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2-ABSORBING PRIMARY VAGUE WEAKLY COMPLETELY IDEALS

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ABSTRACT. In commutative vague algebra, the primary ideals are the remarkably weighty structures. Gau et al. proposed the idea of vague sets as an extension of fuzzy set theory. The aim of this work is to introduce and characterize 2-absorbing primary vague weakly completely ideals of commutative rings as a generalization of primary vague ideals and study their properties. Firstly, we give the definitions prime vague weakly completely ideals, primary vague weakly completely ideals and 2-absorbing vague weakly completely ideals of a commutative ring $\Re$. Then, we introduce the notion of prime K-vague ideal, primary K-vague ideal, 2-absorbing K-vague ideal. Also, we give the notion of 2-absorbing K-vague ideals and 2-absorbing primary K-vague ideals of commutative rings. Moreover, we investigate vague quotient ring of $\Re$ induced by the 2-absorbing vague weakly completely ideal which is a 2-absorbing ring. Finally, we acquire a schema which transitions between definitions of these concepts.

1. Introduction and Preliminaries

Vague sets provide a further distinctive qualification to deal with fuzzy data than fuzzy sets and human cognition is often described as a dynamic process. Shortly afterward, the question of need to describe a vague idea and accurately assess its ambiguity is an interesting topic worth exploring. Gau and Buehrer [**12**] first assigned the vague set's goal as an expansion of fuzzy set theory and situationally fuzzy sets are commonly referred to as vague sets. Let X be an initial universe set, $X = \{x_1, x_2, \ldots, x_n\}$. A truth- membership function t_v and a false- membership function f_v are used to describe a vague set over X such that $t_\eta\colon X \to [0,1]$, $f_\eta\colon X \to [0,1]$, where $t_\eta(x_i)$ is a lower bound on membership's grade of x_i based on available evidence for x_i, $f_\eta(x_i)$ is a lower bound on the negation of x_i based on available evidence against x_i, and $t_\eta(x_i) + f_\eta(x_i) \leq 1$. The membership's grade of x_i in the vague set is bounded to a subinterval $[t_\eta(x_i), 1 - f_\eta(x_i)]$ of $[0,1]$. The vague value $[t_\eta(x_i), 1 - f_\eta(x_i)]$ clearly demonstrates that exact membership's grade $\mu_\eta(x_i)$ of x_i may be uncertain, however, it is bounded by

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$t_\eta(u_i) \leq \mu_\eta(x_i) \leq 1 - f_\eta(x_i)$, where $t_\eta(x_i) + f_\eta(x_i) \leq 1$. Assume the universe X is continuous, a vague set η could be expressed as

$$\eta = \int_X [t_\eta(x_i), 1 - f_\eta(x_i)]/x_i, x_i \in X.$$

If the universe X is discrete, a vague set V [**12**] could be expressed as

$$\eta = \sum_{i=1}^{k} [t_\eta(x_i), 1 - f_\eta(x_i)]/x_i, x_i \in X.$$

The union of two vague sets η and ω is a vague set Q, presented as $Q = \eta \cup \omega$, for which the truth- membership and false- membership functions are linked to that of η and ω by

$$t_Q = (t_\eta \vee t_\omega),$$
$$1 - f_Q = ((1 - f_\eta) \vee (1 - f_\omega)) = 1 - (f_\eta \wedge f_\omega).$$

The intersection of two vague sets η and ω is a vague set Q, given as $Q = \eta \cap \omega$ [**16**], whose truth- membership and false- membership functions are related to those of η and ω by

$$t_Q = (t_\eta \wedge t_\omega),$$
$$1 - f_Q = ((1 - f_\eta) \wedge (1 - f_\omega)) = 1 - (f_\eta \vee f_\omega).$$

Let $\Re$ be a ring and η be a vague set on $\Re$. Then η is called a vague ideal over $\Re$ [**2**] when the following axioms are met for all $x, y \in \Re$,

$$\eta(x - y) \geq \eta(x) \wedge \eta(y) \quad \text{and} \quad \eta(x \cdot y) \geq \eta(x) \vee \eta(y).$$

For more information on vague sets, see [**1, 7, 8, 13, 16**].

Badawi proposed the idea of a 2-absorbing ideal, which is actually the general form of the prime ideal in [**4**], and furthermore, introduced it in cite percent [**3, 5**]. A proper ideal I of a commutative ring $\Re$ is called a 2-absorbing ideal of $\Re$ if whenever $x, y, z \in \Re$ and $xyz \in I$, then $xy \in I$ or $xz \in I$ or $yz \in I$. At present, the deliberations on the theory of the 2-absorbing ideal are rapidly making up for lost time. A large number of authors contributed (e.g., [**6, 9, 14**]) on this direction. Darani [**10**] introduced the concept of L-fuzzy 2-absorbing ideals and obtained interesting results. Then, Darani et al. [**11**] developed the idea of L-fuzzy 2-absorbing ideals in semirings and presented stimulating results based on it. 2-absorbing primary fuzzy ideals of commutative rings were characterized by Sönmez [**15**], and connections between 2-absorbing primary fuzzy ideals and 2-absorbing primary ideals were established.

During this study, $L = [0, 1]$ stands for a complete lattice and the ring $\Re$ is commutative with $1 \neq 0$ for purity. In commutative vague algebra, the primary ideals are the remarkably weighty structures and in this paper, a fancy algebraic structure of prime vague ideals of the commutative ring is abandoned by the theory of 2-absorbing weakly completely prime ideals. We clarify the notion of 2-absorbing primary vague weakly completely ideal of a ring and study part of its classification of algebraic situations. Furthermore, we introduce and characterize the 2-absorbing primary K-vague ideal of a ring. We determine the image and the inverse image of the 2-absorbing primary vague weakly completely ideals of a ring and the 2-absorbing primary K-vague ideals of a commutative ring. Besides, we construct a schema that connects the relations between these terms in the framework of these descriptions.

2. Main results

In this section, we introduce and study the notion of 2-absorbing primary vague weakly completely ideals of a commutative ring $\Re$, shortly $2APVWCI(\Re)$. Firstly, we give the definitions of prime vague weakly completely ideals ($PVWCI(\Re)$), primary vague weakly completely ideals ($PRVWCI(\Re)$) and 2-absorbing vague weakly completely ideals of a commutative ring $\Re(2AVWCI(\Re))$. Let η be a vague ideal of $\Re(VI(\Re))$ and η is called as a $PVWCI(\Re)$ if

$$\eta(xy) \leq \eta(x) \text{ or } \eta(xy) \leq \eta(y),$$

i.e.,

$$\begin{aligned} t_\eta(xy) \leq t_\eta(x) \quad &\text{and} \quad 1 - f_\eta(xy) \leq 1 - f_\eta(x), \quad \text{or} \\ t_\eta(xy) \leq t_\eta(y) \quad &\text{and} \quad 1 - f_\eta(xy) \leq 1 - f_\eta(y) \end{aligned}$$

for all $x, y \in \Re$.

Proposition 2.1. *Let $\eta \in VI(\Re)$. It is called a $PVWCI(\Re)$ if η is a non-constant function and for all $x, y \in \Re$,*

$$\begin{aligned} t_\eta(xy) &= \max\{t_\eta(x), t_\eta(y)\} \quad \textit{and} \\ 1 - f_\eta(xy) &= \max\{1 - f_\eta(x), 1 - f_\eta(y)\}. \end{aligned}$$

Let $\eta \in VI(\Re)$. Then $\sqrt{\eta}$ is referred to as the radical of η and characterized by

$$\sqrt{\eta} = \left(\sqrt{t_\eta}, \sqrt{f_\eta}\right),$$

where

$$\sqrt{t_\eta}(x) = \bigvee_{k \geq 1} t_\eta\left(x^k\right) \quad \text{and} \quad \sqrt{f_\eta}(x) = \bigwedge_{k \geq 1} f_\eta\left(x^k\right).$$

Now, we give the definition of $PRVWCI(\Re)$.

Definition 2.2. Let $\eta \in VI(\Re)$. η is called $PRVWCI(\Re)$ if, for all $x, y \in \Re$,

$$\eta(xy) \leq \eta(x) \quad \text{or} \quad \eta(xy) \leq \sqrt{\eta}(y),$$

i.e.,

$$\begin{aligned} t_\eta(xy) \leq t_\eta(x) \quad &\text{and} \quad 1 - f_\eta(xy) \leq f_\eta(x), \quad \text{or} \\ t_\eta(xy) \leq \sqrt{t_\eta}(y) \quad &\text{and} \quad 1 - f_\eta(xy) \leq 1 - \sqrt{f_\eta}(y). \end{aligned}$$

Proposition 2.3. *Let $\eta \in VI(\Re)$. It is called a $PRVWCI(\Re)$ if η is a non-constant function and for all $x, y \in \Re$,*

$$t_\eta(xy) = \max\left\{t_\eta(x), \sqrt{t_\eta}(y)\right\} \qquad \text{and}$$
$$1 - f_\eta(xy) = \max\left\{1 - f_\eta(x), 1 - \sqrt{f_\eta}(y)\right\}.$$

Now, we define the notion of $2AVWCI(\Re)$.

Definition 2.4. Let $\eta \in VI(\Re)$. η is called 2AVWCI($\Re$) if for all $x, y \in \Re$,

$$\eta(xyz) = \eta(xy) \quad \text{or} \quad \eta(xyz) = \eta(xz) \quad \text{or} \quad \eta(xyz) = \eta(yz),$$

i.e.,

$$\begin{aligned} t_\eta(xyz) &= t_\eta(xy), & 1 - f_\eta(xyz) &= 1 - f_\eta(xy), \quad \text{or} \\ t_\eta(xyz) &= t_\eta(xz), & 1 - f_\eta(xyz) &= 1 - f_\eta(xz), \quad \text{or} \\ t_\eta(xyz) &= t_\eta(yz), & 1 - f_\eta(xyz) &= 1 - f_\eta(yz). \end{aligned}$$

Proposition 2.5. *Let η be a non-constant $VI(\Re)$. Then, for all $x, y, z \in \Re$, $\eta \in 2AVWCI(\Re)$ iff*

$$\begin{aligned} t_\eta(xyz) &= \max\{t_\eta(xy), t_\eta(xz), t_\eta(yz)\}, \\ i_\eta(xyz) &= \max\{i_\eta(xy), i_\eta(xz), i_\eta(yz)\}, \\ f_\eta(xyz) &= \min\{f_\eta(xy), f_\eta(xz), f_\eta(yz)\}. \end{aligned}$$

Theorem 2.6. *Let $\eta \in PVWCI(\Re)$. Then $\eta \in 2AVWCI(\Re)$.*

Proof. Let $\eta \in PVWCI(\Re)$. Then for all $x, y, z \in \Re$, we get

$$t_\eta(xyz) = t_\eta(x) \quad \text{or} \quad t_\eta(xyz) = t_\eta(y) \quad \text{or} \quad t_\eta(xyz) = t_\eta(z),$$
$$1 - f_\eta(xyz) = 1 - f_\eta(x) \ \text{ or } \ 1 - f_\eta(xyz) = 1 - f_\eta(y) \ \text{ or } \ 1 - f_\eta(xyz) = 1 - f_\eta(z).$$

Assume that $t_\eta(xyz) = t_\eta(x)$ and $1 - f_\eta(xyz) = 1 - f_\eta(x)$. By $t_\eta(xyz) \geq t_\eta(xy) \geq t_\eta(x)$ and $1 - f_\eta(xyz) \geq 1 - f_\eta(xy) \geq 1 - f_\eta(x)$, it follows that $t_\eta(xyz) = t_\eta(xy)$ and $1 - f_\eta(xyz) = 1 - f_\eta(xy)$. Likewise, we can demonstrate that if $t_\eta(xyz) = t_\eta(y)$ or $t_\eta(xyz) = t_\eta(z)$, and $1 - f_\eta(xyz) = 1 - f_\eta(y)$ or $1 - f_\eta(xyz) = 1 - f_\eta(z)$, then $t_\eta(xyz) = t_\eta(yz)$ or $t_\eta(xyz) = t_\eta(xz)$, and $1 - f_\eta(xyz) = 1 - f_\eta(yz)$ or $1 - f_\eta(xyz) = 1 - f_\eta(xz)$. It implies that $\eta \in 2AVWCI(\Re)$. □

Now, we introduce the definition of 2-absorbing primary vague weakly completely ideals ($2APVWCI(\Re)$).

Definition 2.7. Let $\eta \in \Re$. η is called $2APVWCI(\Re)$ if for all $x, y, z \in \Re$,

$$\eta(xyz) = \eta(xy) \quad \text{or} \quad \eta(xyz) = \sqrt{\eta}(xz) \quad \text{or} \quad \eta(xyz) = \sqrt{\eta}(yz),$$

i.e.,

$$\begin{aligned} t_\eta(xyz) &= t_\eta(xy), & 1 - f_\eta(xyz) &= 1 - f_\eta(xy), \quad \text{or} \\ t_\eta(xyz) &= \sqrt{t_\eta}(xz), & 1 - f_\eta(xyz) &= 1 - \sqrt{f_\eta}(xz), \quad \text{or} \\ t_\eta(xyz) &= \sqrt{t_\eta}(yz), & 1 - f_\eta(xyz) &= 1 - \sqrt{f_\eta}(yz). \end{aligned}$$

Proposition 2.8. *Let η be a non-constant $VI(\Re)$. Then, for all $x, y, z \in \Re$, $\eta \in 2APVWCI(\Re)$ iff*

$$t_\eta(xyz) = \max\left\{t_\eta(xy), \sqrt{t_\eta}(xz), \sqrt{t_\eta}(yz)\right\},$$
$$1 - f_\eta(xyz) = \max\left\{1 - f_\eta(xy), 1 - \sqrt{f_\eta}(xz), 1 - \sqrt{f_\eta}(yz)\right\}.$$

Theorem 2.9. *Let $\eta \in 2AVWCI(\Re)$. Then $\eta \in 2APVWCI(\Re)$.*

Proof. The proof is straightforward. □

The reverse of Theorem 2.9 is not always true as indicated by the given example.

Example 2.10. Let $\Re = \mathbb{Z}$ and take $\eta \in VI(\mathbb{Z})$ as

$$\eta(x) = \begin{cases}(1,1), & x \in 27\mathbb{Z},\\ (0,0), & x \notin 27\mathbb{Z}.\end{cases}$$

Assume that $\eta(xyz) > \eta(xy)$ such that there exist $x, y, z \in \mathbb{Z}$. Hence, since $\eta(xyz) = (1,1)$ and $\eta(xy) = (0,0)$, then it follows that $xyz \in 27\mathbb{Z}$ and $xy \notin 27\mathbb{Z}$. As $27\mathbb{Z}$ is a primary ideal of $\mathbb{Z}$, we infer that $z \in 3\mathbb{Z}$. From the definition of radical η, we get

$$\sqrt{\eta}(x) = \begin{cases}(1,1), & x \in 3\mathbb{Z},\\ (0,0), & x \notin 3\mathbb{Z},\end{cases}$$

$\sqrt{\eta}(xz) = \sqrt{\eta}(yz) = (1,0)$. Thus, $\sqrt{\eta}(xz) \geq \eta(xyz)$ or $\sqrt{\eta}(yz) \geq \eta(xyz)$. Therefore, $\eta \in 2APVWCI(\mathbb{Z})$. However, since $\eta(3.3.3) = (1,1) > (0,0) = \eta(3.3)$, then it follows that $\eta \notin 2AVWCI(\mathbb{Z})$.

Theorem 2.11. *Let $\eta \in PRVWCI(\Re)$. Then $\eta \in 2APVWCI(\Re)$.*

Proof. Let $\eta \in PRVWCI(\Re)$. Then, for all $x, y, z \in \Re$,

$$t_\eta(xyz) = t_\eta(x) \text{ or } t_\eta(xyz) = \sqrt{t_\eta}(y) \text{ or } t_\eta(xyz) = \sqrt{t_\eta}(z),$$
$$1 - f_\eta(xyz) = f_\eta(x) \text{ or } 1 - f_\eta(xyz) = 1 - \sqrt{f_\eta}(y) \text{ or } 1 - f_\eta(xyz) = 1 - \sqrt{f_\eta}(z).$$

Assume that $t_\eta(xyz) = t_\eta(x)$ and $1 - f_\eta(xyz) = 1 - f_\eta(x)$. By $t_\eta(xyz) \geq t_\eta(xy) \geq t_\eta(x)$ and $1 - f_\eta(xyz) \geq 1 - f_\eta(xy) \geq 1 - f_\eta(x)$, it follows that $t_\eta(xyz) = t_\eta(xy)$, and $1 - f_\eta(xyz) = 1 - f_\eta(xy)$. Likewise, we can clearly indicate that if $t_\eta(xyz) = \sqrt{t_\eta}(y)$ or $t_\eta(xyz) = \sqrt{t_\eta}(z)$ and $1 - f_\eta(xyz) = 1 - \sqrt{f_\eta}(y)$ or $1 - f_\eta(xyz) = 1 - \sqrt{f_\eta}(z)$, then $t_\eta(xyz) = \sqrt{t_\eta}(yz)$ or $t_\eta(xyz) = \sqrt{t_\eta}(xz)$, and $1 - f_\eta(xyz) = 1 - \sqrt{f_\eta}(yz)$ or $1 - f_\eta(xyz) = 1 - \sqrt{f_\eta}(xz)$. It implies that $\eta \in 2APVWCI(\Re)$. □

Theorem 2.12. *Let $\eta \in VI(\Re)$. The following statements hold true:*

1. *$\eta \in 2APVWCI(\Re)$.*
2. *For every $\alpha, \beta \in [0,1]$, the level subset $\eta_{(\alpha,\beta)}$ of $\eta \in 2API(\Re)$.*

Proof. (1) ⇒ (2) Let us suppose that $\eta \in 2APVWCI(\Re)$. $x, y, z \in \Re$, and there exist $\alpha, \beta \in [0,1]$ with $xyz \in \eta_{(\alpha,\beta)}$. Then,

$$\max\left\{t_\eta(xy), \sqrt{t_\eta}(xz), \sqrt{t_\eta}(yz)\right\} = t_\eta(xyz) \geq \alpha,$$

$$\max\left\{1 - f_\eta(xy), 1 - \sqrt{f_\eta}\,(xz)\,, 1 - \sqrt{f_\eta}\,(yz)\right\} = f_\eta\,(xyz) \geq \beta.$$

It means that

$$t_\eta(xy) \geq \alpha \quad \text{or} \quad \sqrt{t_\eta}\,(xz) \geq \alpha \quad \text{or} \quad \sqrt{t_\eta}\,(yz) \geq \alpha, \quad \text{and}$$
$$1 - f_\eta(xy) \geq \beta \quad \text{or} \quad 1 - \sqrt{f_\eta}\,(xz) \geq \beta \quad \text{or} \quad 1 - \sqrt{f_\eta}\,(yz) \geq \beta,$$

which infer that $xy \in \eta_{(\alpha,\beta)}$ or $xz \in \sqrt{\eta}_{(\alpha,\beta)}$ or $yz \in \sqrt{\eta}_{(\alpha,\beta)}$. Thus $\eta_{(\alpha,\beta)} \in 2API\,(\Re)$.

(2) $\Rightarrow$ (1) : Assume that $\eta_{(\alpha,\beta)} \in 2API\,(\Re)$ for every $\alpha, \beta \in [0, 1]$. For $x, y, z \in \Re$, let $t_\eta\,(xyz) = \alpha$ and $1 - f_\eta\,(xyz) = \beta$. Then $xyz \in \eta_{(\alpha,\beta)}$ and $\eta_{(\alpha,\beta)} \in 2API\,(\Re)$. It gives $xy \in \eta_{(\alpha,\beta)}$ or $xz \in \sqrt{\eta}_{(\alpha,\beta)}$ or $yz \in \sqrt{\eta}_{(\alpha,\beta)}$. Thus $t_\eta(xy) \geq \alpha$ or $\sqrt{t_\eta}\,(xz) \geq \alpha$ or $\sqrt{t_\eta}\,(yz) \geq \alpha$, and $1-$ $f_\eta(xy) \geq \beta$ or $1 - \sqrt{f_\eta}\,(xz) \geq \beta$ or $1 - \sqrt{f_\eta}\,(yz) \geq \beta$. It follows that $\max\{t_\eta\,(xy)\,, \sqrt{t_\eta}\,(xz)\,, \sqrt{t_\eta}\,(yz)\} \geq \alpha = t_\eta\,(xyz)$ and $\max\left\{1 - f_\eta(xy), 1 - \sqrt{f_\eta}\,(xz)\,, 1 - \sqrt{f_\eta}\,(yz)\right\} \geq \beta = 1 - f_\eta\,(xyz)$. Furthermore, it η is a primary vague ideal of $\Re$, we have

$$t_\eta\,(xyz) \geq \max\left\{t_\eta(xy), \sqrt{t_\eta}\,(xz)\,, \sqrt{t_\eta}\,(yz)\right\},$$
$$1 - f_\eta\,(xyz) \geq \max\left\{1 - f_\eta(xy), 1 - \sqrt{f_\eta}\,(xz)\,, 1 - \sqrt{f_\eta}\,(yz)\right\}.$$

Hence,

$$t_\eta\,(xyz) = \max\left\{t_\eta(xy), \sqrt{t_\eta}\,(xz)\,, \sqrt{t_\eta}\,(yz)\right\} \quad \text{and}$$
$$1 - f_\eta\,(xyz) = \max\left\{1 - f_\eta(xy), 1 - \sqrt{f_\eta}\,(xz)\,, 1 - \sqrt{f_\eta}\,(yz)\right\}.$$

Therefore, $\eta \in 2APVWCI\,(\Re)$. □

Theorem 2.13. *If $\eta \in 2APVWCI(\Re)$, then $\sqrt{\eta} \in 2AVWCI(\Re)$.*

Proof. If $\eta \in 2APVWCI(\Re)$, then according to the prior theorem, we have that $\eta_{(\alpha,\beta)} \in 2API\,(\Re)$ for any $\alpha, \beta \in [0, 1]$. Since $\eta_{(\alpha,\beta)} \in 2API\,(\Re)$, then $\sqrt{\eta_{(\alpha,\beta)}} = \sqrt{\eta}_{(\alpha,\beta)} \in 2AI(\Re)$. As $\sqrt{\eta}_{(\alpha,\beta)} \in 2AI(\Re)$, we have that $\sqrt{\eta} \in 2AVWCI(\Re)$. Thus, we deduce that $\sqrt{\eta} \in 2AVWCI(\Re)$. □

Let $\phi\colon \Re \to S$ be a ring homomorphism and $\eta \in VI\,(\Re)$ such that η is constant on $\operatorname{Ker}\phi$ and $\omega \in VI\,(S)$. Then,

$$\sqrt{\phi\,(\eta)} = \phi\,(\sqrt{\eta}) \quad \text{and} \quad \sqrt{\phi^{-1}\,(\omega)} = \phi^{-1}\left(\sqrt{\omega}\right).$$

Theorem 2.14. *Let $\phi\colon \Re \to S$ be a surjective ring homomorphism. If $\eta \in 2APVWCI\,(\Re)$ which is constant on* $\operatorname{Ker}\phi$*, then $\phi\,(\eta) \in 2APVWCI\,(S)$.*

Proof. Assume that there exist $x, y, z \in S$ with $\phi\,(\eta)\,(xyz) \neq \phi\,(\eta)\,(xy)$. As ϕ is a surjective ring homomorphism, then

$$\phi\,(a) = x, \quad \phi\,(b) = y, \quad \phi\,(c) = z, \qquad \text{there exist } a, b, c \in \Re.$$

Thus

$$\phi\,(t_\eta)\,(xyz) = \phi\,(t_\eta)\,(\phi\,(a)\,\phi\,(b)\,\phi\,(c)) = \phi\,(t_\eta)\,(\phi\,(abc))$$

$$\neq \phi(t_\eta)(xy) = \phi(t_\eta)(\phi(a)\phi(b)) = \phi(t_\eta)(\phi(ab)) \quad \text{and}$$

$$\phi(1-f_\eta)(xyz) = \phi(1-f_\eta)(\phi(a)\phi(b)\phi(c)) = \phi(1-f_\eta)(\phi(abc)) \neq \phi(1-f_\eta)(xy) = \phi(1-f_\eta)(\phi(a)\phi(b)) = \phi(1-f_\eta)(\phi(ab)).$$

Since η is constant on $\operatorname{Ker}\phi$,

$$\phi(t_\eta)(\phi(abc)) = t_\eta(abc) \quad \text{and} \quad \phi(t_\eta)(\phi(ab)) = t_\eta(ab),$$
$$\phi(1-f_\eta)(\phi(abc)) = 1-f_\eta(abc) \quad \text{and} \quad \phi(1-f_\eta)(\phi(ab)) = 1-f_\eta(ab).$$

It follows that

$$\phi(t_\eta)(\phi(abc)) = t_\eta(abc) \neq t_\eta(ab) = \phi(t_\eta)(\phi(ab)),$$
$$\phi(1-f_\eta)(\phi(abc)) = 1-f_\eta(abc) \neq 1-f_\eta(ab) = \phi(1-f_\eta)(\phi(ab)).$$

By $\eta \in 2APVWCI(\Re)$, we have

$$\begin{aligned} t_\eta(abc) &= \phi(t_\eta)(\phi(a)\phi(b)\phi(c)) = \phi(t_\eta)(xyz) \\ &= \sqrt{t_\eta}(ac) = \phi\left(\sqrt{t_\eta}\right)(\phi(ac)) \\ &= \phi\left(\sqrt{t_\eta}\right)(\phi(a)\phi(c)) = \phi\left(\sqrt{t_\eta}\right)(xz) \quad \text{and} \\ 1-f_\eta(abc) &= \phi(1-f_\eta)(\phi(a)\phi(b)\phi(c)) = \phi(1-f_\eta)(xyz) \\ &= 1-\sqrt{f_\eta}(ac) = \phi\left(1-\sqrt{f_\eta}\right)(\phi(ac)) \\ &= \phi\left(1-\sqrt{f_\eta}\right)(\phi(a)\phi(c)) = \phi\left(1-\sqrt{f_\eta}\right)(xz). \end{aligned}$$

Besides, we get

$$\phi(t_\eta)(xyz) = \phi\left(\sqrt{t_\eta}\right)(xz) = \sqrt{\phi(t_\eta)}(xz) \quad \text{and}$$
$$\phi(1-f_\eta)(xyz) = \phi\left(1-\sqrt{f_\eta}\right)(xz) = \sqrt{\phi(1-f_\eta)}(xz),$$

or

$$\begin{aligned} t_\eta(abc) &= \phi(t_\eta)(\phi(a)\phi(b)\phi(c)) = \phi(t_\eta)(xyz) = \sqrt{t_\eta}(bc) \\ &= \phi\left(\sqrt{t_\eta}\right)(\phi(bc)) = \phi\left(\sqrt{t_\eta}\right)(\phi(b)\phi(c)) = \phi\left(\sqrt{t_\eta}\right)(yz) \quad \text{and} \\ 1-f_\eta(abc) &= \phi(1-f_\eta)(\phi(a)\phi(b)\phi(c)) = \phi(1-f_\eta)(xyz) \\ &= 1-\sqrt{f_\eta}(bc) = \phi\left(1-\sqrt{f_\eta}\right)(\phi(bc)) \\ &= \phi\left(1-\sqrt{f_\eta}\right)(\phi(b)\phi(c)) = \phi\left(1-\sqrt{f_\eta}\right)(yz), \end{aligned}$$

and also $\phi(t_\eta)(xyz) = \phi(\sqrt{t_\eta})(yz) = \sqrt{\phi(t_\eta)}(yz)$ and $\phi(1-f_\eta)(xyz) = \phi(1-f_\eta)(yz) = \sqrt{\phi(1-f_\eta)}(yz)$. Consequently, $\phi(\eta) \in 2APVWCI(S)$. □

Theorem 2.15. *Let $\phi\colon \Re \to S$ be a ring homomorphism. If $\omega = \langle t_\omega, f_\omega\rangle \in 2APVWCI(S)$, then $\phi^{-1}(\omega) \in 2APVWCI(\Re)$.*

Proof. Assume that there exist $x, y, z \in \Re$ such that $\phi^{-1}(t_\omega)(xyz) > \phi^{-1}(t_\omega)(xy)$ and $\phi^{-1}(1-f_\omega)(xyz) > \phi^{-1}(1-f_\omega)(xy)$. Then,

$$\begin{aligned}\phi^{-1}(t_\omega)(xyz) &= t_\omega(\phi(xyz)) = t_\omega(\phi(x)\phi(y)\phi(z)) \\ &> \phi^{-1}(t_\omega)(xy) = t_\omega(\phi(xy)) = t_\omega(\phi(x)\phi(y)) \qquad \text{and} \\ \phi^{-1}(1-f_\omega)(xyz) &= 1-f_\omega(\phi(xyz)) = 1-f_\omega(\phi(x)\phi(y)\phi(z)) \\ &> \phi^{-1}(1-f_\omega)(xy) = 1-f_\omega(\phi(xy)) = 1-f_\omega(\phi(x)\phi(y)).\end{aligned}$$

By $\omega \in 2APVWCI(S)$, we have

$$\begin{aligned}\phi^{-1}(t_\omega)(xyz) &= t_\omega(\phi(x)\phi(y)\phi(z)) \leq \sqrt{t_\omega}(\phi(x)\phi(z)) \\ &= \sqrt{t_\omega}(\phi(xz)) = \sqrt{\phi^{-1}(t_\omega)}(xz) \qquad \text{and} \\ \phi^{-1}(1-f_\omega)(xyz) &= 1-f_\omega(\phi(x)\phi(y)\phi(z)) \leq 1-\sqrt{f_\omega}(\phi(x)\phi(z)) \\ &= 1-\sqrt{f_\omega}(\phi(xz)) = \sqrt{\phi^{-1}(1-f_\omega)}(xz),\end{aligned}$$

or

$$\begin{aligned}\phi^{-1}(t_\omega)(xyz) &= t_\omega(\phi(x)\phi(y)\phi(z)) \leq \sqrt{t_\omega}(\phi(y)\phi(z)) \\ &= \sqrt{t_\omega}(\phi(yz)) = \sqrt{\phi^{-1}(t_\omega)}(yz) \qquad \text{and} \\ \phi^{-1}(1-f_\omega)(xyz) &= 1-f_\omega(\phi(x)\phi(y)\phi(z)) \leq 1-\sqrt{f_\omega}(\phi(y)\phi(z)) \\ &= 1-\sqrt{f_\omega}(\phi(yz)) = \sqrt{\phi^{-1}(1-f_\omega)}(yz).\end{aligned}$$

Therefore, $\phi^{-1}(\omega) \in 2APVWCI(\Re)$. □

Corollary 2.16. *Let ϕ be a ring homomorphism from $\Re$ onto S. ϕ elicits a one to one inclusion providing correspondence between $2APVWCI(S)$ in such a way that if $\eta \in 2APVWCI(\Re)$ which is constant on* $\operatorname{Ker}\phi$*, then $\phi(\eta) \in 2APVWCI(S)$, and if $\omega \in 2APVWCI(S)$, then $\phi^{-1}(\omega) \in 2APVWCI(\Re)$.*

Now, we introduce the notion of prime K-vague ideals ($PKVI$), primary K-vague ideals ($PRKVI$), 2-absorbing K-vague ideals ($2AKVI$), and 2-absorbing primary K-vague ideals ($2APKVI$) of a commutative ring $\Re$.

Let $\eta \in VI(\Re)$. η is called $PKVI(\Re)$ if for all $x, y \in \Re$,

$$\eta(xy) = \eta(0) \quad \text{implies that} \quad \eta(x) = \eta(0) \ \text{ or } \ \eta(y) = \eta(0).$$

Let $\eta \in VI(\Re)$. η is called $PRKVI(\Re)$, if for all $x, y \in \Re$

$$\eta(xy) = \eta(0) \quad \text{implies that} \quad \eta(x) = \eta(0) \ \text{ or } \ \sqrt{\eta}(y) = \eta(0).$$

Definition 2.17. Let $\eta \in VI(\Re)$. η is called $2AKVI(\Re)$, if for all $x, y, z \in \Re$

$$\eta(xyz) = \eta(0) \quad \text{indicates} \quad \eta(xy) = \eta(0) \ \text{ or } \ \eta(xz) = \eta(0) \ \text{ or } \ \eta(yz) = \eta(0).$$

Definition 2.18. Let $\eta \in VI(\Re)$. η is called $2APKVI(\Re)$ if for all $x, y, z \in \Re$,

$$\eta(xyz) = \eta(0) \quad \text{asserts} \quad \eta(xy) = \eta(0) \ \text{ or } \ \sqrt{\eta}(xz) = \eta(0) \ \text{ or } \ \sqrt{\eta}(yz) = \eta(0).$$

Theorem 2.19. *Let $\eta \in 2AVWCI(\Re)$, then $\eta \in 2AKVI(\Re)$.*

Proof. Assume that $\eta \in 2AVWCI(\Re)$. If $\eta\,(xyz) = \eta(0)$ for all $x, y, z \in \Re$, then we have

$$\begin{aligned} \eta(0) &= \eta\,(xyz) \leq \eta(xy) \leq \eta(0) \qquad \text{or} \\ \eta(0) &= \eta\,(xyz) \leq \eta\,(xz) \leq \eta(0) \qquad \text{or} \\ \eta(0) &= \eta\,(xyz) \leq \eta\,(yz) \leq \eta(0) \end{aligned}$$

since $\eta \in 2AVWCI(\Re)$. For this reason, the following outcome is acquired

$$\eta(xy) = \eta(0) \quad \text{or} \quad \eta\,(xz) = \eta(0) \quad \text{or} \quad \eta\,(yz) = \eta(0).$$

We deduce that $\eta \in 2AKVI(\Re)$. □

The converse of the above theorem is not true, as it is shown in the following illustration.

Example 2.20. Let $\Re = \mathbb{Z}$ and define $\eta \in VI\,(\mathbb{Z})$ by

$$\eta(x) = \begin{cases} (1,1) & \text{if } \ x = 0, \\ (1/3, 2/3) & \text{if } x \in 27\mathbb{Z} \setminus \{0\}, \\ (1/4, 3/4) & \text{if } \ x \in \mathbb{Z} \setminus 27\mathbb{Z}. \end{cases}$$

Then, $\eta \in 2AKVI(\mathbb{Z})$. However, we have

$$1 - f_\eta\,(3.3.3) = 2/3 < 3/4 = 1 - f_\eta\,(3.3)\,.$$

Thus, $\eta \notin 2AVWCI(\mathbb{Z})$.

Now we describe the vague quotient ring of $\Re$ induced by $2AVWCI(\Re)$. We define the notion of vague quotient ring induced by VI($\Re$). Let $\eta \in VI\,(\Re)$ and for any $x, y \in \Re$, define a binary relation $\sim$ on $\Re$ which is a congruence relation of $\Re$ by $x \sim y$ if and only if

$$\eta(x - y) = \eta(0).$$

Let $\eta\,[x] = \{y \in \Re \mid y \sim x\}$ be the equivalence class containing x and $\Re/\eta = \{\eta\,[x] \mid x \in \Re\}$ the set of all equivalence classes of $\Re$. Define two operations by

$$\eta\,[x] + \eta\,[y] = \eta\,[x + y] \qquad \text{and} \qquad \eta\,[x]\,\eta\,[y] = \eta\,[xy]$$

for $x, y \in \Re$. Then, $\Re/\eta$ is a vague ring with two operations and it is called a vague quotient ring of $\Re$ induced by $\eta \in VI\,(\Re)$.

Theorem 2.21. *Let $\eta \in VI\,(\Re)$ be such that it is non-constant. Then, $\eta \in 2AKVI(\Re)$ if and only if $\Re/\eta$ is a 2-absorbing ring.*

Proof. Suppose that $\eta \in 2AKVI(\Re)$ and let $\eta[x], \eta[y], \eta[z] \in \Re/\eta$ be such that $\eta[x]\eta[y]\eta[z] = \eta[0]$.

By $\eta[x]\eta[y]\eta[z] = \eta\,[xyz]$, we have

$$\eta\,(xyz) = \eta\,(xyz - 0) = (1, 1) = \eta(0).$$

Since $\eta \in 2AKVI(\Re)$, then

$$\eta(xy) = \eta(0) = (1, 1) \quad \text{or} \quad \eta\,(xz) = \eta(0) = (1, 1) \quad \text{or} \quad \eta\,(yz) = \eta(0) = (1, 1)\,.$$

It implies that

$$\begin{aligned} \eta\,[xy] &= \eta[x]\eta[y] = \eta\,[0] \qquad \text{or} \\ \eta\,[xz] &= \eta[x]\eta[z] = \eta\,[0] \qquad \text{or} \\ \eta\,[yz] &= \eta[y]\eta[z] = \eta\,[0]\,. \end{aligned}$$

Therefore, $\Re/\eta$ is a 2-absorbing ring. Otherwise, assume that $\Re/\eta$ is a 2-absorbing ring and let $\eta\,(xyz) = \eta\,(0) = (1,1)$ for $x, y, z \in \Re$. Then we have

$$\eta[x]\eta[y]\eta[z] = \eta\,[xyz] = \eta\,[0]\,.$$

As $\Re/\eta$ is a 2-absorbing ring, then

$$\eta\,[xy] = \eta\,[0] \text{ or } \eta\,[xz] = \eta\,[0] \text{ or } \eta\,[yz] = \eta\,[0]\,,$$

which implies that $\eta \in 2AKVI(\Re)$. □

Corollary 2.22. *If $\eta \in 2AVWCI(\Re)$, then $\Re/\eta$ is a 2-absorbing ring.*

Theorem 2.23. *Assume $\eta \in PKVI(\Re)$. Then $\eta \in 2AVWCI(\Re)$.*

Proof. Let $\eta \in PKVI(\Re)$. Then for all $x, y, z \in \Re$,

$$\eta\,(xyz) = \eta(0) \quad \text{implies} \quad \eta\,(x) = \eta(0) \;\text{ or }\; \eta(y) = \eta(0) \;\text{ or }\; \eta\,(z) = \eta(0).$$

Suppose that $\eta(x) = \eta(0)$. Then by

$$\eta(0) = \eta(x) \leq \eta(xy) \leq \eta\,(xyz) = \eta(0),$$

we obtain $\eta(xy) = \eta(0)$. In a similar way, we can deduce that $\eta\,(xz) = \eta(0)$ or $\eta\,(yz) = \eta(0)$. Finally, $\eta \in 2AVWCI(\Re)$. □

Corollary 2.24. *Let $\eta \in 2APVWCI(\Re)$. Then $\eta \in 2APKVI(\Re)$.*

It is worth mentioning that the $2APKVI(\Re)$ does not seem to be $2APVWCI(\Re)$ in the following example.

Example 2.25. Let $\Re = \mathbb{Z}$ and define $\eta \in VI(\mathbb{Z})$ as

$$\eta(x) = \begin{cases} (1,1)\,, & x = 0, \\ \left(\frac{1}{2}, \frac{2}{3}\right), & x \in 105\mathbb{Z} \smallsetminus \{0\}\,, \\ \left(\frac{1}{3}, \frac{1}{2}\right), & x \in \mathbb{Z} \smallsetminus 105\mathbb{Z}. \end{cases}$$

Then, $\eta \in 2APKVI(\mathbb{Z})$. However, since

$$\eta\,(3.5.7) = \left(\frac{1}{2}, \frac{2}{3}\right) > \bigvee \{\eta\,(3.5)\,, \eta\,(3.7)\,, \eta\,(5.7)\} = \left(\frac{1}{3}, \frac{1}{2}\right)$$

or

$$\eta\,(3.5.7) = \left(\frac{1}{2}, \frac{2}{3}\right) > \bigvee \{\sqrt{\eta}\,(3.5)\,, \sqrt{\eta}\,(3.7)\,, \sqrt{\eta}\,(5.7)\} = \left(\frac{1}{3}, \frac{1}{2}\right),$$

then $\eta \notin 2APVWCI(\mathbb{Z})$.

Corollary 2.26. *If $\eta \in 2AKVI(\Re)$, then $\eta \in 2APKVI(\Re)$.*

The converse of Corollary 2.26 is not always true as it is shown in the following example.

Example 2.27. Define $\eta \in VI(\mathbb{Z})$ by

$$\eta(x) = \begin{cases} (1,1), & x \in 27\mathbb{Z}, \\ (0,0), & x \notin 27\mathbb{Z}. \end{cases}$$

Then $\eta \in 2APKVI(\mathbb{Z})$. However, since

$$\eta(3.3.3) = \sqrt{\eta}(3.3) = \bigvee_{k \geq 1} \eta\left(x^k\right) = (1,1) = \eta(0)$$

and

$$\eta(3.3.3) = (1,1) = \eta(0) \neq \eta(3.3) = \eta(9) = (0,0),$$

then it follows that $\eta \notin 2AKVI(\mathbb{Z})$.

Theorem 2.28. *Let $\phi : \Re \to S$ be a surjective ring homomorphism. If $\eta \in 2APKVI(\Re)$ which is constant on $\operatorname{Ker}\phi$, then $\phi(\eta) \in 2APKVI(S)$.*

Proof. The proof is parallel to Theorem 2.14's proof, hence it is omitted. □

Theorem 2.29. *Let $\phi\colon \Re \to S$ be a ring homomorphism. If $\eta_1 \in 2APKVI(S)$, then $\phi^{-1}(\eta_1) \in 2APKVI(\Re)$.*

Proof. The proof is omitted as it is parallel to Theorem 2.15's proof. □

Corollary 2.30. *Let ϕ be a ring homomorphism from $\Re$ onto S. ϕ elicits a one to one inclusion providing correspondence between $2APKVI(S)$ in such a way that if $\eta \in 2APKVI(\Re)$ which is constant on $\operatorname{Ker}\phi$, then $\phi(\eta)$ is the corresponding $2APKVI(S)$, and if $\eta_1 \in 2APKVI(S)$, then $\phi^{-1}(\eta_1)$ is the corresponding $2APKVI(\Re)$.*

Remark. The followings schema simplifies implications of $2AVWCI(\Re)$.

$$\begin{array}{ccccc} \text{VWCI} & \Longrightarrow & \text{2AVWCI} & \Longrightarrow & \text{2APVWCI} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{KVI} & \Longrightarrow & \text{2AKVWCI} & \Longrightarrow & \text{2APKVWCI} \end{array}$$

3. Conclusion

In this work, we have proposed the theory of $2APVWCI(\Re)$ and $2APKVI(\Re)$. We have determined the image and the inverse image of $2APVWCI(\Re)$ and $2APKVI(\Re)$. We have demonstrated a scheme for the transition between these algebraic structures. Scientists have been integrated into this coherent approach to produce a plenty of important results about $2APVWCI(\Re)$ and $2APKVI(\Re)$. Based on our work, we propose some open problems to researchers:

(1) To introduce and study 2-absorbing semiprimary vague weakly completely ideals,

(2) To introduce and study 2-absorbing δ-primary vague weakly completely ideals,
(3) To introduce and study 2-absorbing δ-semiprimary vague weakly completely ideals.

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