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JENSEN-TYPE INEQUALITIES FOR LOG-CONVEX FUNCTIONS

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Abstract. The main result of this paper is to give refinement and reverse the celebrated Jensen's inequality. We also present a stronger estimate for the first inequality in the Hermite-Hadamard inequality. We directly apply our results to establish several operator inequalities.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. As customary, we reserve m, M for scalars and $\mathbf{1}_{\mathcal{H}}$ for the identity operator on $\mathcal{H}$. A self-adjoint operator A is said to be positive (written $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ holds for all $x \in \mathcal{H}$, and also an operator A is said to be strictly positive (denoted by $A > 0$) if A is positive and invertible. If A and B are self-adjoint, we write $A \leq B$ in case $0 \leq B - A$. The Gelfand map $f(t) \mapsto f(A)$ is an isometrical $*$-isomorphism between the C^*-algebra $C(\sigma(A))$ of continuous functions on the spectrum $\sigma(A)$ of a selfadjoint operator A and the C^*-algebra generated by A and the identity operator $\mathbf{1}_{\mathcal{H}}$. If $f, g \in C(\sigma(A))$, then $f(t) \geq g(t)$ $(t \in \sigma(A))$ implies that $f(A) \geq g(A)$.

A linear map $\Phi\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is said to be unital if $\Phi(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{K}}$, where $\mathbf{1}_{\mathcal{H}}$ is the identity operator on Hilbert space $\mathcal{K}$. A continuous function f defined on the interval $J \subseteq \mathbb{R}$ is called an operator convex function if $f((1-v)A + vB) \leq (1-v)f(A) + vf(B)$ for every $0 < v < 1$ and for every pair of bounded self-adjoint operators A and B whose spectra are both in J.

The well-known Jensen's inequality states that if f is a convex function on the interval $[m, M]$, then

$$f\left(\sum_{i=1}^{n} w_i a_i\right) \leq \sum_{i=1}^{n} w_i f(a_i) \tag{1}$$

for all $a_i \in [m, M]$ and $w_i \in [0,1]$ $(i = 1, \ldots, n)$ with $\sum_{i=1}^{n} w_i = 1$.

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A lot of literature is devoted to Jensen's inequality concerning different generalizations, refinements, and converse results; see, for example, [**2, 6, 10**].

Mond and Pečarić [**5**] gave an operator extension of the Jensen's inequality as follows: Let $A \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator with $\sigma(A) \subseteq [m, M]$, and let $f(t)$ be a convex function on $[m, M]$. Then for any unit vector $x \in \mathcal{H}$,

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle .$$

Choi [**1**] showed that if $f\colon J \to \mathbb{R}$ is an operator convex function, A is a self-adjoint operator with the spectra in J, and $\Phi\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is unital positive linear mapping, then

$$f(\Phi(A)) \leq \Phi(f(A)) . \tag{2}$$

Though in the case of a convex function, the inequality (2) does not hold in general, we have the following estimate [**3**, Lemma 2.1]:

$$f(\langle \Phi(A)x, x\rangle) \leq \langle \Phi(f(A))x, x\rangle \tag{3}$$

for any unit vector $x \in \mathcal{K}$.

We here cite [**4**] and [**11**] as pertinent references to inequalities of types (2) and (3). We refer the reader to [**3, 7, 8**] for other recent results treating the Jensen's operator inequality.

The current paper gives extensions of Jensen-type inequalities for logarithmically convex functions. Recall that the function $f\colon J \to (0, \infty)$ is called a log-convex if it satisfies the following inequality:

$$f((1-t)a + tb) \leq f^{1-t}(a) f^{t}(b) \qquad (0 \leq t \leq 1) .$$

Our results have been employed to obtain new estimates related to operator means and Tsallis relative operator entropy.

2. Main results

For our purpose, we need the following well-known result. See, for example, [**7**].

Lemma 2.1. *Let $f\colon J \to \mathbb{R}$ be a convex function and let $a, b \in J$. Then*

$$f((1-t)a + tb) \leq (1-t)f(a) + tf(b) - 2r\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right)$$

and

$$(1-t)f(a) + tf(b) \leq f((1-t)a + tb) + 2R\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right),$$

where $r = \min\{1-t, t\}$, $R = \max\{1-t, t\}$, and $0 \leq t \leq 1$.

In our first result, we present a refinement of the Jensen's inequality for log-convex functions.

Theorem 2.2. *Let $f\colon J\to(0,\infty)$ be a log-convex function and $x_1,x_2,\dots,x_n\in J$. If $w_1,w_2,\dots,w_n$ are positive numbers with $\sum_{i=1}^n w_i=1$, then*

$$f\Bigg(\sum_{i=1}^n w_ix_i\Bigg)\le\frac{\prod_{i=1}^n f^{w_i}(x_i)}{\left(\frac{\sqrt{f\left(\sum_{i=1}^n w_ix_i\right)\prod_{i=1}^n f^{w_i}(x_i)}}{\prod_{i=1}^n f^{w_i}\left(\frac{\sum_{j=1}^n w_jx_j+x_i}{2}\right)}\right)^2}\le\prod_{i=1}^n f^{w_i}(x_i).$$

Proof. Assume that f is a convex function. From Lemma 2.1, we infer that

$$\frac{f(a+t(b-a))-f(a)}{t}\le f(b)-f(a)-\frac{2\min\{t,1-t\}}{t}\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right)$$

or

$$\frac{f(a+t(b-a))-f(a)}{t}\le f(b)-f(a)-\frac{1-|2t-1|}{t}\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right).$$

Now by letting $t\to0$, we get

$$f(a)+f'(a)(b-a)\le f(b)-2\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right). \tag{4}$$

Since for any convex function

$$f\left(\frac{a+b}{2}\right)\le\frac{f(a)+f(b)}{2},$$

we have

$$f(a)+f'(a)(b-a)\le f(b)-2\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right)\le f(b).$$

Putting $a=\sum_{i=1}^n w_ix_i$, we get

$$\begin{aligned}&f\Bigg(\sum_{i=1}^n w_ix_i\Bigg)+bf'\Bigg(\sum_{i=1}^n w_ix_i\Bigg)-f'\Bigg(\sum_{i=1}^n w_ix_i\Bigg)\sum_{i=1}^n w_ix_i\\&\quad\le f(b)-2\left(\frac{f\left(\sum_{i=1}^n w_ix_i\right)+f(b)}{2}-f\left(\frac{\sum_{i=1}^n w_ix_i+b}{2}\right)\right)\le f(b).\end{aligned}$$

By replacing $b=x_i$, and then multiplying by w_i and summing from 1 to n, we get

$$\begin{aligned}f\Bigg(\sum_{i=1}^n w_ix_i\Bigg)&\le\sum_{i=1}^n w_if(x_i)-2\Bigg(\frac{f\left(\sum_{i=1}^n w_ix_i\right)+\sum_{i=1}^n w_if(x_i)}{2}\\&\qquad-\sum_{i=1}^n w_if\Bigg(\frac{\sum_{j=1}^n w_jx_j+x_i}{2}\Bigg)\Bigg)\le\sum_{i=1}^n w_if(x_i)\end{aligned}$$

for any convex function $f\colon J\to\mathbb{R}$. If f is log-convex, then $\log f$ is convex. Therefore, from the above inequality, we have

$$\begin{aligned}\log f\left(\sum_{i=1}^n w_ix_i\right)&\le\sum_{i=1}^n w_i\log f(x_i)-2\Bigg(\frac{\log f\left(\sum_{i=1}^n w_ix_i\right)+\sum_{i=1}^n w_i\log f(x_i)}{2}\\&\qquad-\sum_{i=1}^n w_i\log f\Bigg(\frac{\sum_{j=1}^n w_jx_j+x_i}{2}\Bigg)\Bigg)\le\sum_{i=1}^n w_i\log f(x_i)\end{aligned}$$

or equivalently,

$$\log f\left(\sum_{i=1}^{n} w_i x_i\right) \le \log \prod_{i=1}^{n} f^{w_i}(x_i) - 2\left(\frac{\log f\left(\sum_{i=1}^{n} w_i x_i\right) + \log \prod_{i=1}^{n} f^{w_i}(x_i)}{2}\right.$$

$$\left. - \log \prod_{i=1}^{n} f^{w_i}\left(\frac{\sum_{j=1}^{n} w_j x_j + x_i}{2}\right)\right)$$

$$= \log \frac{\prod_{i=1}^{n} f^{w_i}(x_i)}{\left(\frac{\sqrt{f\left(\sum_{i=1}^{n} w_i x_i\right)\prod_{i=1}^{n} f^{w_i}(x_i)}}{\prod_{i=1}^{n} f^{w_i}\left(\frac{\sum_{j=1}^{n} w_j x_j + x_i}{2}\right)}\right)^2} \le \log \prod_{i=1}^{n} f^{w_i}(x_i).$$

We deduce the desired result by applying exp from both sides of the above inequality. □

As a direct consequence of Theorem 2.2, we can obtain the following corollary.

Corollary 2.3. *Let $f\colon J \to (0,\infty)$ be a log-convex function and $x_1, x_2, \ldots, x_n \in J$. If $w_1, w_2, \ldots, w_n$ are positive numbers with $\sum_{i=1}^{n} w_i = 1$, then*

$$f\left(\sum_{i=1}^{n} w_i x_i\right) \le \prod_{i=1}^{n} f^{w_i}\left(\frac{\sum_{j=1}^{n} w_j x_j + x_i}{2}\right)$$

$$\le \sqrt{f\left(\sum_{i=1}^{n} w_i x_i\right)\prod_{i=1}^{n} f^{w_i}(x_i)} \le \prod_{i=1}^{n} f^{w_i}(x_i).$$

Proof. Assume that f is a convex function. It follows from Theorem 2.2 (see also [**9**]),

$$\sum_{i=1}^{n} w_i f\left(\frac{\sum_{j=1}^{n} w_j x_j + x_i}{2}\right) \le \frac{f\left(\sum_{i=1}^{n} w_i x_i\right) + \sum_{i=1}^{n} w_i f(x_i)}{2}.$$

Now, by the Jensen's inequality for the convex function, we have

$$f\left(\sum_{i=1}^{n} w_i x_i\right) \le \sum_{i=1}^{n} w_i f\left(\frac{\sum_{j=1}^{n} w_j x_j + x_i}{2}\right)$$

$$\le \frac{f\left(\sum\limits_{i=1}^{n} w_i x_i\right) + \sum\limits_{i=1}^{n} w_i f(x_i)}{2} \le \sum_{i=1}^{n} w_i f(x_i).$$

Next, if f is log-convex, then

$$\log f\left(\sum_{i=1}^{n} w_i x_i\right) \le \sum_{i=1}^{n} w_i \log f\left(\frac{\sum_{j=1}^{n} w_j x_j + x_i}{2}\right)$$

$$\le \frac{\log f\left(\sum_{i=1}^{n} w_i x_i\right) + \sum_{i=1}^{n} w_i \log f(x_i)}{2} \le \sum_{i=1}^{n} w_i \log f(x_i).$$

As an alternative,

$$\log f\left(\sum_{i=1}^{n} w_i x_i\right) \le \log \prod_{i=1}^{n} f^{w_i}\left(\frac{\sum_{j=1}^{n} w_j x_j + x_i}{2}\right)$$
$$\le \log \sqrt{f\left(\sum_{i=1}^{n} w_i x_i\right) \prod_{i=1}^{n} f^{w_i}(x_i)} \le \log \prod_{i=1}^{n} f^{w_i}(x_i).$$

We deduce the desired result by applying exp from both sides of the above inequality. □

The following theorem gives a converse of the Jensen's inequality for log-convex functions.

Theorem 2.4. *Let $f\colon J \to (0,\infty)$ be a differentiable log-convex function, $x_1, x_2, \dots, x_n \in J$, and let $w_1, w_2, \dots, w_n$ be positive numbers with $\sum_{i=1}^{n} w_i = 1$. Then*

$$\prod_{i=1}^{n} f^{w_i}(x_i) \le \frac{f\left(\sum_{i=1}^{n} w_i x_i\right)}{\left(\frac{\sqrt{\prod_{i=1}^{n} f^{w_i}(x_i) f\left(\sum_{i=1}^{n} w_i x_i\right)}}{\prod_{i=1}^{n} f^{w_i}\left(\frac{x_i + \sum_{j=1}^{n} w_j x_j}{2}\right)}\right)^2}$$
$$+ \exp\left(\sum_{i=1}^{n} w_i x_i \frac{f'(x_i)}{f(x_i)} - \sum_{i=1}^{n} w_i \frac{f'(x_i)}{f(x_i)}\right)$$
$$\le f\left(\sum_{i=1}^{n} w_i x_i\right) + \exp\left(\sum_{i=1}^{n} w_i x_i \frac{f'(x_i)}{f(x_i)} - \sum_{i=1}^{n} w_i \frac{f'(x_i)}{f(x_i)}\right).$$

Proof. Assume that f is a convex function. As we have shown in the proof of Theorem 2.2,

$$f(a) + f'(a)(b-a) \le f(b) - 2\left(\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right).$$

By replacing $b = \sum_{i=1}^{n} w_i x_i$, we get

$$f(a) + \sum_{i=1}^{n} w_i x_i f'(a) - a f'(a)$$
$$\le f\left(\sum_{i=1}^{n} w_i x_i\right) - 2\left(\frac{f(a) + f\left(\sum_{i=1}^{n} w_i x_i\right)}{2} - f\left(\frac{a + \sum_{i=1}^{n} w_i x_i}{2}\right)\right) \le f\left(\sum_{i=1}^{n} w_i x_i\right).$$

By setting $a = x_i$, and then multiplying by w_i and summing from 1 to n, we get

$$\sum_{i=1}^{n} w_i f(x_i) + \sum_{i=1}^{n} w_i x_i \sum_{i=1}^{n} w_i f'(x_i) - \sum_{i=1}^{n} w_i x_i f'(x_i)$$
$$\le f\left(\sum_{i=1}^{n} w_i x_i\right) - 2\left(\frac{\sum_{i=1}^{n} w_i f(x_i) + f\left(\sum_{i=1}^{n} w_i x_i\right)}{2} - \sum_{i=1}^{n} w_i f\left(\frac{x_i + \sum_{j=1}^{n} w_j x_j}{2}\right)\right)$$
$$\le f\left(\sum_{i=1}^{n} w_i x_i\right).$$

Now, if f is log-convex, then the above inequality gives

$$\begin{aligned}
&\sum_{i=1}^{n} w_i \log f(x_i) + \sum_{i=1}^{n} w_i (\log f(x_i))' - \sum_{i=1}^{n} w_i x_i (\log f(x_i))' \\
&\leq \log f\left(\sum_{i=1}^{n} w_i x_i\right) \\
&\quad - 2\left(\frac{\sum_{i=1}^{n} w_i \log f(x_i) + \log f\left(\sum_{i=1}^{n} w_i x_i\right)}{2} - \sum_{i=1}^{n} w_i \log f\left(\frac{x_i + \sum_{j=1}^{n} w_j x_j}{2}\right)\right) \\
&\leq \log f\left(\sum_{i=1}^{n} w_i x_i\right).
\end{aligned}$$

In other words,

$$\begin{aligned}
&\log \prod_{i=1}^{n} f^{w_i}(x_i) + \sum_{i=1}^{n} w_i \frac{f'(x_i)}{f(x_i)} - \sum_{i=1}^{n} w_i x_i \frac{f'(x_i)}{f(x_i)} \\
&\leq \log \frac{f\left(\sum\limits_{i=1}^{n} w_i x_i\right)}{\left(\frac{\sqrt{\prod_{i=1}^{n} f^{w_i}(x_i) f\left(\sum_{i=1}^{n} w_i x_i\right)}}{\prod_{i=1}^{n} f^{w_i}\left(\frac{x_i + \sum_{j=1}^{n} w_j x_j}{2}\right)}\right)^2} \leq \log f\left(\sum_{i=1}^{n} w_i x_i\right).
\end{aligned}$$

By applying exp, we infer that

$$\begin{aligned}
&\prod_{i=1}^{n} f^{w_i}(x_i) \\
&\leq \exp\left(\log \frac{f\left(\sum\limits_{i=1}^{n} w_i x_i\right)}{\left(\frac{\sqrt{\prod_{i=1}^{n} f^{w_i}(x_i) f\left(\sum_{i=1}^{n} w_i x_i\right)}}{\prod_{i=1}^{n} f^{w_i}\left(\frac{x_i + \sum_{j=1}^{n} w_j x_j}{2}\right)}\right)^2} + \sum_{i=1}^{n} w_i x_i \frac{f'(x_i)}{f(x_i)} - \sum_{i=1}^{n} w_i \frac{f'(x_i)}{f(x_i)}\right) \\
&= \frac{f\left(\sum\limits_{i=1}^{n} w_i x_i\right)}{\left(\frac{\sqrt{\prod_{i=1}^{n} f^{w_i}(x_i) f\left(\sum_{i=1}^{n} w_i x_i\right)}}{\prod_{i=1}^{n} f^{w_i}\left(\frac{x_i + \sum_{j=1}^{n} w_j x_j}{2}\right)}\right)^2} + \exp\left(\sum_{i=1}^{n} w_i x_i \frac{f'(x_i)}{f(x_i)} - \sum_{i=1}^{n} w_i \frac{f'(x_i)}{f(x_i)}\right) \\
&\leq \exp\left(\log f\left(\sum_{i=1}^{n} w_i x_i\right) + \sum_{i=1}^{n} w_i x_i \frac{f'(x_i)}{f(x_i)} - \sum_{i=1}^{n} w_i \frac{f'(x_i)}{f(x_i)}\right) \\
&= f\left(\sum_{i=1}^{n} w_i x_i\right) + \exp\left(\sum_{i=1}^{n} w_i x_i \frac{f'(x_i)}{f(x_i)} - \sum_{i=1}^{n} w_i \frac{f'(x_i)}{f(x_i)}\right).
\end{aligned}$$

This completes the proof of the theorem. □

We close this section by providing a new refinement and a reverse for the first inequality in the Hermite-Hadamard inequality for log-convex functions.

Theorem 2.5. *Let $f\colon J \to (0,\infty)$ be a log-convex function and let $a, b \in J$. Then*

$$f\left(\frac{a+b}{2}\right)+\exp\Bigg(\int_0^1 \frac{1-t}{1+|2t-1|}\log f\left((1-t)\,a+tb\right)\mathrm{d}t + \int_0^1 \frac{t}{1+|2t-1|}\log f\left((1-t)\,b+ta\right)\mathrm{d}t - \int_0^1 \frac{1}{1+|2t-1|}\log f\left(2\,(a-b)\left(t^2-t\right)+a\right)\mathrm{d}t\Bigg) \le \exp\left(\int_0^1 \log f\left((1-t)\,a+tb\right)\mathrm{d}t\right)$$

and

$$\exp\left(\int_0^1 \log f\left((1-t)\,a+tb\right)\mathrm{d}t\right) \le f\left(\frac{a+b}{2}\right)+\exp\Bigg(\int_0^1 \frac{1}{1-|1-2t|}\log f\left(2\,(a-b)\left(t^2-t\right)+a\right)\mathrm{d}t - \int_0^1 \frac{1-t}{1-|1-2t|}\log f\left((1-t)\,a+tb\right)\mathrm{d}t - \int_0^1 \frac{t}{1-|1-2t|}\log f\left((1-t)\,b+ta\right)\mathrm{d}t\Bigg).$$

Proof. We prove the first inequality. Assume that f is a convex function. It follows from Lemma 2.1 that

$$f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2} - \frac{1}{2R}\left((1-t)\,f(a)\,+tf(b)-f\left((1-t)\,a+tb\right)\right).$$

Now, by replacing $a=(1-t)\,a+tb$ and $b=(1-t)\,b+ta$, we get

$$f\left(\frac{a+b}{2}\right) \le \frac{f\left((1-t)\,a+tb\right)+f\left((1-t)\,b+ta\right)}{2} - \frac{1}{2R}\Big((1-t)\,f\left((1-t)\,a+tb\right) + tf\left((1-t)\,b+ta\right)-f\left(2\,(a-b)\left(t^2-t\right)+a\right)\Big). \tag{5}$$

Thus,

$$f\left(\frac{a+b}{2}\right) \le \frac{f\left((1-t)\,a+tb\right)+f\left((1-t)\,b+ta\right)}{2} - \left(\frac{1-t}{1+|2t-1|}f\left((1-t)\,a+tb\right)+\frac{t}{1+|2t-1|}f\left((1-t)\,b+ta\right) - \frac{1}{1+|2t-1|}f\left(2\,(a-b)\left(t^2-t\right)+a\right)\right).$$

By taking integral over $0 \leq t \leq 1$, we get

$$f\left(\frac{a+b}{2}\right) \leq \int_0^1 f\left((1-t)\,a+tb\right)\mathrm{d}t$$
$$-\left(\int_0^1 \frac{1-t}{1+|2t-1|} f\left((1-t)\,a+tb\right)\mathrm{d}t + \int_0^1 \frac{t}{1+|2t-1|} f\left((1-t)\,b+ta\right)\mathrm{d}t\right.$$
$$\left.-\int_0^1 \frac{1}{1+|2t-1|} f\left(2\,(a-b)\left(t^2-t\right)+a\right)\mathrm{d}t\right).$$

Next, if f is log-convex, then we get

$$\log f\left(\frac{a+b}{2}\right) + \int_0^1 \frac{1-t}{1+|2t-1|}\log f\left((1-t)\,a+tb\right)\mathrm{d}t$$
$$+\int_0^1 \frac{t}{1+|2t-1|}\log f\left((1-t)\,b+ta\right)\mathrm{d}t$$
$$-\int_0^1 \frac{1}{1+|2t-1|}\log f\left(2\,(a-b)\left(t^2-t\right)+a\right)\mathrm{d}t$$
$$\leq \int_0^1 \log f\left((1-t)\,a+tb\right)\mathrm{d}t.$$

By applying exp, we obtain

$$f\left(\frac{a+b}{2}\right) + \exp\left(\int_0^1 \frac{1-t}{1+|2t-1|}\log f\left((1-t)\,a+tb\right)\mathrm{d}t\right.$$
$$+\int_0^1 \frac{t}{1+|2t-1|}\log f\left((1-t)\,b+ta\right)\mathrm{d}t$$
$$\left.-\int_0^1 \frac{1}{1+|2t-1|}\log f\left(2\,(a-b)\left(t^2-t\right)+a\right)\mathrm{d}t\right)$$
$$\leq \exp\left(\int_0^1 \log f\left((1-t)\,a+tb\right)\mathrm{d}t\right).$$

We prove the second inequality. Assume that f is a convex function. By Lemma 2.1, we have

$$\frac{f\left(a\right)+f\left(b\right)}{2} \leq f\left(\frac{a+b}{2}\right) + \frac{1}{2r}\left((1-t)\,f\left(a\right)+tf\left(b\right)-f\left((1-t)\,a+tb\right)\right).$$

Therefore,

$$\begin{aligned}
&\int_0^1 f\left(\left(1-t\right)a+tb\right)\mathrm{d}t \\
&\le f\left(\frac{a+b}{2}\right)+\int_0^1 \frac{1-t}{1-\left|1-2t\right|} f\left(\left(1-t\right)a+tb\right)\mathrm{d}t \\
&\quad+\int_0^1 \frac{t}{1-\left|1-2t\right|} f\left(\left(1-t\right)b+ta\right)-\int_0^1 \frac{1}{1-\left|1-2t\right|} f\left(2\left(a-b\right)\left(t^2-t\right)+a\right)\mathrm{d}t.
\end{aligned}$$

If f is log-convex, then the above inequality implies

$$\begin{aligned}
&\int_0^1 \log f\left(\left(1-t\right)a+tb\right)\mathrm{d}t \\
&\le \log f\left(\frac{a+b}{2}\right)+\int_0^1 \frac{1}{1-\left|1-2t\right|}\log f\left(2\left(a-b\right)\left(t^2-t\right)+a\right)\mathrm{d}t \\
&\quad-\int_0^1 \frac{1-t}{1-\left|1-2t\right|}\log f\left(\left(1-t\right)a+tb\right)\mathrm{d}t-\int_0^1 \frac{t}{1-\left|1-2t\right|}\log f\left(\left(1-t\right)b+ta\right)\mathrm{d}t.
\end{aligned}$$

Now, if we apply exp, we infer that

$$\begin{aligned}
&\exp\left(\int_0^1 \log f\left(\left(1-t\right)a+tb\right)\mathrm{d}t\right) \\
&\le f\left(\frac{a+b}{2}\right)+\exp\left(\int_0^1 \frac{1}{1-\left|1-2t\right|}\log f\left(2\left(a-b\right)\left(t^2-t\right)+a\right)\mathrm{d}t\right. \\
&\quad\left.-\int_0^1 \frac{1-t}{1-\left|1-2t\right|}\log f\left(\left(1-t\right)a+tb\right)\mathrm{d}t-\int_0^1 \frac{t}{1-\left|1-2t\right|}\log f\left(\left(1-t\right)b+ta\right)\mathrm{d}t\right),
\end{aligned}$$

as desired. □

3. Applications

Lemma 2.1 is a powerful tool for obtaining inequalities related to convex functions. In the following, we give examples showing how Lemma 2.1 affects getting operator inequalities.

For positive and invertible operators $A, B \in \mathcal{B}(\mathcal{H})$, define the geometric mean as

$$A\sharp_t B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^t A^{\frac{1}{2}} \qquad \left(0 \le t \le 1\right).$$

Let $x \in \mathcal{H}$ be a unit vector. The function $f\left(t\right) = \left\langle A\sharp_t Bx, x\right\rangle$ is a convex function on $[0,1]$ (see [**7**]).

- It follows from the inequality (5) that

$$f\left(\frac{1}{2}\right) \le \frac{f(t)+f(1-t)}{2} - \frac{1}{2R}\left((1-t)f(t)+tf(1-t)-f\left(2\left(t-t^2\right)\right)\right), \tag{6}$$

provided that $f\colon J \to \mathbb{R}$ is a convex function and $R=\max\{t,1-t\}$ with $0<t<1$.

The first inequality in Lemma 2.1 ensures that

$$\frac{f(a)+f(b)}{2} \le f\left(\frac{a+b}{2}\right) + \frac{1}{2r}\left((1-t)f(a)+tf(b)-f((1-t)a+tb)\right),$$

where $r=\min\{t,1-t\}$. By replacing $a=(1-t)a+tb$ and $b=(1-t)b+ta$, we get

$$\begin{aligned}&\frac{f((1-t)a+tb)+f((1-t)b+ta)}{2}\\ &\le f\left(\frac{a+b}{2}\right)+\frac{1}{2r}\Big((1-t)f((1-t)a+tb)\\ &\quad +tf((1-t)b+ta)-f\left(2(a-b)\left(t^2-t\right)+a\right)\Big),\end{aligned}$$

which is an interesting inequality in itself. The above inequality implies

$$\frac{f(t)+f(1-t)}{2} \le f\left(\frac{1}{2}\right) + \frac{1}{2r}\left((1-t)f(t)+tf(1-t)-f\left(2\left(t-t^2\right)\right)\right). \tag{7}$$

Thus, by (6) and (7), we have

$$\begin{aligned}&\frac{A\sharp_t B + A\sharp_{1-t}B}{2} - \frac{1}{2r}\left((1-t)A\sharp_t B + tA\sharp_{1-t}B - A\sharp_{2(t-t^2)}B\right) \le A\sharp B\\ &\le \frac{A\sharp_t B + A\sharp_{1-t}B}{2} - \frac{1}{2R}\left((1-t)A\sharp_t B + tA\sharp_{1-t}B - A\sharp_{2(t-t^2)}B\right)\end{aligned}$$

where $r=\min\{t,1-t\}$, $R=\max\{t,1-t\}$, and $0<t<1$. Notice that, in general, we have

$$A\sharp B \le \frac{A\sharp_t B + A\sharp_{1-t}B}{2};\ \ (0\le t\le 1).$$

- By Lemma 2.1, we can write

$$\begin{aligned}&(1-t)f(0)+tf(1)-2R\left(\frac{f(0)+f(1)}{2}-f\left(\frac{1}{2}\right)\right)\\ &\le f(t)\\ &\le (1-t)f(0)+tf(1)-2r\left(\frac{f(0)+f(1)}{2}-f\left(\frac{1}{2}\right)\right),\end{aligned}$$

which in turn demonstrates that

$$\begin{aligned}(1-t)A+tB-2R\left(\frac{A+B}{2}-A\sharp B\right) &\le A\sharp_t B\\ &\le (1-t)A+tB-2r\left(\frac{A+B}{2}-A\sharp B\right).\end{aligned}$$

Consequently,

$$B - A - \frac{R}{t}\left(A + B - 2\left(A \sharp B\right)\right) \leq T_t\left(A|B\right) \leq B - A - \frac{r}{t}\left(A + B - 2\left(A \sharp B\right)\right),$$

where

$$T_t\left(A|B\right) = \frac{A \sharp_t B - A}{t} \qquad (0 \leq t \leq 1)$$

is called the operator Tsallis relative operator entropy.

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