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BOUNDEDNESS OF MARCINKIEWICZ INTEGRAL WITH ROUGH KERNEL AND THEIR COMMUTATOR ON WEIGHTED HERZ SPACE WITH VARIABLE EXPONENT

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Abstract. In this paper, we study the boundedness of Marcinkiewicz integral with rough kernel and their commutators generated by Marcinkiewicz integral and $\mathrm{Lip}_\gamma(\mathbb{R}^n)$ function on weighted Herz spaces with variable exponent.

1. Introduction

Variable exponent spaces have become a popular research area in harmonic analysis in the last few decades due to their diverse applications, as highlighted in [**6, 9, 21**]. Variable exponents have been studied extensively in Lebesgue, Hardy, Herz, and Morrey spaces, yielding fundamental results in both pure and applied mathematics, such as in magnetorheology, fluid modeling, and image restoration. The main challenge in proving theorems in this field is establishing the boundedness of the Marcinkiewicz operator on variable Lebesgue spaces and other spaces. Necessary conditions and critical studies on variable exponents have been obtained, including log-Hölder conditions and Muckenhoupt type conditions, as mentioned in the reports [**2, 3, 4, 5, 10, 17, 18, 24, 26**].

Assume that $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $\mathrm{d}\sigma$. Let $\Omega \in L^q(\mathbb{S}^{n-1})$ with $1 < q \leq \infty$ be a homogeneous function of degree zero

$$\int_{\mathbb{S}^{n-1}} \Omega(x')\mathrm{d}\sigma(x') = 0, \tag{1}$$

where $x' = x/|x|$ for any $x \neq 0$.

The Marcinkiewicz integral operator is defined as

$$\mu_\Omega(g)(x) = \left(\int_0^\infty |F_{\Omega,t}(g)(x)|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2}, \tag{2}$$

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where

$$F_{\Omega,t}(g)(x)=\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}g(y)\mathrm{d}y. \tag{3}$$

Suppose $b\in \mathrm{Lip}_\gamma$ is a locally integrable function on $\mathbb{R}^n$. The commutators generated by the Marcinkiewicz integral μ_Ω and b are defined by

$$[b,\mu_\Omega](g)(x)=\left(\int_0^\infty\left|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]g(y)\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{1/2}. \tag{4}$$

In 1938, Marcinkiewicz presented the Marcinkiewicz integral in [**20**]. In 1958, Stein incorporated the Marcinkiewicz integral operator μ_Ω, which corresponds to the Littlewood-Paley g-function. Stein demonstrated that if Ω is continuous and satisfies a $\mathrm{Lip}_\alpha(0<\alpha\le 1)$ condition on $\mathbb{S}^{n-1}$, then μ_Ω is of type (p,p) for $1<p\le 2$ and of weak type $(1,1)$ in [**22**]. In [**23**], Torchinsky and Wang discussed the boundedness of the commutator generated by the Marcinkiewicz integral μ_Ω and $BMO(\mathbb{R}^n)$ function on the Lebesgue space $L^p(\mathbb{R}^n)$, in order to prove its boundedness.

Cui and Li established the boundedness of the Marcinkiewicz integral with rough kernel and its commutators. They proved that it is bounded on the generalized weighted Morrey space for the commutator generated by a $BMO(\mathbb{R}^n)$ function [**8**]. To prove the boundedness of Marcinkiewicz integrals and their commutators on the space mentioned in Definition 1.4, we rely on the previous results and the following ones.

In their work [**15**], Izuki and Noi studied the boundedness of fractional integral operators on weighted Herz spaces with variable exponents. They also applied weighted Herz spaces with variable exponents to their investigations. Additionally, Abdalmonem and Scapellato studied the boundedness of the homogeneous fractional integral operator and showed that it is bounded on weighted Herz spaces with variable exponents [**1**].

In this work, we aim to explain the boundedness of the Marcinkiewicz integral with rough kernel and its commutator on weighted Herz spaces with variable exponent $\dot{K}^{\alpha,q}_{p(\cdot)}(w)$.

Let E be a Lebesgue measurable set in $\mathbb{R}^n$ with measure $|E|>0$, χ_E means its characteristic function. Now we should review some definitions.

Definition 1.1 ([**6**])**.** If $p(\cdot)\colon E\to[1,\infty)$ is a measurable function, the variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(E)=\left\{g \text{ is measurable}: \int_E\left(\frac{|g(x)|}{\eta}\right)^{p(x)}\mathrm{d}x<\infty \text{ for some constant } \eta>0\right\}. \tag{5}$$

The space $L^{p(\cdot)}_{\mathrm{loc}}(E)$ is defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(E)=\left\{g \text{ is measurable}: g\in L^{p(\cdot)}(K) \text{ for all compact } K\subset E\right\}. \tag{6}$$

The Lebesgue space $L^{p(\cdot)}(E)$ is a Banach space with the norm defined by

$$\|g\|_{L^{p(\cdot)}(E)} = \inf\Big\{\eta > 0 : \int_E \left(\frac{|g(x)|}{\eta}\right)^{p(x)} \mathrm{d}x \le 1\Big\}. \tag{7}$$

We denote $p_- = \operatorname{ess\,inf}\{p(x) : x \in E\}$, $p_+ = \operatorname{ess\,sup}\{p(x) : x \in E\}$, then $\mathcal{P}(E)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$.

Definition 1.2 ([**13**, Definition 2.4])**.** Let w be a weight function, and let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. We define the weighted Lebesgue space with variable exponent $L^{p(\cdot)}(w)$ to be the set of all complex-valued measurable functions g such that $gw^{1/p(\cdot)} \in L^{p(\cdot)}(\mathbb{R}^n)$. The space $L^{p(\cdot)}(w)$ is a Banach space equipped with the norm

$$\|g\|_{L^{p(\cdot)}(w)} = \|gw^{1/p(\cdot)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{8}$$

Definition 1.3 ([**7**, Theorem 2.4])**.** If w be a weight, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and the maximal operator is bounded on $L^{p(\cdot)}(w)$ and $L^{p'(\cdot)}(w^{-1})$, we state that $(p(\cdot), w)$ is an M-pair.

Next, we define the weighted Herz space with variable exponent. For all $k \in \mathbb{Z}$, we denote $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $C_k = B_k \backslash B_{k-1}$, $\chi_k = \chi_{C_k}$.

Definition 1.4 ([**15**, Definition 5])**.** Suppose that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < q \le \infty$, $\alpha \in \mathbb{R}$. The homogeneous weighted Herz space with variable exponent $\dot{K}^{\alpha,q}_{p(\cdot)}(w)$ is the collection of $g \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n\backslash\{0\}, w)$ such that

$$\|g\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(w)} := \left(\sum_{k=-\infty}^{\infty} 2^{\alpha q k}\|g\chi_k\|^q_{L^{p(\cdot)}(w)}\right)^{1/q} < \infty. \tag{9}$$

It is easy to see that if $w = 1$, then $\dot{K}^{\alpha,q}_{p(\cdot)}(w) = \dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ is the Herz space with variable exponent defined in [**12**]. If $w = 1$ and $p(\cdot) = p$, then $\dot{K}^{\alpha,q}_{p(\cdot)}(w) = \dot{K}^{\alpha,q}_{p}(\mathbb{R}^n)$ is the classical Herz space introduced in [**19**]. If $p(\cdot) = p$, then $\dot{K}^{\alpha,q}_{p(\cdot)}(w) = \dot{K}^{\alpha,q}_{p}(w)$ is the weighted Herz space introduced in [**17**].

2. Lemmas and Corollaries

This part includes some definitions, corollaries, and lemmas that may be required to prove our principal theorems. We only describe the relevant parts of the results we need. First, we outline the Lebesgue space with variable exponent.

Definition 2.1 ([**16**, Lemma 5])**.** The weighted Banach function space $X(\mathbb{R}^n, W)$ is a Banach function space equipped with the norm $\|g\|_{X(\mathbb{R}^n,W)} := \|gW\|_X$. The associated space of $X(\mathbb{R}^n, W)$ is a Banach function space and equals $X'(\mathbb{R}^n, W^{-1})$.

Definition 2.2 ([**7**, Definition 2.9])**.** Let $0 < \lambda < n$, $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and $\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\lambda}{n}$. A weight ω is said to be an $A_{(p_1(\cdot),p_2(\cdot))}$ weight if there exists $C > 0$ such that

$$\|\omega\chi_B\|_{L^{p_2(\cdot)}}\|\omega^{-1}\chi_B\|_{L^{p_1'(\cdot)}} \leq C|B|^{\frac{n-\lambda}{n}}, \tag{10}$$

holds for all balls $B \subset \mathbb{R}^n$.

Lemma 2.3 ([**25**, Lemma 1.1])**.** *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be a globally log-Hölder continuity, and satisfies:*

(1) $|p(x) - p(y)| \leq \dfrac{C}{-\log(|x-y|)}, \quad x, y \in \mathbb{R}^n,\ |x-y| \leq 1/2,$

(2) $|p(x) - p_\infty| \leq \dfrac{C}{\log(|x| + \mathrm{e})}, \quad x \in \mathbb{R}^n,\ |y| \geq |x|,$

for some real constant p_∞. The set $LH(\mathbb{R}^n)$ consists of all globally log-Hölder continuous functions.

Then the Hard-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ written as

$$Mf(x) = \sup_{B \ni X} \frac{1}{|B|}\int_B |f(y)|\mathrm{d}y,$$

Let $\mathcal{M}$ be the set of all complex-valued measurable functions on $\mathbb{R}^n$ defined in [**15**, Definition 6].

Lemma 2.4 ([**4**, Theorem 4.2])**.** *Suppose that $X \subset \mathcal{M}$ is a Banach function space.*

(1) *(The generalized Hölder's inequality) For all $f \in X$ and $g \in X'$, we have*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\mathrm{d}x \leq \|f\|_X\|g\|_{X'}.$$

(2) *For all $g \in X'$, we have*

$$\|g\|_{X'} = \sup\left\{\left|\int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x\right| : f \in X, \|f\|_X \leq 1\right\}.$$

In particular, space $(X')' = X$.

As an application of generalized Hölder's inequality above, we have the following Lemma.

Lemma 2.5 ([**15**, Lemma 3])**.** *Let X be a Banach function space. We have:*

$$1 \leq \frac{1}{|B|}\|\chi_B\|_X\|\chi_B\|_{X'},$$

holds for all balls B.

Lemma 2.6 ([**6**, Lemma Ǵ,H])**.** *Let X be a Banach function space. If the Hardy-Littlewood maximal operator M is weakly bounded on X, that is*

$$\|\chi_{\{Mg>\lambda\}}\|_X \leq \lambda^{-1}\|g\|_X,$$

and it holds for all $f \in X$ and $\lambda > 0$, then we have

$$\sup_{B:Ball} \frac{1}{|B|} \|\chi_B\|_X \|\chi_B\|_{X'} < \infty.$$

Lemma 2.7 ([**15**, Remark 2]). *If $p_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, by comparing the definition of the weighted Banach function space with the weighted variable Lebesgue space, we can make the following observations:*

(1) *If $X = L^{p_1(\cdot)}(\mathbb{R}^n)$ and $W = w$, then we obtain $L^{p_1(\cdot)}(\mathbb{R}^n, w) = L^{p_1(\cdot)}(w^{p_1(\cdot)})$.*

(2) *If $X = L^{p_1'(\cdot)}(\mathbb{R}^n)$ and $W = w^{-1}$. Then we can use* Lemma 2.4 *to obtain*

$$L^{p_1'(\cdot)}(\mathbb{R}^n, w^{-1}) = L^{p_1'(\cdot)}(w^{-p_1'(\cdot)}) = (L^{p_1(\cdot)}(w^{p_1(\cdot)}))'.$$

Lemma 2.8 ([**14**, Lemma 6]). *Suppose that X is a Banach space. Let M be a bounded on the associated space X'. Then there exists a constant $0 < \delta < 1$ such that*

$$\frac{\|\chi_E\|_X}{\|\chi_B\|_X} \le \left(\frac{|E|}{|B|}\right)^{\delta},$$

holds for all balls B and all measurable sets $E \subset B$.

Lemma 2.9 ([**1**, Lemma 2.11]). *Suppose that $p_1(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and $w^{p_1(\cdot)} \in A_1$. Let M be a bounded on $L^{p_1(\cdot)}(w^{p_1(\cdot)})$ and $L^{p_1'(\cdot)}(w^{-p_1'(\cdot)})$. Then there exist constants $\delta_1, \delta_2 \in (0,1)$ such that*

$$\frac{\|\chi_S\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}}{\|\chi_B\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_1}, \qquad \frac{\|\chi_S\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}{\|\chi_B\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2},$$

hold for all balls B and all measurable sets $S \subset B$, where the set of $p(\cdot)$ satisfying (1) *and* (2) *in* Lemma 2.3 *is denoted by $LH(\mathbb{R}^n)$.*

Lemma 2.10 ([**11**, Theorem A]). *Suppose that $\Omega \in L^r(\mathbb{S}^{n-1})$, $1 < r \le \infty$. Then for every $r' < r < \infty$ and $\omega \in A_{p/r'}$, there is a constant C independent of g such that*

$$\|\mu_\Omega(g)\|_{L^p(\mathbb{R}^n,\omega)} \le C\|g\|_{L^p(\mathbb{R}^n,\omega)}.$$

Lemma 2.11 ([**7**, Lemma 2.13]). *Assume that p_0, q_0, $1 < p_0 \le q_0 < \infty$, and every $w_0 \in A_{p_0,q_0}$, $\mathcal{F}$ stand for a family of functions*

$$\left(\int_{\mathbb{R}^n} f(x)^{q_0} w_0(x)^{q_0} \mathrm{d}x\right)^{1/q_0} \le \left(\int_{\mathbb{R}^n} g(x)^{q_0} w_0(x)^{q_0} \mathrm{d}x\right)^{1/q_0}, \qquad (f,g) \in \mathcal{F}.$$

Given $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, let us assume that

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}.$$

We define $\sigma \ge 1$, by $(\frac{1}{\sigma}) = (\frac{1}{p_0}) + (\frac{1}{q_0})$. If $w \in A_{p_0,q_0}$ for any M-pair $(p(\cdot), w)$, then

$$\|f\|_{L^{q(\cdot)}(w)} \le C\|g\|_{L^{p(\cdot)}(w)}, \qquad (f,g) \in \mathcal{F},$$

holds for $p_0 = 1$ and the maximal operator is bounded on $L^{p'(\cdot)}(w^{-1})$. We know the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}(w)$ and $L^{p'(\cdot)}(w^{-p'(\cdot)})$

(see [**15**]*). Incorporating* Lemma 2.10 *and* Lemma 2.11*, we obtain the following conclusion.*

Corollary 2.12. *Let $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, $\Omega \in L^r(\mathbb{S}^{n-1})(r \geq 1)$, and $w \in A_{p(\cdot)}$. Then the Marcinkiewicz integral operator μ_Ω is bounded on $L^{p(\cdot)}(w)$. Here $A_{p(\cdot)}$ is the classical Muckenhoupt weight, see* [**15**] *for more information.*

3. Estimate of Marcinkiewicz integral operator

In this section, we examine the boundedness of μ_Ω on $\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})$.

Theorem 3.1. *Suppose that $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, $0 < q_1 \leq q_2 < \infty$, $\lambda > 2$, and $\Omega \in L^r(\mathbb{S}^{n-1})(r \geq 1)$ satisfies* Lemma 2.3*. If $w^{p(\cdot)} \in A_1$ and $-n\delta_1 < \alpha < n\delta_2 - \frac{n}{r}$, where δ_1, δ_2 are the constants in* Lemma 2.9*, then the Marcinkiewicz operator μ_Ω is bounded from $\dot{K}^{\alpha,q_2}_{p_1(\cdot)}(w^{p_1(\cdot)})$ to $\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})$.*

Proof. Let $g \in \dot{K}^{\alpha,q_2}_{p_1(\cdot)}(w^{p_1(\cdot)})$, and let $g_j := g\chi_j$ for each $j \in \mathbb{Z}$. Then we have $g := \sum_{j=-\infty}^{\infty} g_j$, and

$$
\begin{aligned}
\|\mu_\Omega(g)\|^{q_1}_{\dot{K}^{\alpha,q_2}_{p_1(\cdot)}(w^{p_1(\cdot)})} &\leq \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \|\mu_\Omega(g)\chi_k\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\leq C \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \Big(\sum_{j=-\infty}^{k-2} \|\mu_\Omega(g_j)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \Big)^{q_1} \\
&\quad + C \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \Big(\sum_{j=k-1}^{k+1} \|\mu_\Omega(g_j)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \Big)^{q_1} \\
&\quad + C \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \Big(\sum_{j=k+2}^{\infty} \|\mu_\Omega(g_j)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \Big)^{q_1} \\
&=: \Upsilon_1 + \Upsilon_2 + \Upsilon_3.
\end{aligned} \tag{11}
$$

First we deduce Υ_1. We examine

$$
\begin{aligned}
|\mu_\Omega(g_j)(x)| &\leq \Big(\int_0^{|x|} \Big| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} g_j(y) \mathrm{d}y \Big|^2 \frac{\mathrm{d}t}{t^3} \Big)^{\frac{1}{2}} \\
&\quad + \Big(\int_{|x|}^{\infty} \Big| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} g_j(y) \mathrm{d}y \Big|^2 \frac{\mathrm{d}t}{t^3} \Big)^{\frac{1}{2}} \\
&=: \rho_1 + \rho_2.
\end{aligned} \tag{12}
$$

We motion that $x \in A_k$, $y \in A_j$, and $j \leq k-2$. Then $|x-y| \sim |x|$ if

$$
\Big| \frac{1}{|x-y|^2} - \frac{1}{|x|^2} \Big| \leq \frac{|y|}{|x-y|^3}. \tag{13}
$$

By (13) and Minkowski's and generalized Hölder's inequalities, we have

$$\begin{aligned}
\rho_1 &\le C\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-1}}|g_j(y)|\bigg(\int_{|x-y|}^{|x|}\frac{\mathrm{d}t}{t^3}\bigg)^{\frac{1}{2}}\mathrm{d}y\\
&\le C\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-1}}|g_j(y)|\left|\frac{1}{|x-y|^2}-\frac{1}{|x|^2}\right|^{\frac{1}{2}}\mathrm{d}y\\
&\le C\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-1}}|g_j(y)|\frac{|y|^{1/2}}{|x-y|^{3/2}}\mathrm{d}y \qquad (14)\\
&\le C\frac{|2|^{j/2}}{|x|^{n+1/2}}\int_{A_j}|\Omega(x-y)||g(y)|\mathrm{d}y\\
&\le C2^{(j-k)/2}2^{-kn}\|g_j(y)\|_{L^{r'}(\mathbb{R}^n)}\|\Omega(x-y)\|_{L^r(\mathbb{R}^n)}.
\end{aligned}$$

Similarly, let us consider ρ_2. Taking note of $|x-y|\sim|x|$, we can use Minkowski's and generalized Hölder's inequalities to obtain

$$\begin{aligned}
\rho_2 &\le C\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-1}}|g_j(y)|\bigg(\int_{|x|}^{\infty}\frac{\mathrm{d}t}{t^3}\bigg)^{\frac{1}{2}}\mathrm{d}y\\
&\le C\int_{A_j}\frac{\Omega(x-y)}{|x-y|^{n-1}}|g(y)|\frac{1}{|x|}\mathrm{d}y \qquad (15)\\
&\le C2^{-kn}\|g_j(y)\|_{L^{r'}(\mathbb{R}^n)}\|\Omega(x-y)\|_{L^r(\mathbb{R}^n)}.
\end{aligned}$$

So, we have

$$|\mu_\Omega(g_j)(x)|\le C2^{-kn}\|g_j(y)\|_{L^{r'}(\mathbb{R}^n)}\|\Omega(x-y)\|_{L^r(\mathbb{R}^n)}. \qquad (16)$$

If $\Omega\in L^r(\mathbb{S}^{n-1})$, we obtain

$$\begin{aligned}
\|\Omega(x-y)\|_{L^r(\mathbb{R}^n)} &\le \bigg[\int_{|x-y|<r}|\Omega(x-y)|^r\mathrm{d}y\bigg]^{1/r} \qquad (17)\\
&\le C^{\frac{kn}{r}}\|\Omega\|_{L^r(\mathbb{S}^{n-1})}\le C^{\frac{kn}{r}}.
\end{aligned}$$

Therefore, applying generalized Hölder's inequality, we have

$$\begin{aligned}
\|g_j(y)\|_{L^{r'}(\mathbb{R}^n)} &\le |B_j|^{-1/r}\|g_j\omega\|_{L^{p_1(\cdot)}}\|\omega^{-1}\chi_j\|_{L^{p_1'(\cdot)}} \qquad (18)\\
&\le 2^{-jn/r}\|g_j\omega\|_{L^{p_1(\cdot)}}\|\omega^{-1}\chi_j\|_{L^{p_1'(\cdot)}}.
\end{aligned}$$

Now, using (15)–(18), we find that

$$|\mu_\Omega(g_j)(x)|\le C2^{-kn}2^{(k-j)n/r}\|g_j\omega\|_{L^{p_1(\cdot)}}\|\omega^{-1}\chi_j\|_{L^{p_1'(\cdot)}}. \qquad (19)$$

Applying Lemma 2.5 and Lemma 2.6, we have

$$\begin{aligned}
\|\mu_\Omega(g_j)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} &\le C2^{-kn}2^{(k-j)n/r}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\
&\quad\times\|\omega^{-1}\chi_j\|_{L^{p_1(\cdot)}}\|\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\
&\le C2^{-kn}2^{(k-j)n/r}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\
&\quad\times\|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{B_k}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}
\end{aligned}$$

$$
\begin{aligned}
&\le C2^{(k-j)n/r}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\
&\quad\times \|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{B_k}\|^{-1}_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\\
&\le C2^{(k-j)n/r}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\frac{\|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}{\|\chi_{B_k}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}\\
&\le C2^{(j-k)(n\delta_2-n/r)}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}\tag{20}
$$

Thus, we get

$$
\begin{aligned}
\Upsilon_1 &\le C\sum_{k=-\infty}^{\infty}2^{\alpha q_1 k}\bigg(\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_2-n/r)}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\bigg)^{q_1}\\
&\le C\sum_{k=-\infty}^{\infty}\bigg(\sum_{j=-\infty}^{k-2}2^{j\alpha}2^{(j-k)(n\delta_2-\frac{n}{r}-\alpha)}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\bigg)^{q_1}.
\end{aligned}\tag{21}
$$

If $1<q_1<\infty$, take $\frac{1}{q_1}+\frac{1}{q_1'}=1$. Since $n\delta_2-\frac{n}{r}-\alpha>0$, by Hölder's inequality, we have

$$
\begin{aligned}
\Upsilon_1 &\le C\sum_{k=-\infty}^{\infty}\bigg(\sum_{j=-\infty}^{k-2}2^{j\alpha}2^{(j-k)(n\delta_2-\frac{n}{r}-\alpha)}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\bigg)^{q_1}\\
&\le C\sum_{k=-\infty}^{\infty}\bigg(\sum_{j=-\infty}^{k-2}2^{j\alpha q_1}2^{(j-k)(n\delta_2-\frac{n}{r}-\alpha)q_1/2}\|g_j\omega\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\bigg)\\
&\qquad\times\bigg(\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_2-\frac{n}{r}-\alpha)q_1'/2}\bigg)^{q_1/q_1'}\\
&\le C\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2}2^{j\alpha q_1}2^{(j-k)(n\delta_2-\frac{n}{r}-\alpha)q_1/2}\|g_j\omega\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\
&\le C\sum_{j=-\infty}^{\infty}2^{j\alpha q_1}\|g_j\omega\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\sum_{k=j+2}^{\infty}2^{(j-k)(n\delta_2-\frac{n}{r}-\alpha)q_1/2}\\
&\le C\|g\|^{q_1}_{\dot K^{\alpha,q_2}_{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}\tag{22}
$$

If $0<q_1\le 1$, we get

$$
\begin{aligned}
\Upsilon_1 &\le C\sum_{k=-\infty}^{\infty}\bigg(\sum_{j=-\infty}^{k-2}2^{j\alpha}2^{(j-k)(n\delta_2-\frac{n}{r}-\alpha)}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\bigg)^{q_1}\\
&\le C\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2}2^{j\alpha q_1}2^{(j-k)(n\delta_2-\frac{n}{r}-\alpha))q_1}\|g_j\omega\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\
&\le C\sum_{j=-\infty}^{\infty}2^{j\alpha q_1}\|g_j\omega\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\sum_{k=j+2}^{\infty}2^{(j-k)(n\delta_2-\frac{n}{r}-\alpha)q_1}\\
&\le C\|g\|^{q_1}_{\dot K^{\alpha,q_2}_{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}\tag{23}
$$

Next, we consider Υ_2. By Corollary 2.12, we know that μ_Ω is bounded on $L^{p(\cdot)}(w)$. Therefore, we have

$$\begin{aligned}\Upsilon_2 &= C\sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k}\Big(\sum_{j=k-2}^{k+2}\|\mu_\Omega(g_j)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\Big)^{q_1}\\ &= C\sum_{k=-\infty}^{\infty}\Big(\sum_{j=k-2}^{k+2}2^{\alpha(k-j)}2^{\alpha j}\|g_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\Big)^{q_1}\\ &\le C\|g\|^{q_1}_{\dot K^{\alpha,q_2}_{p_1(\cdot)}(w^{p_1(\cdot)})}.\end{aligned}\tag{24}$$

Finally, we calculate Υ_3. Take note of that $x\in A_k$, $y\in A_j$, and $j\ge k+2$, so we have $|y-x|\sim|y|$. Let us examine

$$\begin{aligned}|\mu_\Omega(g_j)(x)| \le& \left(\int_0^{|x|}\left|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}g_j(y)\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{\frac12}\\ &+\left(\int_{|x|}^{\infty}\left|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}g_j(y)\mathrm{d}y\right|^2\frac{\mathrm{d}t}{t^3}\right)^{\frac12}\\ =:&\ \Lambda_1+\Lambda_2.\end{aligned}\tag{25}$$

In a similar way to ρ_1, we have

$$\Lambda_1\le C2^{(k-j)/2}2^{-jn}\|g_j(y)\|_{L^{r'}(\mathbb{R}^n)}\|\Omega(x-y)\|_{L^r(\mathbb{R}^n)}.\tag{26}$$

Similarly, for ρ_2, we have

$$\Lambda_2\le C2^{-jn}\|g_j(y)\|_{L^{r'}(\mathbb{R}^n)}\|\Omega(x-y)\|_{L^r(\mathbb{R}^n)}.\tag{27}$$

Furthermore, if $\Omega\in L^r(\mathbb{S}^{n-1})$, we obtain

$$\|\Omega(x-y)\|_{L^r(\mathbb{R}^n)}\le C2^{jn/r}\|\Omega\|_{L^r(\mathbb{S}^{n-1})}\le C2^{jn/r}.\tag{28}$$

Therefore, we get

$$\begin{aligned}\|g_j(y)\|_{L^{r'}(\mathbb{R}^n)} &\le |B_j|^{-1/r}\|g_j\omega\|_{L^{p_1(\cdot)}}\|\omega^{-1}\chi_j\|_{L^{p_1'(\cdot)}}\\ &\le 2^{-jn/r}\|g_j\omega\|_{L^{p_1(\cdot)}}\|\omega^{-1}\chi_j\|_{L^{p_1'(\cdot)}}.\end{aligned}\tag{29}$$

By a similar argument to (19), we obtain

$$|\mu_\Omega(g_j)(x)|\le C2^{-jn}\|g_j\omega\|_{L^{p_1(\cdot)}}\|\omega^{-1}\chi_j\|_{L^{p_1'(\cdot)}}.\tag{30}$$

Applying Lemma 2.5 and Lemma 2.9, we have

$$\begin{aligned}\|\mu_\Omega(g_j)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} &\le C2^{-jn}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\omega^{-1}\chi_j\|_{L^{p_1(\cdot)}}\|\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\ &\le C2^{-jn}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\ &\quad\times\|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{B_k}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\ &\le C\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_{B_k}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_{B_j}\|^{-1}_{L^{p(\cdot)}(w^{p(\cdot)})}\end{aligned}\tag{31}$$

$$\leq C\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\frac{\|\chi_{B_k}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}}{\|\chi_{B_j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}}$$
$$\leq C2^{(k-j)n\delta_1}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}.$$

Therefore, we get

$$\begin{aligned}\Upsilon_3 &\leq C\sum_{k=-\infty}^{\infty}2^{\alpha q_1 k}\Bigg(\sum_{j=k+2}^{\infty}2^{(k-j)n\delta_1}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\Bigg)^{q_1}\\ &\leq C\sum_{k=-\infty}^{\infty}\Bigg(\sum_{j=k+2}^{\infty}2^{j\alpha}2^{(k-j)(n\delta_1+\alpha)}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\Bigg)^{q_1}.\end{aligned}\tag{32}$$

If $1 < q_1 < \infty$, take $\frac{1}{q_1}+\frac{1}{q_1'}=1$. Since $n\delta_1+\alpha>0$ by Hölder's inequality, we have

$$\begin{aligned}\Upsilon_3 &\leq C\sum_{k=-\infty}^{\infty}\Bigg(\sum_{j=k+2}^{\infty}2^{j\alpha}2^{(k-j)(\alpha+n\delta_1)}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\Bigg)^{q_1}\\ &\leq C\sum_{k=-\infty}^{\infty}\Bigg(\sum_{j=k+2}^{\infty}2^{j\alpha q_1}2^{(k-j)(n\delta_1+\alpha)q_1/2}\|g_j\omega\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\Bigg)\\ &\qquad\times\Bigg(\sum_{j=k+2}^{\infty}2^{(k-j)(n\delta_1+\alpha)q_1'/2}\Bigg)^{q_1/q_1'}\\ &\leq C\sum_{k=-\infty}^{\infty}\sum_{j=k+2}^{\infty}2^{j\alpha q_1}2^{(k-j)(n\delta_1+\alpha)q_1/2}\|g_j\omega\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\ &\leq C\sum_{j=-\infty}^{\infty}2^{j\alpha q_1}\|g_j\omega\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\sum_{k=-\infty}^{j-2}2^{(k-j)(n\delta_1+\alpha)q_1/2}\\ &\leq C\|g\|^{q_1}_{\dot{K}^{\alpha,q_2}_{p_1(\cdot)}(w^{p_1(\cdot)})}.\end{aligned}\tag{33}$$

If $0 < q_1 \leq 1$, we obtain

$$\begin{aligned}\Upsilon_3 &\leq C\sum_{k=-\infty}^{\infty}\Bigg(\sum_{j=k+2}^{\infty}2^{j\alpha}2^{(k-j)(n\delta_1+\alpha)}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\Bigg)^{q_1}\\ &\leq C\sum_{k=-\infty}^{\infty}\sum_{j=k+2}^{\infty}2^{j\alpha q_1}2^{(k-j)(\alpha+n\delta_1)q_1}\|g_j\omega\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\ &\leq C\sum_{j=-\infty}^{\infty}2^{j\alpha q_1}\|g_j\omega\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\sum_{k=-\infty}^{j-2}2^{(k-j)(\alpha-n\delta_2)q_1}\\ &\leq C\|g\|^{q_1}_{\dot{K}^{\alpha,q_2}_{p_1(\cdot)}(w^{p_1(\cdot)})}.\end{aligned}\tag{34}$$

Therefore, the proof is completed. □

4. Lipschitz estimate for the commutator of Marcinkiewicz integral operator

In this section, we examine the boundedness of $[b, \mu_\Omega]$ on $\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})$.

Definition 4.1. For all $0 < \gamma \leq 1$, the Lipschitz space $\mathrm{Lip}_\gamma(\mathbb{R}^n)$ is defined by

$$\mathrm{Lip}_\gamma = \left\{ g : \|g\|_{\mathrm{Lip}_\gamma} = \sup_{x,y\in\mathbb{R}^n; x\neq y} \frac{|g(x)-g(y)|}{|x-y|^\gamma} < \infty \right\}. \tag{35}$$

Let $0 < \gamma \leq 1$, $b \in \mathrm{Lip}_\gamma(\mathbb{R}^n)$, $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$. It is easy to see that $|b(x) - b(y)| \leq \|b\|_{\mathrm{Lip}_\gamma} |x-y|^\gamma$.

Corollary 4.2. *Let $0 < \gamma \leq 1$, $b \in \mathrm{Lip}_\gamma(\mathbb{R}^n)$, and $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ such that it satisfies* Lemma 2.3, *and let $p(\cdot)$ be a variable exponent. Then $[b, \mu_\Omega]$ is bounded on $L^{p(\cdot)}(w^{p(\cdot)})$. This means that there exists a constant $C > 0$ such that*

$$\|[b, \mu_\Omega] g\|_{L^{p(\cdot)}(w^{p(\cdot)})} \leq \|b\|_{\mathrm{Lip}_\gamma} \|g\|_{L^{p(\cdot)}(w^{p(\cdot)})}. \tag{36}$$

Proof.

$$\begin{aligned} |[b, \mu_\Omega] g| &\leq C \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |b(x) - b(y)| |g(y)| \left(\int_{|x-y|}^\infty \frac{\mathrm{d}t}{t^3} \right)^{\frac{1}{2}} \mathrm{d}y \\ &\leq C \|b\|_{\mathrm{Lip}_\gamma} \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |x-y|^\gamma |g(y)| \frac{1}{|x-y|} \mathrm{d}y \\ &\leq C \|b\|_{\mathrm{Lip}_\gamma} \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\gamma}} |g(y)| \mathrm{d}y. \end{aligned} \tag{37}$$

We know that $T_{|\Omega|,\gamma}|g|$ is bounded on $L^{p(\cdot)}(w^{p(\cdot)})$ (see[**1**]), so we obtain

$$\begin{aligned} \|[b, \mu_\Omega] g\|_{L^{p(\cdot)}(w^{p(\cdot)})} &\leq C \|b\|_{\mathrm{Lip}_\gamma} \|T_{|\Omega|,\gamma}|g|\|_{L^{p(\cdot)}(w^{p(\cdot)})} \\ &\leq C \|b\|_{\mathrm{Lip}_\gamma} \|g\|_{L^{p\cdot)}(w^{p(\cdot)})}. \end{aligned} \tag{38}$$

□

Theorem 4.3. *Let $b \in \mathrm{Lip}_\gamma(\mathbb{R}^n)$ with $0 < \gamma \leq 1$, $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, and $\lambda > 2$ satisfying* Lemma 2.3. *Let $\Omega \in L^r(\mathbb{S}^{n-1})$, $0 < q_1 \leq q_2 < \infty$, and $\frac{n}{r} + \gamma - n\delta_2 < \alpha \leq \frac{n}{r} + n\delta_1$, where δ_1 and δ_2 are the constants in* Lemma 2.9. *If $w^{p(\cdot)} \in A_1$, then the commutator of the Marcinkiewicz operator $[b, \mu_\Omega]$ is bounded from $\dot{K}^{\alpha,q_2}_{p_1(\cdot)}(w^{p_1(\cdot)})$ to $\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})$.*

Proof. Suppose that $g \in \dot{K}^{\alpha,q_2}_{p_1(\cdot)}(w^{p_1(\cdot)})$, we have

$$g(x) = \sum_{j=-\infty}^{\infty} g\chi_j(x) = \sum_{j=-\infty}^{\infty} g_j(x),$$

then

$$
\begin{aligned}
\|[b,\mu_\Omega](g)\|^{q_1}_{\dot{K}^{\alpha,q_2}_{p_1(\cdot)}(w^{p_1(\cdot)})} &= \Big[\sum_{k=-\infty}^{\infty} 2^{\alpha q_2 k}\|[b,\mu_\Omega](g)\chi_k\|^{q_2}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\Big]^{q_1/q_2} \\
&\le \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k}\|[b,\mu_\Omega](g_j)\chi_k\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\le C\sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k}\Big(\sum_{j=-\infty}^{k-2}\|[b,\mu_\Omega](g_j)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\Big)^{q_1} \qquad (39)\\
&\quad + C\sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k}\Big(\sum_{j=k-2}^{k+2}\|[b,\mu_\Omega](g_j)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\Big)^{q_1} \\
&\quad + C\sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k}\Big(\sum_{j=k+2}^{\infty}\|[b,\mu_\Omega](g_j)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\Big)^{q_1} \\
&=: \Gamma_1+\Gamma_2+\Gamma_3.
\end{aligned}
$$

Now we estimate Γ_1. Note that when $x\in A_k$, it follows from $j\le k-2$ that $|x-y|\sim|x|$. By utilizing generalized Hölder's inequality, we will examine a function that we can use in our proof

$$
\begin{aligned}
&\|[b,\mu_\Omega](g_j)(x)\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\le C\Big(\int_0^{|x|}\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]g_j(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big)^{1/2} \qquad (40)\\
&\quad + C\Big(\int_{|x|}^{\infty}\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]g_j(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big)^{1/2} \\
&=: \Theta_1+\Theta_2.
\end{aligned}
$$

Noting that $j\le k-2$, then $|x-y|\sim|x|$. Using (13) and Minkowski's inequality, we have

$$
\begin{aligned}
\Theta_1 &\le C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|b(x)-b(y)||g_j(y)|\Big(\int_{|x-y|}^{|x|}\frac{\mathrm{d}t}{t^3}\Big)^{1/2}\mathrm{d}y \\
&\le C\int_{A_j}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|b(x)-b(y)||g_j(y)|\Big|\frac{1}{|x-y|^2}-\frac{1}{|x|^2}\Big|^{1/2}\mathrm{d}y \qquad (41)\\
&\le C\int_{A_j}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|b(x)-b(y)||g_j(y)|\frac{|y|^{1/2}}{|x-y|^{3/2}}\mathrm{d}y \\
&\le C2^{(j-k)/2}2^{-k(n-\gamma)}\|b\|_{\mathrm{Lip}_\gamma}\int_{A_j}|\Omega(x-y)||g_j(y)|\mathrm{d}y.
\end{aligned}
$$

Similarly to the way we considered Θ_1 under the same conditions, we now examine Θ_2

$$
\begin{aligned}
\Theta_2 &\leq C\int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1}}|b(x)-b(y)||g_j(y)|\bigg(\int_{|x|}^{\infty}\frac{\mathrm{d}t}{t^3}\bigg)^{1/2}\mathrm{d}y \\
&\leq C\int_{A_j} \frac{|\Omega(x-y)|}{|x-y|^{n-1}}|b(x)-b(y)||g_j(y)|\frac{1}{|x|}\mathrm{d}y \\
&\leq C\int_{A_j} \frac{|\Omega(x-y)|}{|x-y|^{n}}|b(x)-b(y)||g_j(y)|\mathrm{d}y \\
&\leq C2^{-k(n-\gamma)}\|b\|_{\mathrm{Lip}_\gamma}\int_{A_j}|\Omega(x-y)||g_j(y)|\mathrm{d}y.
\end{aligned}
\tag{42}
$$

Using the generalized Hölder's inequality, we have

$$
\begin{aligned}
|[b,\mu_\Omega](g_j)(x)| &\leq C2^{-k(n-\gamma)}\|b\|_{\mathrm{Lip}_\gamma}\int_{A_j}|\Omega(x-y)||g(y)|\mathrm{d}y \\
&\leq C2^{-k(n-\gamma)}\|b\|_{\mathrm{Lip}_\gamma}\|\Omega(x-y)\|_{L^r(\mathbb{R}^n)}\|g_j\|_{L^{r'}(\mathbb{R}^n)}.
\end{aligned}
\tag{43}
$$

Under generalized Hölder's inequality, we get

$$
\begin{aligned}
\|g_j\|_{L^{r'}} &\leq |B_j|^{-1/r}\|g_j\omega\|_{L^{p_1(\cdot)}}\|\omega^{-1}\chi_j\|_{L^{p_1'(\cdot)}} \\
&\leq 2^{-nj/r}\|g_j\omega\|_{L^{p_1(\cdot)}}\|\omega^{-1}\chi_j\|_{L^{p_1'(\cdot)}}.
\end{aligned}
\tag{44}
$$

Furthermore, if $\Omega\in L^r(\mathbb{S}^{n-1})$ is true according to (17), then we obtain the following inequality

$$
|[b,\mu_\Omega](g_j)(x)| \leq C2^{-k(n-\gamma)}\|b\|_{\mathrm{Lip}_\gamma}2^{(k-j)\frac{n}{r}}\|g_j\omega\|_{L^{p_1(\cdot)}}\|\omega^{-1}\chi_j\|_{L^{p_1'(\cdot)}}.
\tag{45}
$$

Now applying the result above, we have

$$
\begin{aligned}
&\|[b,\mu_\Omega](g_j)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\quad\leq C2^{-k(n-\gamma)}\|b\|_{\mathrm{Lip}_\gamma}2^{(k-j)\frac{n}{r}}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\omega^{-1}\chi_j\|_{L^{p_1(\cdot)}}\|\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\quad\leq C2^{-k(n-\gamma)}2^{(k-j)\frac{n}{r}}\|g_j\omega\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{B_k}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}
\tag{46}
$$

By Lemmas 2.5 and 2.9, we get

$$
\begin{aligned}
&\|[b,\mu_\Omega](g_j)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\leq C2^{k\gamma}2^{(k-j)\frac{n}{r}}\|g_j\omega\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}2^{-kn}\|\chi_{B_k}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\leq C2^{k\gamma}2^{(k-j)\frac{n}{r}}\|g_j\omega\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{B_k}\|^{-1}_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \\
&\leq C2^{k\gamma}2^{(k-j)\frac{n}{r}}\|b\|_{\mathrm{Lip}_\gamma}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\quad\times\|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{B_j}\|^{-1}_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\frac{\|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}{\|\chi_{B_k}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}} \\
&\leq C2^{k\gamma}2^{n\delta_2(j-k)}2^{(k-j)\frac{n}{r}}\|b\|_{\mathrm{Lip}_\gamma}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\quad\times\|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{B_j}\|^{-1}_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}.
\end{aligned}
\tag{47}
$$

Using Lemma 2.6,

$$
\begin{aligned}
\|\chi_{B_j}\|^{-1}_{(L^{p_1(\cdot)}(w^{p(\cdot)}))'} &\le 2^{-\gamma j}\|\chi_{B_j}\|^{-1}_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \\
&\le C2^{-\gamma}2^{-nj}\|\chi_{B_j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\le C2^{-j(\gamma+n)}\|\chi_{B_j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}
\tag{48}
$$

Therefore, we get

$$
\begin{aligned}
&\|[b,\mu_\Omega](g_j)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\le C2^{k\gamma}2^{n\delta_2(j-k)}2^{(k-j)\frac{n}{r}}\|b\|_{\mathrm{Lip}_\gamma}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\quad\times\|\chi_{B_k}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{B_k}\|^{-1}_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \\
&\le C2^{k\gamma}2^{n\delta_2(j-k)}2^{(k-j)\frac{n}{r}}\|b\|_{\mathrm{Lip}_\gamma}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\quad\times\|\chi_{B_k}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}2^{-j(\gamma+n)}\|\chi_{B_j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\le C2^{\gamma(k-j)}2^{-n\delta_2(k-j)}2^{(k-j)\frac{n}{r}}\|b\|_{\mathrm{Lip}_\gamma} \\
&\quad\times\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\left[2^{-jn}\|\chi_{B_k}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{B_j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right] \\
&\le C2^{(j-k)[n\delta_2-\gamma-\frac{n}{r}]}\|b\|_{\mathrm{Lip}_\gamma}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}
\tag{49}
$$

Thus, we obtain

$$
\begin{aligned}
\Gamma_1 &\le C\sum_{k=-\infty}^{\infty}2^{\alpha q_1 k}\|[b,\mu_\Omega](g_j)\chi_k\|^{q_1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\
&\le C\sum_{k=-\infty}^{\infty}2^{\alpha q_1 k}\left(\sum_{j=-\infty}^{k-2}\|[b,\mu_\Omega](g_j)\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\right)^{q_1} \\
&\le C\|b\|^{q_1}_{\mathrm{Lip}_\gamma}\sum_{k=-\infty}^{\infty}2^{\alpha q_1 k}\left(\sum_{j=-\infty}^{k-2}2^{(j-k)[n\delta_2-\gamma-\frac{n}{r}]}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{q_1} \\
&\le C\|b\|^{q_1}_{\mathrm{Lip}_\gamma}\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}2^{\alpha j}2^{(j-k)[n\delta_2-\gamma-\frac{n}{r}+\alpha]}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{q_1}.
\end{aligned}
\tag{50}
$$

Now, when $1 < q_1 < \infty$, take $\frac{1}{q_1}+\frac{1}{q_1'}=1$. Since $n\delta_2-\gamma-\frac{n}{r}+\alpha > 0$, using Hölder's inequality, we find

$$
\begin{aligned}
\Gamma_1 &\le C\|b\|^{q_1}_{\mathrm{Lip}_\gamma}\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}2^{\alpha j}2^{(k-j)(n\delta_2-\gamma-\frac{n}{r}+\alpha)}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{q_1} \\
&\le C\|b\|^{q_1}_{\mathrm{Lip}_\gamma}\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}2^{q_1\alpha j}2^{(k-j)q_1/2(\frac{n}{r}-n\delta_2+\alpha)}\|g_j\omega\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right) \\
&\qquad\times\left(\sum_{j=-\infty}^{k-2}2^{n\delta_2-\gamma-\frac{n}{r}+\alpha)q_1'/2}\right)^{q_1/q_1'}
\end{aligned}
$$

$$\leq C\|b\|^{q_1}_{\mathrm{Lip}_\gamma}\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}2^{q_1\alpha j}2^{(k-j)q_1/2(n\delta_2-\gamma-\frac{n}{r}+\alpha)}\|g_j\omega\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\Big) \tag{51}$$

$$= C\|b\|^{q_1}_{\mathrm{Lip}_\gamma}\sum_{j=-\infty}^{\infty}2^{q_1\alpha j}\|g_j\omega\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\Big(\sum_{k=j+2}^{\infty}2^{(k-j)q_1/2(n\delta_2-\gamma-\frac{n}{r}+\alpha)}\Big)$$

$$= C\|b\|^{q_1}_{\mathrm{Lip}_\gamma}\sum_{j=-\infty}^{\infty}2^{q_1\alpha j}\|g_j\omega\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}$$

$$\leq C\|b\|^{q_1}_{\mathrm{Lip}_\gamma}\|g\|_{\dot K^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}.$$

If $0<q_1\leq 1$, then we get

$$\Gamma_1 \leq C\|b\|^{q_1}_{\mathrm{Lip}_\gamma}\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}2^{\alpha j}2^{(k-j)(n\delta_2-\gamma-\frac{n}{r}+\alpha)}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\Big)^{q_1}$$

$$= C\|b\|^{q_1}_{\mathrm{Lip}_\gamma}\sum_{j=-\infty}^{\infty}2^{q_1\alpha j}\|g_j\omega\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\Big(\sum_{k=j+2}^{\infty}2^{(k-j)q_1/2(n\delta_2-\gamma-\frac{n}{r}+\alpha)}\Big) \tag{52}$$

$$= C\|b\|^{q_1}_{\mathrm{Lip}_\gamma}\sum_{j=-\infty}^{\infty}2^{q_1\alpha j}\|g_j\omega\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}$$

$$\leq C\|b\|^{q_1}_{\mathrm{Lip}_\gamma}\|g\|_{\dot K^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}.$$

Now we consider Γ_2. According to Corollary 4.2, we recognize that $[b,\mu_\Omega]$ is bounded on $L^{p(\cdot)}(w)$, so we can write

$$\Gamma_2 = C\sum_{k=-\infty}^{\infty}2^{\alpha q_1 k}\Big(\sum_{j=k-2}^{k+2}\|[b,\mu_\Omega](g_j)\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\Big)^{q_1}$$

$$= \sum_{k=-\infty}^{\infty}\Big(\sum_{j=k-2}^{k+2}2^{\alpha(k-j)}2^{\alpha j}\|g_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\Big)^{q_1} \tag{53}$$

$$\leq C\|g\|^{q_1}_{\dot K^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}.$$

Finally, let us calculate Γ_3. If $x\in A_k, y\in A_j$ and $j\geq k+2$, we have

$$\begin{aligned}
&\|[b,\mu_\Omega](g_j)(x)\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\
&\leq C\Big(\int_0^{|x|}\Big|\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]g_j(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big)^{1/2}\\
&\quad + C\Big(\int_{|x|}^{\infty}\Big|\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]g_j(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big)^{1/2}\\
&= \Phi_1+\Phi_2.
\end{aligned} \tag{54}$$

By Minkowski's inequality, we get

$$
\begin{aligned}
\Phi_1 &\le C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|b(x)-b(y)||g_j(y)|\bigg(\int_{|x-y|}^{|x|}\frac{\mathrm{d}t}{t^3}\bigg)^{1/2}\mathrm{d}y\\
&\le C\int_{A_j}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|b(x)-b(y)||g_j(y)|\bigg|\frac{1}{|x-y|^2}-\frac{1}{|x|^2}\bigg|^{1/2}\mathrm{d}y\\
&\le C\int_{A_j}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|b(x)-b(y)||g_j(y)|\frac{|y|^{1/2}}{|x-y|^{3/2}}\mathrm{d}y\\
&\le C2^{(k-j)/2}2^{-j(n-\gamma)}\|b\|_{\mathrm{Lip}_\gamma}\int_{A_j}|\Omega(x-y)||g_j(y)|.
\end{aligned}\tag{55}
$$

Similarly to the way we considered Φ_1, we consider Φ_2

$$
\begin{aligned}
\Phi_2 &\le C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|b(x)-b(y)||g_j(y)|\bigg(\int_{|x|}^{\infty}\frac{\mathrm{d}t}{t^3}\bigg)^{1/2}\mathrm{d}y\\
&\le C\int_{A_j}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}|b(x)-b(y)||g_j(y)|\frac{1}{|x|}\mathrm{d}y\\
&\le C\int_{A_j}\frac{|\Omega(x-y)|}{|x-y|^{n}}|b(x)-b(y)||g_j(y)|\mathrm{d}y\\
&\le C2^{-j(n-\gamma)}\|b\|_{\mathrm{Lip}_\gamma}\int_{A_j}|\Omega(x-y)||g_j(y)|.
\end{aligned}\tag{56}
$$

Now, by the generalized Hölder's inequality, we obtain

$$
\begin{aligned}
|[b,\mu_\Omega](g_j)(x)| &\le C2^{-j(n-\gamma)}\|b\|_{\mathrm{Lip}_\gamma}\int_{A_j}|\Omega(x-y)||g_j(y)|\\
&\le C2^{-j(n-\gamma)}\|b\|_{\mathrm{Lip}_\gamma}\|\Omega(x-y)\|_{L^r(\mathbb{R}^n)}\|g_j\|_{L^{r'}(\mathbb{R}^n)}.
\end{aligned}\tag{57}
$$

Then, for any $\Omega\in L^r(\mathbb{S}^{n-1})$, by using (55)–(57), we have the following inequality

$$
|[b,\mu_\Omega](g_j)(x)|\le C2^{-j(n-\gamma)}\|b\|_{\mathrm{Lip}_\gamma}2^{(k-j)\frac{n}{r}}\|g_j\omega\|_{L^{p_1(\cdot)}}\|\omega^{-1}\chi_j\|_{L^{p_1'(\cdot)}}.\tag{58}
$$

Now using Lemmas 2.5 and 2.9, we get

$$
\begin{aligned}
&\|[b,\mu_\Omega](g)(x)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\
&\le C2^{-j(n-\gamma)}\|b\|_{\mathrm{Lip}_\gamma}2^{(k-j)\frac{n}{r}}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\
&\quad\times\|\omega^{-1}\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_{B_k}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\
&\le C2^{-j(n-\gamma)}\|b\|_{\mathrm{Lip}_\gamma}2^{(k-j)\frac{n}{r}}\|g_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\
&\quad\times\|\omega^{-1}\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_{B_j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\frac{\|\chi_{B_k}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}}{\|\chi_{B_j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}}\\
&\le C2^{-j(n-\gamma)}\|b\|_{\mathrm{Lip}_\gamma}2^{(k-j)\frac{n}{r}}2^{(k-j)n\delta_1}\|f_j\omega\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\
&\quad\times\|\omega^{-1}\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_{B_j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}\tag{59}
$$

From the definition of $A_{p_1(\cdot),p_2(\cdot)}$ (see Definition 2.2), we get

$$\begin{aligned}\|\omega^{-1}\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_{B_j}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} &\le \|\omega^{-1}\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\omega\chi_{B_j}\|_{L^{p_1(\cdot)}} \\ &\le |B_j|^{1-\frac{\gamma}{n}} = C2^{j(n-\gamma)}.\end{aligned} \tag{60}$$

Hence, we have

$$\|[b,\mu_\Omega](g)(x)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \le C2^{(k-j)(\frac{n}{r}+n\delta_1)}\|b\|_{\mathrm{Lip}_\gamma}\|g_j\omega\|_{L^{p_1(\cdot)}}. \tag{61}$$

Then, we get

$$\begin{aligned}\Gamma_3 &\le C\sum_{k=-\infty}^{\infty}2^{\alpha q_1 k}\Bigg(\sum_{j=k+2}^{\infty}\|[b,\mu_\Omega](g_j)\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\Bigg)^{q_1} \\ &\le C\sum_{k=-\infty}^{\infty}2^{\alpha qk}\Bigg(\sum_{j=k+2}^{\infty}C\|b\|_{\mathrm{Lip}_\gamma}2^{(k-j)(\frac{n}{r}+n\delta_1)}\|g_j\omega\|_{L^{p_1(\cdot)}}\Bigg)^{q_1} \\ &\le C\|b\|^{q_1}_{\mathrm{Lip}_\gamma}\sum_{k=-\infty}^{\infty}\Bigg(\sum_{j=k+2}^{\infty}2^{\alpha j}2^{(k-j)(\frac{n}{r}+n\delta_1+\alpha)}\|g_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\Bigg)^{q_1}.\end{aligned} \tag{62}$$

Now, when $1<q_1<\infty$, take $\frac{1}{q_1}+\frac{1}{q_1'}=1$. Since $\frac{n}{r}+n\delta_1+\alpha>0$, using Hölder's inequality, we get

$$\begin{aligned}\Gamma_3 &\le C\|b\|^{q_1}_{\mathrm{Lip}_\gamma}\sum_{k=-\infty}^{\infty}\Bigg(\sum_{j=k+2}^{\infty}2^{\alpha j}2^{(k-j)(\frac{n}{r}+n\delta_1+\alpha)}\|g_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\Bigg)^{q_1} \\ &\le C\|b\|^{q_1}_{\mathrm{Lip}_\gamma}\sum_{k=-\infty}^{\infty}\Bigg(\sum_{j=k+2}^{\infty}2^{\alpha q_1 j}\|g_j\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}2^{(k-j)(\frac{n}{r}+n\delta_1+\alpha)q_1/2}\Bigg) \\ &\qquad\times\Bigg(\sum_{j=k+2}^{\infty}2^{(k-j)(\frac{n}{r}+n\delta_1+\alpha)q_1'/2}\Bigg)^{q_1/q_1'} \\ &\le C\|b\|^{q_1}_{\mathrm{Lip}_\gamma}\sum_{k=-\infty}^{\infty}\Bigg(\sum_{j=k+2}^{\infty}2^{\alpha qj}\|g_j\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}2^{(k-j)(\frac{n}{r}+n\delta_1+\alpha)q_1/2}\Bigg) \\ &\le C\|b\|^{q_1}_{\mathrm{Lip}_\gamma}\sum_{j=-\infty}^{\infty}2^{\alpha q_1 j}\|g_j\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\Bigg(\sum_{k=-\infty}^{j-2}2^{(k-j)(\frac{n}{r}+n\delta_1+\alpha)q_1/2}\Bigg) \\ &\le C\|b\|^{q_1}_{\mathrm{Lip}_\gamma}\sum_{j=-\infty}^{\infty}2^{\alpha q_1 j}\|g_j\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\ &\le C\|b\|^{q_1}_{\mathrm{Lip}_\gamma}\|g_j\|_{\dot K^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}.\end{aligned} \tag{63}$$

If $0 < q_1 \leq 1$, we have

$$
\begin{aligned}
\Gamma_3 &\leq C\|b\|^{q_1}_{\mathrm{Lip}_\gamma} \sum_{k=-\infty}^{\infty} \left(\sum_{j=k+2}^{\infty} 2^{\alpha j} 2^{(k-j)(\frac{n}{r}+n\delta_1+\alpha)} \|g_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1} \\
&\leq C\|b\|^{q_1}_{\mathrm{Lip}_\gamma} \sum_{k=-\infty}^{\infty} \left(\sum_{j=k+2}^{\infty} 2^{q_1 \alpha j} 2^{q_1(k-j)(\frac{n}{r}+n\delta_1+\alpha)} \|g_j\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right) \\
&\leq C\|b\|^{q_1}_{\mathrm{Lip}_\gamma} \sum_{j=-\infty}^{\infty} 2^{\alpha q_1 j} \|g_j\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \left(\sum_{k=-\infty}^{j-2} 2^{(k-j)(\frac{n}{r}+n\delta_1+\alpha) q_1} \right) \qquad (64) \\
&\leq C\|b\|^{q_1}_{\mathrm{Lip}_\gamma} \sum_{j=-\infty}^{\infty} 2^{\alpha q_1 j} \|g_j\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\leq C\|b\|^{q_1}_{\mathrm{Lip}_\gamma} \|g_j\|_{\dot{K}^{\alpha, q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}
$$

Therefore, the proof is completed. □

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