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TOPOLOGICAL AND GEOMETRIC APPROACH TO THE FIXED-POINT THEORY WITH LEAKLY RECTIFIED LINEAR UNIT APPLICATION

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Abstract. In this paper, we focus on the Banach contraction principle on S_b-metric spaces. We present an alternative proof to the Banach contraction principle on S_b-metric spaces. Also, we investigate some geometric properties of the fixed-point set of a given self-mapping modifying the Banach contractive condition with an illustrative example. Finally, we obtain an application to Leakly rectified linear unit activation functions.

1. Introduction and background

What is the notion of a fixed point in mathematics?

A fixed point u of a self-mapping $T\colon U \to U$ is an element such that

$$Tu = u,$$

that is, the self-mapping's domain is mapped to itself by T.

Also, the following method can be considered as a fixed point method:

Let $T\colon U \to U$ be a self-mapping and $S\colon U \to U$ a function defined as

$$Su = Tu - u$$

for all $u \in U$. If $z \in U$ is the solution of

$$Su = 0,$$

then z is a fixed point of T.

A notion of a fixed point can be discussed by the geometric approach. For this purpose, let u be a fixed point of T. Then the point (u, Tu) is on the line $v = u$. For example, let us define the self-mapping $T\colon \mathbb{R} \to \mathbb{R}$ as

$$Tu = u^3$$

for all $u \in \mathbb{R}$. Then we get the fixed point set of T such as

$$\mathrm{Fix}(T) = \{u \in \mathbb{R} : Tu = u\} = \{-1, 0, 1\},$$

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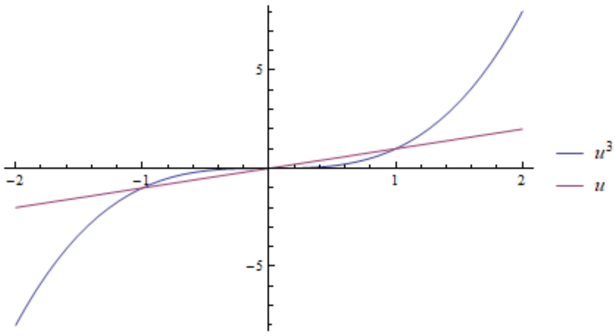
as seen in the following figure drawn by [**31**].



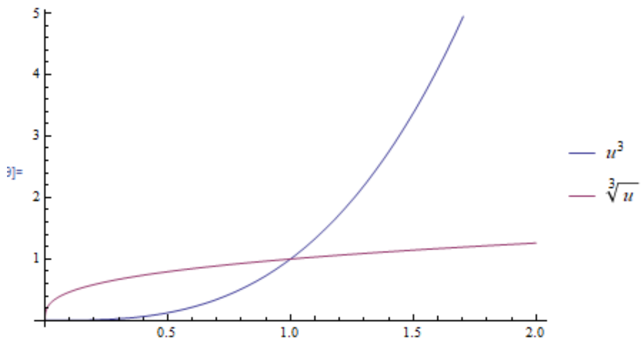
On the other hand, the fixed-point set of $T \colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ defined by $Tu = u^3$ can be found as

$$\mathrm{Fix}(T) = \left\{u \in \mathbb{R} : Tu = T^{-1}u\right\}.$$

Now, let us consider the above example. Then we get

$$\mathrm{Fix}(T) = \left\{u \in \mathbb{R} : u^3 = \sqrt[3]{u}\right\} = \{0, 1\},$$

as seen in the following figure drawn by [**31**].



The fixed-point theory was started with the Banach contraction principle [**4**]. This principle guarantees the existence and uniqueness of fixed points of a self-mapping. But there are some examples of a self-mapping which does not satisfy the condition of the Banach contractive condition or has more than one fixed points. In this case, this principle has been generalized with different methods. One of these methods is to generalize either the used contractive condition or the used metric space such as an S-metric, an S_b-metric, a G-metric, a G_b-metric etc (for example, see [**2, 7, 8, 9, 10, 12, 13**] and the references therein).

Let U be a nonempty set, $b \geq 1$ a given real number, and $\mathcal{M} \colon U \times U \times U \to [0, \infty)$ a function satisfying the following conditions for all $u, v, w, a \in U$:

(a)	$\mathcal{M}(u,v,w)=0$ if $u=v=w$,
(b)	$0<\mathcal{M}(u,u,v)$ with $u\neq v$,
(c)	$\mathcal{M}(u,u,v)\leq \mathcal{M}(u,v,w)$ with $v\neq w$,
(d)	$\mathcal{M}(u,v,w)=\mathcal{M}(u,w,v)=\mathcal{M}(v,w,u)=\ldots$,
(e)	$\mathcal{M}(u,v,w)\leq \mathcal{M}(u,a,a)+\mathcal{M}(a,v,w)$,
(f)	$\mathcal{M}(u,v,w)\leq b\,[\mathcal{M}(u,a,a)+\mathcal{M}(a,v,w)]$,
(g)	$\mathcal{M}(u,v,w)=0$ if and only if $u=v=w$,
(h)	$\mathcal{M}(u,v,w)\leq \mathcal{M}(u,u,a)+\mathcal{M}(v,v,a)+\mathcal{M}(w,w,a)$,
(i)	$\mathcal{M}(u,v,w)\leq b\,[\mathcal{M}(u,u,a)+\mathcal{M}(v,v,a)+\mathcal{M}(w,w,a)]$.

Using the properties (a)–(i), we get four known notions as generalizations of metric spaces. If the properties (a)–(e) hold true, then the pair $(U,\mathcal{M})$ is a G-metric space on U [**18**]. If the properties (a)–(d) and (f) hold true, then the pair $(U,\mathcal{M})$ is a G_b-metric space on U [**1**]. If the properties (g) and (h) hold true, then the pair $(U,\mathcal{M})$ is an S-metric space on U [**27**]. If the properties (g) and (i) hold true, then the pair $(U,\mathcal{M})$ is an S_b-metric space on U [**28**]. The notion of a G_b-metric is a generalization of both a metric and a G-metric. The notion of an S_b-metric is a generalization of both a metric and an S-metric. Also, the notions of an S-metric and a G-metric are distinct [**6**]. Thereby, the notions of an S_b-metric and a G_b-metric are distinct.

In this paper, since we study our results on S_b-metric spaces, we recall some basic notions related to S_b-metric spaces.

Definition 1.1 ([**28**])**.** Let U be a nonempty set and $b\geq 1$ a given real number. A function $S_b\colon U\times U\times U\to[0,\infty)$ is said to be S_b-metric if and only if for all $u,v,w,a\in X$, the following conditions are satisfied:

(S_b1) $S_b(u,v,w)=0$ if and only if $u=v=w$,

(S_b2) $S_b(u,v,w)\leq b[S_b(u,u,a)+S_b(v,v,a)+S_b(w,w,a)]$.

The pair (U,S_b) is called an S_b-metric space.

We note that S_b-metric spaces are the generalizations of S-metric spaces since every S-metric is an S_b-metric with $b=1$. But the converse statement is not always true (see [**28**] and [**30**] for more details).

Definition 1.2 ([**30**])**.** Let (U,S_b) be an S_b-metric space and $b>1$. An S_b-metric S_b is called symmetric if

$$S_b(u,u,v)=S_b(v,v,u)$$

for all $u,v\in U$.

Definition 1.3 ([**28**])**.** Let (U,S_b) be an S_b-metric space.

1. A sequence $\{u_n\}$ in U converges to u if and only if $S_b(u_n,u_n,u)\to 0$ as $n\to\infty$, that is, for each $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$, $S_b(u_n,u_n,u)<\varepsilon$. It is denoted by
$$\lim_{n\to\infty}u_n=u.$$

2. A sequence $\{u_n\}$ in U is called a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S_b(u_n, u_n, u_m) < \varepsilon$ for each $n, m \geq n_0$.
3. The S_b-metric space (U, S_b) is said to be complete if every Cauchy sequence is convergent.

Lemma 1.4 ([**28**]). *Let (U, S_b) be an S_b-metric space with $b \geq 1$, then we have*

$$S_b(u, u, v) \leq bS_b(v, v, u) \quad \text{and} \quad S_b(v, v, u) \leq bS_b(u, u, v).$$

Recently, the fixed-circle problem (see [**21**]) and the fixed-figure problem (see [**22**]) have been studied as a geometric approach to the fixed-point theory. When the number of fixed points is more than one, investigating some solutions to these problems is important. For this reason, in the literature, there exist some studies related to these recent problems (for example, see [**11, 14, 24, 25, 26**], and [**29**]).

In the light of that motivated by the above reasons, at first, we prove the Banach contraction principle using the Palais method (for more details, see [**17**] and [**23**]) on S_b-metric spaces. Since the notion of an S_b-metric is a generalization of a metric (resp. b-metric and S-metric), then studying on an S_b-metric space is gained importance. Also, we modify the Banach contractive condition and investigate some geometric properties of fixed-point set of a self-mapping T on S_b-metric spaces. Finally, we give an application to Leakly rectified linear unit (Leakly ReLU) activation functions.

2. A topological approach to the Banach contraction principle

In this section, we prove the "Banach Contraction Principle" using the Palais method on S_b-metric spaces.

Let (U, S_b) be an S_b-metric space with $b \geq 1$ and $T\colon U \to U$ self-mapping.

Assume that $\{u_n\}$ is a Picard sequence by $u_n = T^n u_0$, and T satisfies the Banach contractive condition on S_b-metric spaces, that is, there exists $h \in [0, 1)$ such that

$$S_b(Tu, Tu, Tv) \leq hS_b(u, u, v) \tag{1}$$

for all $u, v \in U$.

If (U, S_b) is a complete S_b-metric space, then T has a unique fixed point of T whenever $h \in [0, 1)$.

An alternative proof of the Banach contraction principle: Let $h \in (0, 1)$ and $\{u_n\}$ be a Picard sequence by $u_n = Tu_{n-1} = T^n u_0$.

If $u_m = u_{m-1}$ for some $m \in \mathbb{N}$, then we obtain that T has a unique fixed point u_{m-1}. Therefore, suppose that $u_n \neq u_{n-1}$ for all $n \in \mathbb{N}$.

At first, we show the existence of a fixed point of T. From the inequality (1), we get

$$S_b(u_n, u_n, u_{n+1}) \leq hS_b(u_{n-1}, u_{n-1}, u_n)$$

for all $n \in \mathbb{N}$, that is,

$$S_b(u_n, u_n, u_{n+1}) \to 0 \qquad \text{as } n \to \infty.$$

Now we prove the sequence $\{u_n\}$ is a Cauchy sequence. To do this, we use the known Palais method (for more details, see [**17**] and [**23**]). For $n < m$, we consider the following inequalities:

$$\begin{aligned}\frac{1}{b}S_b(u_n,u_n,u_m) &\leq 2S_b(u_n,u_n,u_{n+1}) + S_b(u_m,u_m,u_{n+1})\\ &\leq 2S_b(u_n,u_n,u_{n+1}) + bS_b(u_{n+1},u_{n+1},u_m)\\ &\leq 2S_b(u_n,u_n,u_{n+1}) + 2b^2S_b(u_{n+1},u_{n+1},u_{m+1})\\ &\quad + b^2S_b(u_m,u_m,u_{m+1}).\end{aligned}$$

Using the inequality (1), we get

$$\frac{1}{b}S_b(u_n,u_n,u_m) \leq 2S_b(u_n,u_n,u_{n+1}) + 2b^2hS_b(u_n,u_n,u_m) + b^2S_b(u_m,u_m,u_{m+1}),$$

and so,

$$\Big(\frac{1}{b} - 2b^2h\Big)S_b(u_n,u_n,u_m) \leq 2S_b(u_n,u_n,u_{n+1}) + b^2S_b(u_m,u_m,u_{m+1}).$$

From the above inequality, we have the following cases:

Case 1: $\frac{1}{b} - 2b^2h > 0$, that is, $0 \leq h < \frac{1}{2b^3}$,

Case 2: $\frac{1}{2b^3} \leq h < 1$.

Using the inequality (1) and the condition

$$S_b(u_n,u_n,u_{n+1}) \to 0 \text{ as } n \to \infty,$$

then under *Case 1*, we get

$$S_b(u_n,u_n,u_m) \leq \frac{b}{1-2b^3h}\left[2S_b(u_n,u_n,u_{n+1}) + b^2S_b(u_m,u_m,u_{m+1})\right] \to 0,$$

as $n \to \infty$.

So, for $0 \leq h < \frac{1}{2b^3}$, we obtain that $\{u_n\}$ is Cauchy. Since (U, S_b) is a complete S_b-metric space, using the similar techniques given in [**30**], we say that T has a unique fixed point z.

For *Case 2*, there exists $s \in \mathbb{N}$ such that $h^s < \frac{1}{2b^3}$ such that

$$S_b(T^su, T^su, T^sv) \leq h^sS_b(u,u,v).$$

Then we have $0 \leq h^s < \frac{1}{2b^3}$, and so from *Case 1*, there is a unique fixed point $z \in U$ such that

$$T^sz = z, \quad \text{ that is, } \quad T^s(Tz) = Tz.$$

Consequently, z is a unique fixed point of T, satisfying

$$\mathrm{Fix}(T) = \{z\}.$$ □

3. A geometric approach to the Banach contraction principle

In this section, we investigate some fixed-figure results modifying the Banach contractive condition on S_b-metric spaces. At first, we recall the following notions given in [**3**] and [**20**]:

Let (U, S_b) be an S_b-metric space with $b \geq 1$ and $u_0, u_1, u_2 \in U$, $\mu \in [0,\infty)$.

- The circle is defined by
$$C^{S_b}_{u_0,\mu} = \{u \in U : S_b(u,u,u_0) = \mu\}.$$
- The disc is defined by
$$D^{S_b}_{u_0,\mu} = \{u \in U : S_b(u,u,u_0) \leq \mu\}.$$
- The ellipse is defined by
$$E^{S_b}_{\mu}(u_1,u_2) = \{u \in U : S_b(u,u,u_1) + S_b(u,u,u_2) = \mu\}.$$
- The hyperbola is defined by
$$H^{S_b}_{\mu}(u_1,u_2) = \{u \in U : |S_b(u,u,u_1) - S_b(u,u,u_2)| = \mu\}.$$
- The Cassini curve is defined by
$$C^{S_b}_{\mu}(u_1,u_2) = \{u \in U : S_b(u,u,u_1)S_b(u,u,u_2) = \mu\}.$$
- The Apollonious circle is defined by
$$A^{S_b}_{\mu}(u_1,u_2) = \left\{u \in U - \{u_2\} : \frac{S_b(u,u,u_1)}{S_b(u,u,u_2)} = \mu\right\}.$$

Also, the notion of a fixed figure was given in [**3**] on S_b-metric spaces as follows:

Let (U,S_b) be an S_b-metric space with $b \geq 1$ and $T\colon U \to U$ a self-mapping. A geometric figure $\mathcal{F}$ contained in the fixed point set $Fix(T)$ is called a fixed figure of T.

Assume that

1. There exists $u_0 \in U$ such that
$$S_b(u,u,Tu) \leq hS_b(u,u,u_0) \tag{2}$$
for all $u \in U$, where $h \in [(0,1)$.
or

2. There exists $u_1, u_2 \in U$ such that
$$S_b(u,u,Tu) \leq h\left[S_b(u,u,u_1) + S_b(u,u,u_2)\right], \tag{3}$$
$$S_b(u,u,Tu) \leq h|S_b(u,u,u_1) - S_b(u,u,u_2)|, \tag{4}$$
$$S_b(u,u,Tu) \leq hS_b(u,u,u_1)S_b(u,u,u_2), \tag{5}$$
and
$$S_b(u,u,Tu) \leq h\frac{S_b(u,u,u_1)}{S_b(u,u,u_2)}, \tag{6}$$
or all $u \in U$, where $h \in [0,1)$.

Let us define the number μ as
$$\mu = \inf\{S_b(u,u,Tu) : u \notin \mathrm{Fix}(T)\}.$$

If T satisfies the inequality (2) (resp., the inequality (3), the inequality (4) with $\mu > 0$, the inequality (5), and the inequality (6), and $Tu_1 = u_1$, $Tu_2 = u_2$, then $D^{S_b}_{u_0,\mu} \subset \mathrm{Fix}(T)$ (resp., $E^{S_b}_{\mu}(u_1,u_2) \subset \mathrm{Fix}(T)$, $H^{S_b}_{\mu}(u_1,u_2) \subset \mathrm{Fix}(T)$,

$C_{\mu}^{S_b}(u_1, u_2) \subset \mathrm{Fix}(T)$, and $A_{\mu}^{S_b}(u_1, u_2) \subset \mathrm{Fix}(T)$). Especially, we have $C_{u_0,\mu}^{S_b} \subset \mathrm{Fix}(T)$.

Proof of the above claim. Let T satisfies the inequality (2) with $u_0 \in U$. At first, we show $Tu_0 = u_0$. To do this, we assume $Tu_0 \neq u_0$. Using the inequality (2), we have

$$S_b(u_0, u_0, Tu_0) \leq hS_b(u_0, u_0, u_0) = 0,$$

a contradiction. Hence it should be $Tu_0 = u_0$.

Let $\mu = 0$. Then we have $D_{u_0,\mu}^{S_b} = (u_0)$ and by the above equality, we get $D_{u_0,\mu}^{S_b} \subset \mathrm{Fix}(T)$. Let $\mu > 0$ and $u \in D_{u_0,\mu}^{S_b}$ such that $u \notin \mathrm{Fix}(T)$. Using the inequality (2), we get

$$S_b(u, u, Tu) \leq hS_b(u, u, u_0) \leq h\mu \leq hS_b(u, u, Tu),$$

a contradiction with $h \in [0, 1)$. Therefore, it should be $u \in \mathrm{Fix}(T)$ and so $D_{u_0,\mu}^{S_b} \subset \mathrm{Fix}(T)$.

Assume that T satisfies the inequality (3) with $u_1, u_2 \in U$. Let $\mu = 0$. Then we have $E_{\mu}^{S_b}(u_1, u_2) = \{u_1\} = \{u_2\}$ and so by the hypothesis, we get $E_{\mu}^{S_b}(u_1, u_2) \subset \mathrm{Fix}(T)$. Let $\mu > 0$, and $u \in E_{\mu}^{S_b}(u_1, u_2)$ such that $u \notin \mathrm{Fix}(T)$. Using the inequality (3), we obtain

$$S_b(u, u, Tu) \leq h\left[S_b(u, u, u_1) + S_b(u, u, u_2)\right] = h\mu \leq hS_b(u, u, Tu),$$

a contradiction with $h \in [0, 1)$. Thereby, it should be $u \in \mathrm{Fix}(T)$ and so $E_{\mu}^{S_b}(u_1, u_2) \subset \mathrm{Fix}(T)$.

If T satisfies the inequalities (4), (5), and (6) with $u_1, u_2 \in U$, then using the similar approaches, we see that $H_{\mu}^{S_b}(u_1, u_2) \subset \mathrm{Fix}(T)$, $C_{\mu}^{S_b}(u_1, u_2) \subset \mathrm{Fix}(T)$ and $A_{\mu}^{S_b}(u_1, u_2) \subset \mathrm{Fix}(T)$. Also, if T satisfies the inequality (2) with $u_0 \in U$, then we get $C_{u_0,\mu}^{S_b} \subset \mathrm{Fix}(T)$.

Consequently, we say T fixes the circle $C_{u_0,\mu}^{S_b}$, the disc $D_{u_0,\mu}^{S_b}$, the ellipse $E_{\mu}^{S_b}(u_1, u_2)$, the hyperbola $H_{\mu}^{S_b}(u_1, u_2)$, the Cassini curve $C_{\mu}^{S_b}(u_1, u_2)$, and the Apollonious circle $A_{\mu}^{S_b}(u_1, u_2)$. □

Now we give the following example:

Let $U = \{-3, -1, 0, 1, 3\}$ be the S_b-metric space with the S_b-metric defined as

$$S_b(u, v, w) = \frac{1}{16}\left(|u - v| + |v - w| + |u - w|\right)^2$$

for all $u, v, w \in U$ [**30**]. Let us define a self-mapping $T \colon U \to U$ as

$$Tu = \begin{cases} u + 1 & \text{if } u = 3, \\ u & \text{if } u \in U - \{3\} \end{cases}$$

for all $u \in U$. Then T satisfies the inequality (2) with $u_0 = 0$ and satisfies (3), (4), (5), (6) with $u_1 = -1$, $u_2 = 1$. Also, we get $\mu = \frac{1}{4}$ and $\mathrm{Fix}(T) = \{-3, -1, 0, 1\}$. Consequently, we have $C_{0,\frac{1}{4}}^{S_b} = \{-1, 1\} \subset \mathrm{Fix}(T)$, $D_{0,\frac{1}{4}}^{S_b} = \{-1, 0, 1\} \subset \mathrm{Fix}(T)$, $E_{\frac{1}{4}}^{S_b}(-1, 1) = \emptyset \subset \mathrm{Fix}(T)$, $H_{\frac{1}{4}}^{S_b}(-1, 1) = \emptyset \subset \mathrm{Fix}(T)$, $C_{\frac{1}{4}}^{S_b}(-1, 1) = \emptyset \subset \mathrm{Fix}(T)$ and $A_{\frac{1}{4}}^{S_b}(-1, 1) = \{-3\} \subset \mathrm{Fix}(T)$.

4. An application to Leakly ReLU activation functions

The notion of an activation function is very important in the artificial neural networks. Some examples of activation functions are Rectified linear unit (ReLU) [**19**], Exponential linear unit (ELU) [**5**], Scaled exponential linear unit (SELU) [**15**], and Leakly rectified linear unit (Leakly ReLU) [**16**].

Activation functions are important for the fixed-circle problem. The obtained fixed-circle results are applicable to various activation functions. Some authors gave some applications related to this problem (see, for example, [**24, 25, 26, 29**]).

In this section, we focus on Leakly rectified linear unit (Leakly ReLU) type activation functions defined as

$$LeaklyRELU(u) = Lu = \begin{cases} 0.01u & \text{if } u < 0, \\ u & \text{if } u \geq 0. \end{cases}$$

Let us consider the S_b-metric defined as

$$S_b(u, v, w) = b\left(|u - w| + |u + w - 2v|\right)$$

for all $u \in \mathbb{R}$, where $b \geq 1$ [**30**].

If attention, the function L does not satisfy the condition of the Banach contraction principle on S_b-metric spaces. Indeed, if we take $u = 1$, $v = 2$, then we have

$$S_b(u, u, v) = S_b(1, 1, 2) = 2b$$

and

$$S_b(Lu, Lu, Lv) = S_b(1, 1, 2) = 2b.$$

Then we get

$$S_b(Lu, Lu, Lv) = 2b \leq h2b = hS_b(u, u, v),$$

a contradiction with $h \in [0, 1)$.

Also, L has more than one fixed point and the fixed-point set of L is

$$\mathrm{Fix}(L) = [0, \infty).$$

On the other hand, this activation function L satisfies the inequality (2) with $u_0 = 0$ and $h = 0.99$. Indeed, we get

$$S_b(u, u, Lu) = 2b\,(0.99u)$$

and

$$S_b(u, u, u_0) = 2bu$$

for all $u \in (-\infty, 0)$.

We have

$$S_b(u, u, Lu) \leq hS_b(u, u, u_0)$$

and we find

$$\mu = \inf\left\{S_b(u, u, Lu) : u \in (-\infty, 0)\right\} = 0.$$

Therefore, the circle $C_{0,0}^{S_b} = \{0\}$ is a fixed circle (or fixed point) of L and L has at least one fixed point $u = 0$.

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