\rho(h) \geq \beta \|h\|$ for every $h \in \text{Sym}(A)$.*

Lemma 3.10. *Let $C(K)$ be the algebra of all continuous functions on a compact K, endowed with the supremum norm. If the cone of positive elements of $C(K)$ is supernormal, then $C(K)$ is finite dimensional.*

Before giving the proof of Lemma 3.10, we need the following definition.

Definition 3.11 ([**2**, Definition 16])**.** A locally convex lattice $E(\tau)$ is of type (M) if its topology can be defined by a family B of semi-norms such that for every $p \in B$, we have:

(1) For all $x, y \in E$, $|x| \leq |y| \implies p(x) \leq p(y)$.
(2) For all $x \geq 0$, $y \geq 0$, $p(x \text{ v } y) = p(x) \text{ v } p(y)$, where v denotes the supremum.

Let us also specify that the term "nuclear cone" used by G. Isac in [**6**], or by O. Bahya in [**2**], and the term "supernormal cone" adopted in this work, refer to the same property.

Proof of Lemma 3.10. Let $C_R(K) = \{f : K \to R : f \text{ continuous}\}$. It is a type of (M) lattice. If the cone of positive elements of $C(K)$ is assumed to be supernormal, it is also supernormal in $C_R(K)$. The latter space is separable, it is a nuclear space, according to [**2**, Theorem 3.4.3]. Hence the result since $C_R(K)$ is normed. □

Theorem 3.12. *Let A be a Banach algebra with involution.*

(i) *A is hermitian if and only if K_0 is spectrally normal.*
(ii) *A is a C^*-algebra if and only if K_0 is normal and spectrally normal.*
(iii) *A is a finite dimensional C^*-algebra if and only if K_0 is supernormal and spectrally normal.*

Proof. (i) According to the definition of K_0, we have $u^2 + v^2 \geq u^2$ for every u, v in $\text{Sym}(A)$. Since the cone K_0 is spectrally normal, then (see also Remark 3.2) there exists $\alpha > 0$ such that $\rho(u^2 + v^2) \geq \alpha\rho(u^2)$ for all $u, v \in \mathrm{Sym}(A)$. Let $x = h + \mathrm{i}k$, where $h, k \in \mathrm{Sym}(A)$ and $hk = kh$, then we have

$$\rho(xx^*) = \rho(h^2 + k^2) \geq \frac{\alpha}{2}[\rho(h^2) + \rho(k^2)]$$
$$\geq \frac{\alpha}{4}[\rho(h) + \rho(k)]^2 \geq \frac{\alpha}{4}[\rho(x)]^2.$$

Replacing x by x^n in the last inequality and using the normality of x, we obtain

$$\rho^2(x) \geq \rho(xx^*) = \left[\rho\left(x^n(x^n)^*\right)\right]^{\frac{1}{n}} \geq \left[\frac{\alpha}{4}(\rho(x^n))^2\right]^{\frac{1}{n}} \geq \left(\frac{\alpha}{4}\right)^{\frac{1}{n}} \rho^2(x).$$

Whence $\rho^2(x) = \rho(xx^*)$ for every normal element x in A. Now let $h \in \mathrm{Sym}(A)$. If $a + \mathrm{i}b \in \mathrm{Sp}(h)$ with a and b real, put $z = h + \mathrm{i}t$ with t a real number. Then z is a normal element of A such that $a + \mathrm{i}(b + t) \in \mathrm{Sp}(z)$. Since $zz^* = h^2 + t^2$, it follows that $a^2 + (b + t)^2 \leq \rho^2(z) = \rho(zz^*) \leq \rho^2(h) + t^2$ for every $t \in \mathbb{R}$. Whence $a^2 + b^2 + 2bt \leq \rho^2(h)$ for every $t \in \mathbb{R}$. This implies that $b = 0$ and prove that A is hermitian. Conversely, if A is hermitian, then $K_0 \subset A_+$, and by Proposition 3.6, K_0 is spectrally normal.

(ii) Observe first that according to (i), A is hermitian. Let $h \in \mathrm{Sym}(A)$ be such that $\rho(h) < 1$. By Ford's Lemma (Lemma 3.8), $\mathrm{e}+h$ and $\mathrm{e}-h$ are both squares of hermitian element, and so $0 \leq h+\mathrm{e} \leq 2\mathrm{e}$, where "$\leq$ " is the order associated to the cone K_0. This cone being normal, there exist $\alpha > 0$ such that $\|h + \mathrm{e}\| \leq 2\alpha\|\mathrm{e}\|$, and so $\|h\| \leq (2\alpha + 1)\|\mathrm{e}\|$. Now, by standard technics, one gets that $\|h\| \leq \beta\rho(h)$ for every $h \in \mathrm{Sym}(A)$ and for $\beta = (2\alpha + 1)\|\mathrm{e}\|$. We conclude by Lemma 3.9.

(iii) Suppose that K_0 is supernormal and spectrally normal. Then it is normal and spectrally normal. It follows, According to (ii) it follows that A is a C^*-algebra for an equivalent norm, and then $K_0 = A_+$ since any positive element of a C^*-algebra is a square of an hermitian element [**4**, p. 15, Proposition 1]. To get that A is finite dimensional, it suffices to show, that for every $h \in \mathrm{Sym}(A)$, the unitary closed subalgebra A_h generated by h is finite dimensional [**1**, p. 72, Theorem 4]. Let $h \in \mathrm{Sym}(A)$. Since $(A_h)_+ = A_+ \cap A_h$, the cone $(A_h)_+$ is supernormal. The result is then obtained by applying Lemma 3.10 to the algebra A_h. Indeed, the latter being a unitary and commutative C^*-algebra, it is isomorphic to the algebra $C(K)$ of all continuous functions on some compact space K [**4**, p. 11, Theorem 1]. □

If the algebra A is assumed to be commutative, then K_0 is stable by product and therefore, according to Proposition 3.3, the normality of K_0 is sufficient to characterize C^*-algebras among involutive Banach algebras.

Theorem 3.13. *Let A be a commutative Banach algebra with involution.*

(i) *A is a C^*-algebra if and only if K_0 is normal.*
(ii) *A is a finite dimensional C^*-algebra if and only if K_0 is supernormal.*

Using [**2**, Proposition 2.3], next we give some applications of the previous theorem.

Theorem 3.14. *Let A be a commutative Banach algebra with continuous involution. Then:*

(i) *A is a C^*-algebra if and only if $\left\|h^2+k^2\right\| \geq \left\|h^2\right\|$ for every $h,k \in \mathrm{Sym}(A)$.*
(ii) *A is a finite dimensional C^*-algebra if and only if there exists $\alpha > 0$ such that $\left\|h^2+k^2\right\| \geq \alpha \left\|h^2\right\| + \left\|k^2\right\|$ for every $h,k \in \mathrm{Sym}(A)$.*

Proof. Observe that since the binomial series for $(1-t)^{\frac{1}{2}}$ converges uniformly and absolutely on $-1 \leq t \leq 1$, if $\|1-h\| \leq 1$, then h has a square root, the root being given by the binomial series in $(1-h)$ [**7**, Remark 3.2] (see also comment before [**7**, Theorem 2.1]). But, since involution is supposed to be continuous, the sub-algebra $\mathrm{Sym}(A)$ is closed and the square root belongs to $\mathrm{Sym}\,(A)$.

(i) Suppose that $\left\|h^2+k^2\right\| \geq \left\|h^2\right\|$ (for all $h,k \in \mathrm{Sym}(A)$). According to Theorem 3.13, we have to prove that K_0 is normal. If we prove that $K_0 = \left\{h^2 : h \in \mathrm{Sym}\,(A)\right\}$, then the above hypothesis is not other than the normality of K_0 (Proposition 3.1). But the proof of this last equality amounts to proving that the sum of two squares of $\mathrm{Sym}(A)$ is also a square of $\mathrm{Sym}(A)$. So, let x,y be two squares in $\mathrm{Sym}(A)$ such that $\|x\|, \|y\| \leq 1$. Then, by the remark above, $1-x, 1-y$ are squares in $\mathrm{Sym}(A)$, and so $\|1-x\|, \|1-y\| \leq 1$ since $\left\|h^2+k^2\right\| \geq \left\|h^2\right\|$ for all $h,k \in \mathrm{Sym}(A)$. Now we have

$$\left\|1-\frac{x+y}{2}\right\| \leq \frac{1}{2}\left(\|1-x\| + \|1-y\|\right) \leq 1.$$

Hence $x+y$ is a square in $\mathrm{Sym}(A)$. Now let x,y be two squares in $\mathrm{Sym}\,(A)$, and take $X = \frac{x}{\|x\|+\|y\|}$ and $Y = \frac{y}{\|x\|+\|y\|}$. Then X,Y are two squares in $\mathrm{Sym}\,(A)$ such that $\|X\|, \|Y\| \leq 1$. According to the above, $X+Y$ is a square in $\mathrm{Sym}\,(A)$, and then it is also for $x+y$.

(ii) It is clear that the condition assumed in (ii) implies that $\left\|h^2+k^2\right\| \geq \left\|h^2\right\|$ for every $h,k \in \mathrm{Sym}\,(A)$. Then by (i), A is a C^*-algebra and $K_0 = \left\{h^2, h \in \mathrm{Sym}(A)\right\}$. By [**2**, Proposition 3.2], the cone K_0 is nuclear, i.e., supernormal. We conclude by Theorem 3.13. □

Since the cone P_+ is normal and spectrally normal, some results of Vidav-Palmer type can be stated as follows.

Theorem 3.15. *Let A be a Banach algebra with involution. If $K_0 \subset P_+$, then A is a C^*-algebra.*

Corollary 3.16. *Let A be a Banach algebra with involution. The following assertions are equivalents:*

(i) *A is a C^*-algebra.*
(ii) *$f(h^2) \subset \mathbb{R}$ for every $h \in \mathrm{Sym}(A)$ and every $f \in V(A)$.*
(iii) *$\left\|\mathrm{e}^{\mathrm{i}th^2}\right\| = 1$ for every $h \in \mathrm{Sym}(A)$ and every $t \in \mathbb{R}$.*

Proof. According to [**3**, p. 55, Definition 12 and Corollary 13], the assertions (ii) and (iii) are equivalent, and it means that $h^2 \in \mathrm{Her}(A)$ for all $h \in \mathrm{Sym}(A)$.

Let us prove that (i) $\Longrightarrow$ (ii). If A is a C^*-algebra for an equivalent norm, then $\mathrm{Sym}(A) = \mathrm{Her}(A)$, and so $h^2 \in \mathrm{Her}(A)$ for every $h \in \mathrm{Sym}(A)$. Conversely, if $h^2 \in \mathrm{Her}(A)$ for every $h \in \mathrm{Sym}(A)$. By [**3**, p. 53, Proposition 6], we get $\mathrm{Sp}(h^2) \subset \mathbb{R}$, and so $\mathrm{Sp}(h) \subset \mathbb{R}$. Now, by [**3**, p. 206, Lemma 3], we obtain $V(h^2) = coSp(h^2) \subset \mathbb{R}_+$. This proves that $K_0 \subset P_+$. We conclude with Theorem 3.15. □

References

1. Aupetit B., *Propriety spectrales des algebres de Banach*, Lecture Notes in Math. 735, Springer-Verlag, Berlin-Heidelberg-New York, 1979.

2. Bahya A. O., *Etude des cônes nucléaires*, Ann. Sci. Math. Québec **15** (1991), 123–133.

3. Bonsall F. F. and Duncan J., *Complete Normed Algebra*, Ergeb. Math. Grenzgeb. (2), vol. 80, Springer Verlag, 1973.

4. Dixmier J., *C*-Algebras*, University of Paris VI. Volume 15, 1977.

5. El Kinani A., Nejjari M. A., Oudadess M., *Normal cones and strictly real algebraic structure*, Rev. Acad. Ci. Exactas Fis. Quim. Nat. Zaragoza II. Ser. **60** (2005), 91–97.

6. Isac G., *Supernormal cones and fixed point theory*, Rocky Mountain J. Math. **17** (1987), 219–226.

7. Kelley J. L. and Vaught R. L., *The positive cone in Banach algebras*, Trans. Amer. Math. Soc. **74** (1953), 44–55.

8. Pták V., *Banach algebras with involution*, Manuscripta Math. **6** (1972), 245–290.

M. A. Nejjari, Centre Régional des Métiers de l'Education et de la Formation (CRMEF), Oujda, Morocco,
e-mail: `m_a_nejjari@hotmail.com`

J. Hlal, Centre Régional des Métiers de l'Education et de la Formation (CRMEF), Oujda, Morocco,
e-mail: `jamalhilal77@gmail.com`