Acta Math. Univ. Comenianae
Vol. XCI, 4 (2022), pp. 365–376

ROUGH STATISTICAL CONVERGENCE OF COMPLEX UNCERTAIN TRIPLE SEQUENCE

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ABSTRACT. We investigate the rough statistical convergence of complex uncertain triple sequences in this research. We show three forms of rough statistically convergent complex uncertain triple sequences and rough λ^3-statistical convergence in measure, as well as other fundamental features.

1. INTRODUCTION

Zygmund [**33**] used the term "almost convergence" to describe the concept of statistical convergence. It was formally presented by Fast [**12**]. Later the idea was associated with summability theory by Fridy [**14**] and many others [**1, 3, 4, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 30, 31**].

The theory of uncertainty plays a vital role not only in pure mathematics. The majority of human decisions are made in the face of uncertainty. A specific sort of mathematical measure can be used to represent the performance of an uncertainty. Fuzziness is another paradigm for uncertainty pioneered by Zadeh [**32**] in 1965 using membership functions. Fuzzy set theory and probability theory are undeniably valuable tools for dealing with uncertainty. However, in real life, natural language expressions such as "middle age", "about 30 kilometers", "about 15 degrees Celsius", and "roughly 6 kilograms" are commonly employed to represent imperfect knowledge or facts. But multiple studies have demonstrated that such utterances are neither random nor fuzzy. These facts encourage the development of uncertainty theory as an axiomatic mathematics branch for representing human uncertainty. To model uncertainty, Liu [**25**] established an uncertainty theory that is a branch of mathematics based on normality, monotonicity, self-duality, countable subadditivity, and product measure axioms. Liu [**25**] defined the concept of uncertain variables as a function from a measurable space to $\mathbb{R}$. If real numbers are rebuilt with a set of complex numbers, it is named as complex uncertain variable which was worked by Peng [**26**]. Recently, various

Received April 24, 2022; revised October 5, 2022.
2020 *Mathematics Subject Classification.* Primary 40A35.
Key words and phrases. Statistical convergence; λ-statistical convergence; rough convergence; uncertain sequence.

researchers have also done significant studies based on complex uncertain variables, see, [**6, 7, 8, 9, 10, 11, 27, 28, 29**]. The conception of rough convergence was first investigated by Phu [**23**] in finite-dimensional normed spaces. Phu [**24**] expanded the results given in [**23**] to infinite-dimensional normed spaces. In [**2**], Aytar investigated rough statistical convergence. The notion of λ-statistical convergence was examined by Mursaleen [**18**]. Das et al. [**5**] expanded these ideas in 2015, including rough λ-statistical convergence in probability. For the purpose of delve deeper into uncertainty theory, we defined rough statistical convergence of complex uncertain triple sequences and worked on some convergence conceptions such as rough statistical convergence in measure, rough λ^3-statistical convergence in measure, rough converges in distribution, and rough statistical convergence in mean of complex uncertain triple sequences, obtaining some inter-relationships between them. In the entire paper, let r be a positive, non-negative real number and (ϖ_{pqr}) be a complex uncertain triple sequence.

2. Main results

Definition 2.1. A complex uncertain triple sequence (ϖ_{pqr}) is called to be rough statistical (rst) convergent almost surely to ϖ with roughness degree r provided that for any event Λ with $\mathcal{M}(\Lambda) = 1$, so that

$$\delta\left((p,q,r) \in \mathbb{N}^3 : \|\varpi_{pqr}(\gamma) - \varpi(\gamma)\| \geq r + \sigma\right) = 0$$

for each $\gamma \in \Lambda$. When the above equation supplies, ϖ is a rough statistical limit point of $\{\varpi_{pqr}\}$, which is generally no more unique. Hence, we contemplate r-statistical limit set of $\{\varpi_{pqr}\}$ determined by

$$st\text{-LIM}^r \varpi_{pqr} := \left\{\varpi : \varpi_{pqr} \xrightarrow{rst} \varpi\right\}.$$

Example 2.2. Take into account the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. It becomes $\Gamma = \{\gamma_1, \gamma_2, \dots\}$ with $\mathcal{M}(\Lambda) = \sum\limits_{\gamma_p,\gamma_q,\gamma_r \in \Lambda} 2^{-(p+q+r)}$. We determine a complex uncertain variable by

$$\varpi_{pqr}(\gamma) = \begin{cases} \mathrm{i}\cdot(-1)^{p+q+r} & \text{when } \gamma = \gamma_{p+q+r}, \\ 0 & \text{otherwise,} \end{cases}$$

for $p \neq k^2$, $q \neq l^2$, $r \neq m^2$ and

$$\varpi_{pqr}(\gamma) = \begin{cases} \mathrm{i}\cdot(p+q+r) & \text{when } \gamma = \gamma_{p+q+r}, \\ 0 & \text{otherwise,} \end{cases}$$

for $p = k^2$, $q = l^2$, $r = m^2$ and $\varpi = 0$. Then, $\varpi_{pqr} \xrightarrow{rst} \varpi$ where

$$st - \text{LIM}^r \varpi_{pqr} = \begin{cases} \emptyset & \text{for } r < 1, \\ \varpi \mathrm{i},\ \varpi \in [1-r, r-1] & \text{for } r \geq 1. \end{cases}$$

Additionally, we get that the sequence (ϖ_{pqr}) is not rough convergent a.s. to ϖ, however, it is rst-convergent a.s. to ϖ for any $r \geq 1$.

Theorem 2.3.

(i) *When* $\varpi_{pqr} \overset{rst\ a.s.}{\longrightarrow} \varpi$, *then* $\beta\varpi_{pqr} \overset{rst\ a.s.}{\longrightarrow} \beta\varpi$, *where* $\beta \in \mathbb{C}$.

(ii) *When* $\varpi_{pqr} \overset{rst\ a.s.}{\longrightarrow} \varpi$ *and* $\vartheta_{pqr} \overset{rst\ a.s.}{\longrightarrow} \vartheta$, *then* $\beta\varpi_{pqr} + \beta\vartheta_{pqr} \overset{rst\ a.s.}{\longrightarrow} \varpi + \vartheta$.

Proof. It is obvious, so omitted. □

Theorem 2.4. *The* r*-statistical limit set of a complex uncertain triple sequence is convex.*

Proof. Presume that $\varpi_0, \varpi_1 \in st\text{-LIM}^r \varpi_{pqr}$ for the complex uncertain triple sequence (ϖ_{pqr}), and let $\sigma > 0$ be given. Determine

$$T_1 := \left\{(p,q,r) \in \mathbb{N}^3 : \|\varpi_{pqr} - \varpi_0\| \geq r + \sigma\right\}$$

and

$$T_2 := \left\{(p,q,r) \in \mathbb{N}^3 : \|\varpi_{pqr} - \varpi_1\| \geq r + \sigma\right\}.$$

Since $\varpi_0, \varpi_1 \in st\text{-LIM}^r \varpi_{pqr}$, we get $\delta(T_1) = \delta(T_2) = 0$. So, we acquire

$$\|\varpi_{pqr} - [(1-\tau)\varpi_0 + \tau\varpi_1]\| = \|(1-\tau)(\varpi_{pqr} - \varpi_0) + \tau(\varpi_{pqr} - \varpi_1)\| < r + \sigma$$

for all $(p,q,r) \in T_1^c \cap T_2^c$ and every $\tau \in [0,1]$. Since $\delta(T_1^c \cap T_2^c) = 1$, we obtain

$$\delta\left\{(p,q,r) \in \mathbb{N}^3 : \|\varpi_{pqr} - [(1-\tau)\varpi_0 + \tau\varpi_1]\| \geq r + \sigma\right\} = 0,$$

namely, $[(1-\tau)\varpi_0 + \tau\varpi_1] \in st\text{-LIM}^r \varpi_{pqr}$, which gives the convexity of the set $st\text{-LIM}^r \varpi_{pqr}$. □

Definition 2.5. A sequence (ϖ_{pqr}) is named to be rough statistical convergent in measure to ϖ with roughness degree r provided that for each $\sigma, \kappa > 0$,

$$\delta\left\{(p,q,r) \in \mathbb{N}^3 : \mathcal{M}\left\{\gamma : \|\varpi_{pqr}(\gamma) - \varpi(\gamma)\| \geq \kappa\right\} \geq r + \sigma\right\} = 0$$

for each $\gamma \in \Lambda$. We write $\varpi_{pqr} \overset{r-S_{\mathcal{M}}^U}{\longrightarrow} \varpi$.

Example 2.6. Contemplate the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \dots\}$ with

$$\mathcal{M}(\Lambda) = \begin{cases} \sup\limits_{\gamma_{p+q+r} \in \Lambda} \dfrac{p+q+r}{2(p+q+r)+1} & \text{when } \sup\limits_{\gamma_{p+q+r} \in \Lambda} \dfrac{p+q+r}{2(p+q+r)+1} < 0.5, \\ 1 - \sup\limits_{\gamma_{p+q+r} \in \Lambda^c} \dfrac{p+q+r}{2(p+q+r)+1} & \text{when } \sup\limits_{\gamma_{p+q+r} \in \Lambda^c} \dfrac{p+q+r}{2(p+q+r)+1} < 0.5, \\ 0.5 & \text{otherwise,} \end{cases}$$

and think about the uncertain variable (ϖ_{pqr}) defined by

$$\varpi_{pqr}(\gamma) = \begin{cases} \mathrm{i} \cdot (p+q+r) & \text{when } \gamma = \gamma_{p+q+r}, \\ 0 & \text{otherwise,} \end{cases}$$

for $p = k^2$, $q = l^2$, $r = m^2$, and $\varpi_{pqr}(\gamma) = 0$, for $p \neq k^2$, $q \neq l^2$, $r \neq m^2$. Also take $\varpi \equiv 0$. Then, we obtain

$$\delta\left\{(p,q,r) \in \mathbb{N}^3 : \mathcal{M}\left\{\gamma : \|\varpi_{pqr}(\gamma) - \varpi(\gamma)\| \geq \kappa\right\} \geq r + \sigma\right\} = 0$$

for $r \geq 0$. This demonstrates that (ϖ_{pqr}) is rst-convergent in measure to ϖ for $r \geq 0$. In addition, for $r \in \left[0, \frac{1}{2}\right)$, the sequence (ϖ_{pqr}) is not rough convergent in measure to ϖ, howewer it is rst-convergent in measure to ϖ.

Theorem 2.7. *When* $\varpi_{pqr} \overset{r_1-S^U_{\mathcal{M}}}{\longrightarrow} \varpi_1$ *and* $\varpi_{pqr} \overset{r_2-S^U_{\mathcal{M}}}{\longrightarrow} \varpi_2$*, then* $\mathcal{M}\left\{\|\varpi_1-\varpi_2\| \geq r_1+r_2\right\}=0.$

Proof. Assume that σ, κ are any two $\mathbb{R}^+$ and take

$$(u,v,w) \in \left\{(p,q,r) \in \mathbb{N}^3 : \mathcal{M}\left(\|\varpi_{pqr}-\varpi_1\| \geq r_1+\frac{\sigma}{2}\right) < \frac{\kappa}{2}\right\}$$
$$\cap \left\{(p,q,r) \in \mathbb{N}^3 : \mathcal{M}\left(\|\varpi_{pqr}-\varpi_2\| \geq r_2+\frac{\sigma}{2}\right) < \frac{\kappa}{2}\right\}$$

(because the asymptotic density of both sets is equal to one, the existence of (u,v,w) is assured). So,

$$\mathcal{M}\left(\|\varpi_1-\varpi_2\| \geq r_1+r_2+\sigma\right)$$
$$\leq \mathcal{M}\left(\|\varpi_{pqr}-\varpi_1\| \geq r_1+\frac{\sigma}{2}\right)+\mathcal{M}\left(\|\varpi_{pqr}-\varpi_2\| \geq r_2+\frac{\sigma}{2}\right) < \kappa.$$

This means that $\mathcal{M}\left\{\|\varpi_1-\varpi_2\| \geq r_1+r_2\right\}=0.$ $\square$

Theorem 2.8.

(i) $\varpi_{pqr} \overset{r_1-S^U_{\mathcal{M}}}{\longrightarrow} \varpi \Leftrightarrow \varpi_{pqr}-\varpi \overset{r_1-S^U_{\mathcal{M}}}{\longrightarrow} 0.$

(ii) $\varpi_{pqr} \overset{r-S^U_{\mathcal{M}}}{\longrightarrow} \varpi \Leftrightarrow \alpha\varpi_{pqr} \overset{r-S^U_{\mathcal{M}}}{\longrightarrow} \alpha\varpi$*, where* $\alpha \in \mathbb{C}$.

(iii) $\varpi_{pqr} \overset{r_1-S^U_{\mathcal{M}}}{\longrightarrow} \varpi$ *and* $\nu_{pqr} \overset{r_2-S^U_{\mathcal{M}}}{\longrightarrow} \nu \Rightarrow \varpi_{pqr}+\nu_{pqr} \overset{(r_1+r_2)-S^U_{\mathcal{M}}}{\longrightarrow} \varpi+\nu.$

(iv) $\varpi_{pqr} \overset{r_1-S^U_{\mathcal{M}}}{\longrightarrow} \varpi$ *and* $\nu_{pqr} \overset{r_2-S^U_{\mathcal{M}}}{\longrightarrow} \nu \Rightarrow \varpi_{pqr}-\nu_{pqr} \overset{(r_1+r_2)-S^U_{\mathcal{M}}}{\longrightarrow} \varpi-\nu.$

(v) $\varpi_{pqr} \overset{r-S^U_{\mathcal{M}}}{\longrightarrow} \varpi$*, then for all* $\sigma>0$*, there is a* $(k,l,m) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ *so that for any* $\kappa>0$,

$$\lim_{u,v,w\to\infty} \frac{1}{uvw}\left|\left\{p \leq u, q \leq v, r \leq w : \mathcal{M}\left(\|\varpi_{pqr}-\varpi_{klm}\| \geq 2r+\sigma\right) \geq \kappa\right\}\right|=0.$$

Proof. Assume that σ, κ be any two $\mathbb{R}^+$. Then:

(i) The proof is self-evident, thus it is removed.

(ii) When $\alpha=0$, then the claim is obvious. So, presuming $\alpha \neq 0$, then

$$\left\{(p,q,r) \in \mathbb{N}^3 : \mathcal{M}\left(\|\alpha\varpi_{pqr}-\alpha\varpi\| \geq |\alpha|\, r+\sigma\right) \geq \kappa\right\}$$
$$=\left\{(p,q,r) \in \mathbb{N}^3 : \mathcal{M}\left(\|\varpi_{pqr}-\varpi\| \geq r+\frac{\sigma}{|\alpha|}\right) \geq \kappa\right\}.$$

As a result, $\alpha\varpi_{pqr} \overset{r-S^U_{\mathcal{M}}}{\longrightarrow} \alpha\varpi$, where $\alpha \in \mathbb{C}$.

(iii)

$$\mathcal{M}\left(\|(\varpi_{pqr}+\nu_{pqr})-(\varpi+\nu)\| \geq r_1+r_2+\sigma\right)$$
$$=\mathcal{M}\left(\|(\varpi_{pqr}-\varpi)+(\nu_{pqr}-\nu)\| \geq r_1+r_2+\sigma\right)$$
$$\leq \mathcal{M}\left(\|\varpi_{pqr}-\varpi\| \geq r_1+\frac{\sigma}{2}\right)+\mathcal{M}\left(\|\nu_{pqr}-\nu\| \geq r_2+\frac{\sigma}{2}\right).$$

This gives

$$\begin{aligned}
&\left\{(p,q,r)\in\mathbb{N}^3 : \mathcal{M}\left(\|(\varpi_{pqr}+\nu_{pqr})-(\varpi+\nu)\|\geq r_1+r_2+\sigma\right)\geq\kappa\right\}\\
&\subseteq\left\{(p,q,r)\in\mathbb{N}^3 : \mathcal{M}\left(\|\varpi_{pqr}-\varpi\|\geq r_1+\frac{\sigma}{2}\right)\geq\frac{\kappa}{2}\right\}\\
&\quad\cup\left\{(p,q,r)\in\mathbb{N}^3 : \mathcal{M}\left(\|\nu_{pqr}-\nu\|\geq r_2+\frac{\sigma}{2}\right)\geq\frac{\kappa}{2}\right\}.
\end{aligned}$$

Hence, $\varpi_{pqr}+\nu_{pqr}\overset{(r_1+r_2)-S^U_{\mathcal{M}}}{\longrightarrow}\varpi+\nu$.

(iv) Similar to the preceding evidence, and hence omitted.

(v) Select $(k,l,m)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ to be such that $\mathcal{M}\left(\|\varpi_{klm}-\varpi\|\geq r+\frac{\sigma}{2}\right)\geq\frac{\kappa}{2}$ (the existence of (k,l,m) is ensured). The assertion is therefore obvious from the inequality

$$\begin{aligned}
\mathcal{M}\left(\|\varpi_{pqr}-\varpi_{klm}\|\geq 2r+\sigma\right)&\leq\mathcal{M}\left(\|\varpi_{pqr}-\varpi\|\geq r+\frac{\sigma}{2}\right)+\mathcal{M}\left(\|\varpi_{klm}-\varpi\|\geq r+\frac{\sigma}{2}\right)\\
&\leq\frac{\kappa}{2}+\mathcal{M}\left(\|\varpi_{pqr}-\varpi\|\geq r+\frac{\sigma}{2}\right).
\end{aligned}$$

Hence, we obtain

$$\lim_{u,v,w\to\infty}\frac{1}{uvw}\left|\left\{p\leq u,q\leq v,r\leq w : \mathcal{M}\left(\|\varpi_{pqr}-\varpi_{klm}\|\geq 2r+\sigma\right)\geq\kappa\right\}\right|=0.\quad\square$$

Theorem 2.9. *Rough statistical convergence in measure does not mean rough statistical convergence a.s.*

Example 2.10. Contemplate the uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to be $\{\gamma_1,\gamma_2,\dots\}$ with

$$\mathcal{M}(\Lambda)=\begin{cases}\sup\limits_{\gamma_{p+q+r}\in\Lambda}\dfrac{p+q+r}{5(p+q+r)+1} & \text{when } \sup\limits_{\gamma_{p+q+r}\in\Lambda}\dfrac{p+q+r}{5(p+q+r)+1}<\dfrac{1}{5},\\ 1-\sup\limits_{\gamma_{p+q+r}\in\Lambda^c}\dfrac{p+q+r}{5(p+q+r)+1} & \text{when } \sup\limits_{\gamma_{p+q+r}\in\Lambda^c}\dfrac{p+q+r}{5\,(p+q+r)+1}<\dfrac{1}{5},\\ 0.5 & \text{otherwise,}\end{cases}$$

and the uncertain variable (ϖ_{pqr}) described by

$$\varpi_{pqr}(\gamma)=\begin{cases}\mathrm{i}\cdot(p+q+r)^3 & \text{when } \gamma=\gamma_{p+q+r},\\ 0 & \text{otherwise,}\end{cases}$$

for $p,q,r=1,2,3,\dots$. Also take $\varpi\equiv 0$. Then, for any $\kappa>0$, we obtain

$$\mathcal{M}\left\{\gamma : \|\varpi_{pqr}(\gamma)-\varpi(\gamma)\|\geq\kappa\right\}=\mathcal{M}\left\{\gamma_{p+q+r}\right\}=\frac{p+q+r}{5(p+q+r)+1}.$$

The sequence (ϖ_{pqr}) is thus rst-convergent to ϖ for $r\geq\frac{1}{5}$. However, it is not rst-convergent a.s. to ϖ.

Definition 2.11. A sequence (ϖ_{pqr}) is called to be rough λ^3-statistical convergence in measure to ϖ with roughness degree r provided that for $\sigma,\kappa>0$,

$$\lim_{k,l,m\to\infty}\frac{1}{\lambda_{klm}}\left|\left\{p\in I_k,q\in J_l,r\in K_m : \mathcal{M}\left(\|\varpi_{pqr}-\varpi\|\geq r+\sigma\right)\geq\kappa\right\}\right|=0,$$

where $I_k=[k-\lambda_k+1,k]$, $J_l=[l-\lambda_l+1,l]$, $K_m=[m-\lambda_m+1,m]$. We write $\varpi_{pqr}\overset{r-S^U_{\lambda^3}}{\longrightarrow}\varpi$ in this situation.

Definition 2.12. If

$$\lim_{k,l,m\to\infty}\frac{1}{\lambda_{klm}}\sum_{p\in I_k,q\in J_l,r\in K_m}\mathcal{M}\left(\|\varpi_{pqr}-\varpi\|\geq r+\sigma\right)=0,$$

then a complex uncertain sequence (ϖ_{pqr}) is called to be rough (V,λ)-summable in measure to ϖ with roughness degree r. We write $\varpi_{pqr}\overset{r-[V,\lambda]^{\mathcal{M}}}{\longrightarrow}\varpi$ in this situation.

Theorem 2.13. *The following are identical for any complex uncertain sequence* (ϖ_{pqr}).

(i) $\varpi_{pqr}\overset{r-S^U_{\lambda^3}}{\longrightarrow}\varpi$.

(ii) $\varpi_{pqr}\overset{r-[V,\lambda]^{\mathcal{M}}}{\longrightarrow}\varpi$.

Proof. (i) $\Rightarrow$ (ii): Assume that $\varpi_{pqr}\overset{r-S^U_{\lambda^3}}{\longrightarrow}\varpi$. Then, we can write

$$\begin{aligned}&\frac{1}{\lambda_{klm}}\sum_{p\in I_k,q\in J_l,r\in K_m}\mathcal{M}\left(\|\varpi_{pqr}-\varpi\|\geq r+\sigma\right)\\&=\frac{1}{\lambda_{klm}}\sum_{\substack{p\in I_k,q\in J_l,r\in K_m\\ \mathcal{M}(\|\varpi_{pqr}-\varpi\|\geq r+\sigma)\geq\frac{\kappa}{2}}}\mathcal{M}\left(\|\varpi_{pqr}-\varpi\|\geq r+\sigma\right)\\&\quad+\frac{1}{\lambda_{klm}}\sum_{\substack{p\in I_k,q\in J_l,r\in K_m\\ \mathcal{M}(\|\varpi_{pqr}-\varpi\|\geq r+\sigma)<\frac{\kappa}{2}}}\mathcal{M}\left(\|\varpi_{pqr}-\varpi\|\geq r+\sigma\right)\\&\leq\frac{1}{\lambda_{klm}}\left|\left\{p\in I_k,q\in J_l,r\in K_m:\mathcal{M}\left(\|\varpi_{pqr}-\varpi\|\geq r+\sigma\right)\geq\frac{\kappa}{2}\right\}\right|+\frac{\kappa}{2}.\end{aligned}$$

As a result, we get $\varpi_{pqr}\overset{r-[V,\lambda]^{\mathcal{M}}}{\longrightarrow}\varpi$.

(ii) $\Rightarrow$ (i): Now, presume that condition (ii) supplies. Then

$$\begin{aligned}&\sum_{p\in I_k,q\in J_l,r\in K_m}\mathcal{M}\left(\|\varpi_{pqr}-\varpi\|\geq r+\sigma\right)\\&\geq\sum_{\substack{p\in I_k,q\in J_l,r\in K_m\\ \mathcal{M}(\|\varpi_{pqr}-\varpi\|\geq r+\sigma)\geq\kappa}}\mathcal{M}\left(\|\varpi_{pqr}-\varpi\|\geq r+\sigma\right)\\&\geq\kappa\left|\left\{p\in I_k,q\in J_l,r\in K_m:\mathcal{M}\left(\|\varpi_{pqr}-\varpi\|\geq r+\sigma\right)\geq\kappa\right\}\right|.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{\lambda_{klm}}\sum_{p\in I_k,q\in J_l,r\in K_m}\mathcal{M}\left(\|\varpi_{pqr}-\varpi\|\geq r+\sigma\right)\\&\geq\frac{1}{\lambda_{klm}}\left|\left\{p\in I_k,q\in J_l,r\in K_m:\mathcal{M}\left(\|\varpi_{pqr}-\varpi\|\geq r+\sigma\right)\geq\kappa\right\}\right|.\end{aligned}$$

As a result, we get $\varpi_{pqr}\overset{r-S^U_{\lambda^3}}{\longrightarrow}\varpi$. $\square$

Definition 2.14. Assume that $\Phi, \Phi_1, \Phi_2, \dots$ be the complex uncertainty distributions of complex uncertain variables ϖ, ϖ_{pqr}, respectively, where $p, q, r \in \mathbb{N}$. A sequence (ϖ_{pqr}) is defined as rough statistical convergence in distribution to ϖ with roughness degree r if for $\sigma > 0$,

$$\delta\left((p,q,r) \in \mathbb{N}^3 : \|\Phi_{pqr}(y) - \Phi(y)\| \geq r + \sigma\right) = 0$$

for each $\gamma \in \Lambda$ and for all y at which $\Phi(y)$ is continuous.

Example 2.15. Contemplate the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \dots\}$ with

$$\mathcal{M}(\Lambda) = \begin{cases} \sup\limits_{\gamma_{p+q+r} \in \Lambda} \dfrac{p+q+r}{3(p+q+r)+1} & \text{when } \sup\limits_{\gamma_{p+q+r} \in \Lambda} \dfrac{p+q+r}{3(p+q+r)+1} < \dfrac{1}{3}, \\ 1 - \sup\limits_{\gamma_{p+q+r} \in \Lambda^c} \dfrac{p+q+r}{3(p+q+r)+1} & \text{when } \sup\limits_{\gamma_{p+q+r} \in \Lambda^c} \dfrac{p+q+r}{3\,(p+q+r)+1} < \dfrac{1}{3}, \\ 0.5 & \text{otherwise,} \end{cases}$$

and the uncertain variable (ϖ_{pqr}) be described by

$$\varpi_{pqr}(\gamma) = \begin{cases} \mathrm{i} \cdot (p+q+r)^2 & \text{when } \gamma = \gamma_{p+q+r}, \\ 0 & \text{otherwise,} \end{cases}$$

for $p = k^2$, $q = l^2$, $r = m^2$, and $\varpi_{pqr}(\gamma) = 0$, for $p \neq k^2$, $q \neq l^2$, $r \neq m^2$. In addition, take $\varpi \equiv 0$. Then, for $p = k^2$, $q = l^2$, $r = m^2$, we get the uncertainty distribution of uncertain variable (ϖ_{pqr}) as

$$\Phi_{pqr}(y) = \Phi_{pqr}\left(u + \mathrm{i}v\right) = \begin{cases} 0 & \text{if } u < 0,\ v < \infty, \\ 0 & \text{if } u \geq 0,\ v < 0, \\ 1 - \dfrac{p+q+r}{3(p+q+r)+1} & \text{if } u \geq 0,\ 0 \leq v < (p+q+r)^2, \\ 1 & \text{if } u \geq 0,\ v \geq (p+q+r)^2. \end{cases}$$

$$\Phi_{pqr}(y) = \begin{cases} 0 & \text{when } u < 0,\ v < \infty, \\ 0 & \text{when } u \geq 0,\ v < 0, \\ 1 & \text{when } u \geq 0,\ v \geq 0. \end{cases} \qquad \Phi(y) = \begin{cases} 0 & \text{when } u < 0,\ v < \infty, \\ 0 & \text{when } u \geq 0,\ v < 0, \\ 1 & \text{when } u \geq 0,\ v \geq 0. \end{cases}$$

is also the complex uncertainty distribution of uncertain variable ϖ. Thus, we obtain

$$\delta\left((p,q,r) \in \mathbb{N}^3 : \|\Phi_{pqr}(y) - \Phi(y)\| \geq r + \sigma\right) = 0$$

for $r \geq 0$. In addition, we find that the sequence (ϖ_{pqr}) is not rough convergent in distribution to ϖ, however it rst-convergent in distribution to ϖ for $r \in \left[0, \frac{1}{3}\right)$.

Theorem 2.16. *In distribution, rough statistical convergence does not imply rough statistical convergence a.s.*

Example 2.17. Contemplate the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \dots\}$ with

$$\mathcal{M}(\Lambda) = \begin{cases} \sup\limits_{\gamma_{p+q+r} \in \Lambda} \dfrac{p+q+r}{3(p+q+r)+1} & \text{when } \sup\limits_{\gamma_{p+q+r} \in \Lambda} \dfrac{p+q+r}{3(p+q+r)+1} < \dfrac{1}{3}, \\ 1 - \sup\limits_{\gamma_{p+q+r} \in \Lambda^c} \dfrac{p+q+r}{3(p+q+r)+1} & \text{when } \sup\limits_{\gamma_{p+q+r} \in \Lambda^c} \dfrac{p+q+r}{3\,(p+q+r)+1} < \dfrac{1}{3}, \\ 0.5 & \text{otherwise,} \end{cases}$$

and contemplate the uncertain variable (ϖ_{pqr}) described by

$$\varpi_{pqr}(\gamma)=\begin{cases} \mathrm{i}\cdot(p+q+r)^2 & \text{when } \gamma=\gamma_{p+q+r},\\ 0 & \text{otherwise,}\end{cases}$$

for $p=k^2$, $q=l^2$, $r=m^2$, and $\varpi_{pqr}(\gamma)=0$. Then, as

$$\Phi_{pqr}(y)=\Phi_{pqr}\left(u+\mathrm{i}v\right)=\begin{cases} 0 & \text{if } u<0,\ v<\infty,\\ 0 & \text{if } u\geq 0,\ v<0,\\ 1-\dfrac{p+q+r}{3(p+q+r)+1} & \text{if } u\geq 0,\ 0\leq v<(p+q+r)^2,\\ 1 & \text{if } u\geq 0,\ v\geq (p+q+r)^2,\end{cases}$$

for $p,q,r\in\mathbb{N}$, we have the uncertainty distribution of uncertain variable (ϖ_{pqr}). In addition, the complex uncertainty distribution of uncertain variable ϖ is

$$\Phi(y)=\begin{cases} 0 & \text{when } u<0,\ v<\infty,\\ 0 & \text{when } u\geq 0,\ v<0,\\ 1 & \text{when } u\geq 0,\ v\geq 0.\end{cases}$$

Thus, we obtain

$$\delta\left((p,q,r)\in\mathbb{N}^3:\|\Phi_{pqr}(y)-\Phi(y)\|\geq r+\sigma\right)=0$$

for $r\geq\frac{1}{3}$. However, it is not rst-convergent a.s. to ϖ.

Definition 2.18. A sequence (ϖ_{pqr}) is said to have rough statistical convergence in mean to ϖ with roughness degree r if and only if

$$\delta\left((p,q,r)\in\mathbb{N}^3:E\left[\|\varpi_{pqr}(\gamma)-\varpi\left(\gamma\right)\|\right]\geq r+\sigma\right)=0$$

for each $\gamma\in\Lambda$ and $\sigma>0$.

Example 2.19. Contemplate the uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to be $\{\gamma_1,\gamma_2,\dots\}$ with

$$\mathcal{M}(\Lambda)=\begin{cases} \sup\limits_{\gamma_{p+q+r}\in\Lambda}\dfrac{p+q+r}{2(p+q+r)^2+1} & \text{if } \sup\limits_{\gamma_{p+q+r}\in\Lambda}\dfrac{p+q+r}{2(p+q+r)^2+1}<0.5,\\ 1-\sup\limits_{\gamma_{p+q+r}\in\Lambda^c}\dfrac{p+q+r}{2(p+q+r)^2+1} & \text{if } \sup\limits_{\gamma_{p+q+r}\in\Lambda^c}\dfrac{p+q+r}{2(p+q+r)^2+1}<0.5,\\ 0.5 & \text{otherwise,}\end{cases}$$

and the uncertain variable (ϖ_{pqr}) be described by

$$\varpi_{pqr}(\gamma)=\begin{cases} (p+q+r)\cdot\mathrm{i} & \text{when } \gamma=\gamma_{p+q+r},\\ 0 & \text{otherwise,}\end{cases}$$

for $p=k^2$, $q=l^2$, $r=m^2$, and $\varpi_{pqr}(\gamma)=0$, for $p\neq k^2$, $q\neq l^2$, $r\neq m^2$. Also take $\varpi\equiv 0$. Then, for $p=k^2$, $q=l^2$, $r=m^2$, we obtain the uncertainty distribution of uncertain variable (ϖ_{pqr}) as

$$\Phi_{pqr}(y)=\Phi_{pqr}\left(u+\mathrm{i}v\right)=\begin{cases} 0 & \text{if } u<0,\ v<\infty,\\ 0 & \text{if } u\geq 0,\ v<0,\\ 1-\dfrac{p+q+r}{2(p+q+r)^2+1} & \text{if } u\geq 0,\ 0\leq v<p+q+r,\\ 1 & \text{if } u\geq 0,\ v\geq p+q+r.\end{cases}$$

$$\Phi_{pqr}(y)=\begin{cases}0 & \text{when } u<0,\ v<\infty,\\ 0 & \text{when } u\geq 0,\ v<0,\\ 1 & \text{when } u\geq 0,\ v\geq 0.\end{cases}$$

In addition, the complex uncertainty distribution of uncertain variable ϖ is

$$\Phi(y)=\begin{cases}0 & \text{if } u<0,\ v<\infty,\\ 0 & \text{if } u\geq 0,\ v<0,\\ 1 & \text{if } u\geq 0,\ v\geq 0.\end{cases}$$

As a result, we acquire for $p=k^2$, $q=l^2$, $r=m^2$,

$$E\left[\|\varpi_{pqr}(\gamma)-\varpi\left(\gamma\right)\|\right]=\frac{(p+q+r)^2}{2(p+q+r)^2+1}$$
$$\implies \quad \delta\left\{(p,q,r)\in\mathbb{N}^3 : E\left[\|\varpi_{pqr}(\gamma)-\varpi\left(\gamma\right)\|\right]\geq r+\sigma\right\}=0$$

for $r\geq 0$. In addition, we get that the sequence (ϖ_{pqr}) is not rough convergent in mean to ϖ, hovewer it is rst-convergent in mean to ϖ for $r\in\left[0,\frac{1}{2}\right)$.

Theorem 2.20. *Rough statistical convergence in mean does not imply rough statistical convergence a.s.*

Example 2.21. Contemplate the uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to be $\{\gamma_1,\gamma_2,\dots\}$ with

$$\mathcal{M}(\Lambda)=\begin{cases}\sup\limits_{\gamma_{p+q+r}\in\Lambda}\dfrac{p+q+r}{2(p+q+r)^2+1} & \text{if } \sup\limits_{\gamma_{p+q+r}\in\Lambda}\dfrac{p+q+r}{2\left(p+q+r\right)^2+1}<0.5,\\ 1-\sup\limits_{\gamma_{p+q+r}\in\Lambda^c}\dfrac{p+q+r}{2(p+q+r)^2+1} & \text{if } \sup\limits_{\gamma_{p+q+r}\in\Lambda^c}\dfrac{p+q+r}{2(p+q+r)^2+1}<0.5,\\ 0.5 & \text{otherwise,}\end{cases}$$

and the uncertain variable (ϖ_{pqr}) be described by

$$\varpi_{pqr}(\gamma)=\begin{cases}(p+q+r)\cdot \mathrm{i} & \text{when } \gamma=\gamma_{p+q+r},\\ 0 & \text{otherwise}\end{cases}$$

for $p=k^2$, $q=l^2$, $r=m^2$ and $\varpi\equiv 0$. Then, as

$$\Phi_{pqr}(y)=\Phi_{pqr}\left(u+iv\right)=\begin{cases}0 & \text{when } u<0,\ v<\infty,\\ 0 & \text{when } u\geq 0,\ v<0,\\ 1-\dfrac{p+q+r}{2(p+q+r)^2+1} & \text{when } u\geq 0,\ 0\leq v<p+q+r,\\ 1 & \text{when } u\geq 0,\ v\geq p+q+r\end{cases}$$

for $p,q,r\in\mathbb{N}$, we get the uncertainty distribution of uncertain variable (ϖ_{pqr}). In addition, the complex uncertainty distribution of uncertain variable ϖ is

$$\Phi(y)=\begin{cases}0 & \text{when } u<0,\ v<\infty,\\ 0 & \text{when } u\geq 0,\ v<0,\\ 1 & \text{when } u\geq 0,\ v\geq 0.\end{cases}$$

Thus, we obtain for $p=k^2$, $q=l^2$, $r=m^2$,

$$lE\left[\|\varpi_{pqr}(\gamma)-\varpi\left(\gamma\right)\|\right]=\frac{(p+q+r)^2}{2(p+q+r)^2+1}$$
$$\implies \quad \delta\left((p,q,r)\in\mathbb{N}^3 : E\left[\|\varpi_{pqr}(\gamma)-\varpi\left(\gamma\right)\|\right]\geq r+\sigma\right)=0$$

for $r \geq 0.5$. As a result, the sequence (ϖ_{pqr}) is roughly convergent in mean to ϖ for $r \geq 0.5$, but not roughly statistically convergent a.s. to ϖ.

Theorem 2.22. *Rough statistical convergence in measure is not synonymous with rough statistical convergence in mean.*

Example 2.23. Contemplate the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \dots\}$ with

$$\mathcal{M}(\Lambda) = \begin{cases} \sup\limits_{\gamma_{p+q+r} \in \Lambda} \dfrac{p+q+r}{2(p+q+r)+1} & \text{if } \sup\limits_{\gamma_{p+q+r} \in \Lambda} \dfrac{p+q+r}{2(p+q+r)+1} < 0.5, \\ 1 - \sup\limits_{\gamma_{p+q+r} \in \Lambda^c} \dfrac{p+q+r}{2(p+q+r)+1} & \text{if } \sup\limits_{\gamma_{p+q+r} \in \Lambda^c} \dfrac{p+q+r}{2\,(p+q+r)+1} < 0.5, \\ 0.5 & \text{otherwise,} \end{cases}$$

and the uncertain variable (ϖ_{pqr}) to be described by

$$\varpi_{pqr}(\gamma) = \begin{cases} (p+q+r) \cdot \mathrm{i} & \text{when } \gamma = \gamma_{p+q+r}, \\ 0 & \text{otherwise} \end{cases}$$

for $p, q, r \in \mathbb{N}$ and $\varpi \equiv 0$. The uncertainty distribution of an uncertain variable (ϖ_{pqr}) is thus obtained as

$$\Phi_{pqr}(y) = \Phi_{pqr}(u + \mathrm{i}v) = \begin{cases} 0 & \text{when } u < 0,\ v < \infty, \\ 0 & \text{when } u \geq 0,\ v < 0, \\ 1 - \dfrac{p+q+r}{2(p+q+r)+1} & \text{when } u \geq 0,\ 0 \leq v < p+q+r \\ 1 & \text{when } u \geq 0,\ v \geq p+q+r, \end{cases}$$

for $p, q, r \in \mathbb{N}$. In addition, the complex uncertainty distribution of uncertain variable ϖ is

$$\Phi(y) = \begin{cases} 0 & \text{when } u < 0,\ v < \infty, \\ 0 & \text{when } u \geq 0,\ v < 0, \\ 1 & \text{when } u \geq 0,\ v \geq 0. \end{cases}$$

Thus, we obtain

$$\delta\left((p,q,r) \in \mathbb{N}^3 : \mathcal{M}\left\{\gamma : \|\varpi_{pqr}(\gamma) - \varpi(\gamma)\| \geq \kappa\right\} \geq r + \sigma\right) = 0,$$

for $r \geq 0.5$, however,

$$\delta\left((p,q,r) \in \mathbb{N}^3 : E\left[\|\varpi_{pqr}(\gamma) - \varpi\,(\gamma)\| \geq \kappa\right] \geq r + \sigma\right) \neq 0.$$

Theorem 2.24. *The presence of rough statistical convergence in the distribution does not imply the presence of rough statistical convergence in the mean.*

Proof. It is quite simple to demonstrate from the preceding example, thus it has been removed. $\square$

3. Conclusion

The notion of rough statistical convergence of complex uncertain sequence was worked by [**10**]. The aim of this study is to extend this notion to the complex uncertain triple sequence. These findings integrate and generalize previous findings.

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