

SEQUENTIAL FRACTIONAL CALCULUS BETWEEN BANACH SPACES AND ALTERNATIVE OSTROWSKI AND GRÜSS TYPE INEQUALITIES

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ABSTRACT. In this article, we present a sequential fractional calculus of Caputo type for functions between Banach spaces. Left and right fractional derivatives are defined on a segment of a Banach space, then we develop our theory. We apply the above to new alternative Ostrowski and Grüss type inequalities at a very abstract level.

1. INTRODUCTION

The problem of estimating the difference of a value of a function and its average is a top one. The answer to it are the Ostrowski type inequalities. Ostrowski type inequalities are very useful among others in Numerical Analysis for approximating integrals. The problem of estimating the difference between the average of a product of functions and the product of their averages is also very important. The answer to it is given by the Grüss type inequalities. Grüss type inequalities are very useful among others in Probability for estimating expected values, etc. There exists a huge amount of literature on Ostrowski and Grüss type inequalities to all possible directions. Mathematical community is very interested in these inequalities due to their applications. So here we derive alternative, very general fractional Ostrowski and Grüss type inequalities at a very abstract level. Our functions are between Banach spaces. First we develop the related abstract sequential fractional calculus, which is based on a Banach space segment. Great sources to support our goal are the books [1, 2].

We are motivated by the following basic results:

Theorem 1 (1938, Ostrowski [4]). *Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) whose derivative $f': (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) ,*

Received October 4, 2022.

2020 *Mathematics Subject Classification.* Primary 26A33, 26D10, 26D15.

Key words and phrases. Abstract sequential fractional calculus; abstract sequential fractional derivative; generalized Ostrowski inequality; generalized Grüss inequality.

i.e., $\|f'\|_\infty^{\sup} := \sup_{t \in (a,b)} |f'(t)| < +\infty$. Then

$$(1) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty^{\sup}$$

for any $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

Theorem 2 (1935, Grüss [3]). Let f, g be integrable functions from $[a, b]$ into $\mathbb{R}$, that satisfy the conditions

$$m \leq f(x) \leq M, \quad n \leq g(x) \leq N, \quad x \in [a, b],$$

where $m, M, n, N \in \mathbb{R}$. Then

$$(2) \quad \left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right) \right| \leq \frac{1}{4} (M-m)(N-n).$$

2. SEQUENTIAL FRACTIONALITY BETWEEN A PAIR OF BANACH SPACES

We make the following remark.

Remark 3. Throughout this article, let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be Banach spaces. Here X^j denotes the j -fold product space $\underbrace{X \times X \times \cdots \times X}_j$ endowed with

the max-norm $\|x\|_{X^j} := \max_{1 \leq \lambda \leq j} \|x_\lambda\|_1$, where $x := (x_1, \dots, x_j) \in X^j$.

Suppose the space $L_j := L_j(X^j, Y)$ of all j -multilinear continuous maps $h: X^j \rightarrow Y$, $j = 1, \dots, m$, which is a Banach space with norm

$$(3) \quad \|h\| = \|h\|_{L_j} := \sup_{(\|x\|_{X^j}=1)} \|h(x)\|_2 = \sup \frac{\|h(x)\|_2}{\|x_1\|_1 \cdots \|x_j\|_1}.$$

Let M be a non-empty convex and compact set of X and $x_0 \in M$ be fixed.

Let O be an open subset of X : $M \subset O$.

Let $f: O \rightarrow Y$ be a continuous function whose Fréchet derivatives ([2, 87–127]) $f^{(j)}: O \rightarrow L_j = L_j(X^j, Y)$ exist and are continuous for $1 \leq j \leq m$, $m \in \mathbb{N}$.

Call $(x - x_0)^j := (x - x_0, \dots, x - x_0) \in X^j$, $x \in M$.

We will work with $f|_M$ and here we set $f^{(0)}(x_0)(x - x_0)^0 = f(x_0)$.

We obtain

$$(4) \quad \left\| \left(f^{(m)}(x_0 + u(x - x_0)) \right) (x - x_0)^m \right\|_2 \leq \left\| f^{(m)}(x_0 + u(x - x_0)) \right\| \|x - x_0\|_1^m.$$

Above $(f|_M)^{(j)}$, $j = 0, 1, \dots, m$, are norm bounded by continuity, and thus they are integrable.

Here

$$(5) \quad L(x_0, x_1) := \{\bar{x} | \bar{x} = \theta x_1 + (1 - \theta) x_0, \quad 0 \leq \theta \leq 1\}$$

is the line segment joining the points x_0 and x_1 (notice that the last $\bar{x} = x_0 + \theta(x_1 - x_0)$).

Denote also $L(x_0, x_1) = \overline{x_0 x_1}$.

Here $(\cdot - x_0)^j$ maps M into X^j and it is continuous, also $f^{(j)}(x_0)$ maps X^j into Y and it is continuous. Hence their composition $f^{(j)}(x_0)(\cdot - x_0)^j$ is continuous from M into Y .

Let us restrict f on the line segment $\overline{x_0 x_1}$. Then for

$$\bar{x}(u) = ux_1 + (1-u)x_0 = x_0 + u(x_1 - x_0), \quad 0 \leq u \leq 1,$$

the abstract function

$$f(u) = f(\bar{x}(u)) = f(x_0 + u(x_1 - x_0))$$

will map $[0, 1]$ into an abstract arc in Y , which starts at $y_0 = f(x_0)$ and ends at $y_1 = f(x_1)$.

By [2, p. 124], we have

$$(6) \quad f^{(k)}(u) = f^{(k)}(x_0 + u(x_1 - x_0))(x_1 - x_0)^k$$

for $k = 1, 2, \dots, m; u \in [0, 1]$.

We need the following definition.

Definition 4. Same as in Remark 3. We define the vector left Caputo-Fréchet fractional derivative of order $\alpha > 0$, $m = \lceil \alpha \rceil$ ($\lceil \cdot \rceil$ ceiling of the number), by

$$(7) \quad \begin{aligned} D_{*0}^\alpha(f(x_0 + u(x_1 - x_0))) &:= J_0^{m-\alpha} \left(\left(f^{(m)}(x_0 + u(x_1 - x_0)) \right) (x_1 - x_0)^m \right) \\ &:= \frac{1}{\Gamma(m-\alpha)} \int_0^u (u-t)^{m-\alpha-1} \left(f^{(m)}(x_0 + t(x_1 - x_0)) \right) (x_1 - x_0)^m dt, \end{aligned}$$

all $0 \leq u \leq 1$, defined via the vector left Riemann-Liouville fractional integral ([1, p. 2]).

In particular, when $0 < \alpha < 1$, we have

$$(8) \quad \begin{aligned} D_{*0}^\alpha(f(x_0 + u(x_1 - x_0))) \\ = \frac{1}{\Gamma(1-\alpha)} \int_0^u (u-t)^{-\alpha} (f'(x_0 + t(x_1 - x_0))) (x_1 - x_0) dt, \end{aligned}$$

all $0 \leq u \leq 1$.

If $\alpha \in \mathbb{N}$, we set $D_{*0}^\alpha f := f^{(\alpha)}$ the ordinary Y -valued derivative, and also set $D_{*0}^0 f := f$.

We make the following remark.

Remark 5. We are again working on the segment $\overline{x_0 x_1}$, where $x_0, x_1 \in M$; $0 \leq u \leq 1$, with

$$\bar{x} = \bar{x}(u) := x_0 + u(x_1 - x_0) \quad \text{and} \quad f(u) = f(\bar{x}(u)) = f(x_0 + u(x_1 - x_0)),$$

i.e., $f: [0, 1] \rightarrow Y$.

When $u = 0$, $f(0) = f(x_0)$, and when $u = 1$, $f(1) = f(x_1)$.

We have (by [1, pp. 121–122]) the abstract Riemann integral:

$$(9) \quad \int_{x_0}^{x_1} f(\bar{x}) d\bar{x} = \int_0^1 f(u) du.$$

We also have ([1, p. 122 and p. 124])

$$f'(u) = f'(x_0 + u(x_1 - x_0))(x_1 - x_0)$$

and

$$(10) \quad f^{(k)}(u) = f^{(k)}(x_0 + u(x_1 - x_0))(x_1 - x_0)^k, \quad k = 1, 2, \dots, m.$$

Let $\alpha > 0$ such that $[\alpha] = m$.

We consider the vector valued right hand side Riemann-Liouville fractional integral ([1, p. 34]),

$$(11) \quad J_{1-}^\alpha f(u) = \frac{1}{\Gamma(\alpha)} \int_u^1 (J - u)^{\alpha-1} f(J) dJ.$$

We define the vector right Caputo-Fréchet fractional derivative of order α , by

$$(12) \quad \begin{aligned} & D_{1-}^\alpha f(x_0 + u(x_1 - x_0)) \\ &= \frac{(-1)^m}{\Gamma(m - \alpha)} \int_u^1 (t - u)^{m-\alpha-1} f^{(m)}(x_0 + t(x_1 - x_0))(x_1 - x_0)^m dt, \end{aligned}$$

all $0 \leq u \leq 1$ (see also [1, p. 42–43]).

In particular, when $0 < \alpha < 1$, we get

$$(13) \quad \begin{aligned} & D_{1-}^\alpha (f(x_0 + u(x_1 - x_0))) \\ &= \frac{-1}{\Gamma(1 - \alpha)} \int_u^1 (t - u)^{-\alpha} f'(x_0 + t(x_1 - x_0))(x_1 - x_0) dt, \end{aligned}$$

all $0 \leq u \leq 1$.

We set $D_{1-}^m f = (-1)^m f^{(m)}$ for $m \in \mathbb{N}$, and $D_{1-}^0 f = f$.

Denote by

$$(14) \quad \int_{\langle x_0 x^* x_1 \rangle} f(x) dx := \int_{x_0}^{x^*} f(x) dx + \int_{x^*}^{x_1} f(x) dx,$$

where $x_0, x^*, x_1 \in M$ and x^* is not necessarily on the line segment $\overline{x_0 x_1}$.

Denote the sequential Caputo-Bochner left and right fractional derivatives ($\alpha > 0$)

$$(15) \quad D_{*a}^{n\alpha} := D_{*a}^\alpha D_{*a}^\alpha \dots D_{*a}^\alpha \quad (n\text{-times}), \quad n \in \mathbb{N}$$

and

$$(16) \quad D_{b-}^{n\alpha} := D_{b-}^\alpha D_{b-}^\alpha \dots D_{b-}^\alpha \quad (n\text{-times}).$$

For the detailed definitions of D_{*a}^α , D_{b-}^α , see [1, pp. 128–129]. In this article, we consider $[a, b] = [0, 1]$.

We mention the following left alternative fractional Taylor's formula.

Theorem 6 ([1, p. 129, Theorem 4.43]). *Let Y a Banach space and $0 < \alpha \leq 1$, $n \in \mathbb{N}$, $f \in C^1([a, b], Y)$.*

*For $k = 1, \dots, n$, we assume that $D_{*a}^{k\alpha} f \in C^1([a, b], Y)$ and $D_{*a}^{(n+1)\alpha} f \in C([a, b], Y)$. Then*

$$(17) \quad \begin{aligned} f(x) &= \sum_{i=0}^n \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} (D_{*a}^{i\alpha} f)(a) \\ &+ \frac{1}{\Gamma((n+1)\alpha)} \int_a^x (x-t)^{(n+1)\alpha-1} \left(D_{*a}^{(n+1)\alpha} f \right)(t) dt \end{aligned}$$

for all $x \in [a, b]$.

We also mention the following right alternative fractional Taylor's formula.

Theorem 7 ([1, p. 129, Theorem 4.44]). *Let Y a Banach space and $0 < \alpha \leq 1$, $n \in \mathbb{N}$, $f \in C^1([a, b], Y)$. For $k = 1, \dots, n$, we assume that $D_{b-}^{k\alpha} f \in C^1([a, b], Y)$ and $D_{b-}^{(n+1)\alpha} f \in C([a, b], Y)$. Then*

$$(18) \quad \begin{aligned} f(x) &= \sum_{i=0}^n \frac{(b-x)^{i\alpha}}{\Gamma(i\alpha+1)} (D_{b-}^{i\alpha} f)(b) \\ &+ \frac{1}{\Gamma((n+1)\alpha)} \int_x^b (t-x)^{(n+1)\alpha-1} \left(D_{b-}^{(n+1)\alpha} f \right)(t) dt \end{aligned}$$

for all $x \in [a, b]$.

We give the corresponding left and right fractional alternative Taylor's formulae on the segment $\overline{x_0 x_1}$.

Theorem 8. *Let Y be a Banach space and $0 < \alpha \leq 1$, $n \in \mathbb{N}$, $f(x_0 + u(x_1 - x_0)) \in C^1([0, 1], Y)$.*

*For $k = 1, \dots, n$, we assume that $D_{*0}^{k\alpha} f(x_0 + u(x_1 - x_0)) \in C^1([0, 1], Y)$ and $D_{*0}^{(n+1)\alpha} f(x_0 + u(x_1 - x_0)) \in C([0, 1], Y)$. Then*

$$(19) \quad \begin{aligned} f(x_0 + u(x_1 - x_0)) &= \sum_{i=0}^n \frac{u^{i\alpha}}{\Gamma(i\alpha+1)} (D_{*0}^{i\alpha} (f(x_0 + u(x_1 - x_0))))(0) \\ &+ \frac{1}{\Gamma((n+1)\alpha)} \int_0^u (u-t)^{(n+1)\alpha-1} \left(D_{*0}^{(n+1)\alpha} (f(x_0 + u(x_1 - x_0))) \right)(t) dt \end{aligned}$$

for all $u \in [0, 1]$.

Proof. By Theorem 6. □

Theorem 9. Let Y be a Banach space and $0 < \alpha \leq 1$, $n \in \mathbb{N}$,
 $f(x_0 + u(x_1 - x_0)) \in C^1([0, 1], Y)$.

For $k = 1, \dots, n$, we assume that $D_{1-}^{k\alpha} f(x_0 + u(x_1 - x_0)) \in C^1([0, 1], Y)$ and
 $D_{1-}^{(n+1)\alpha} f(x_0 + u(x_1 - x_0)) \in C([0, 1], Y)$. Then

$$(20) \quad \begin{aligned} f(x_0 + u(x_1 - x_0)) &= \sum_{i=0}^n \frac{(1-u)^{i\alpha}}{\Gamma(i\alpha+1)} (D_{1-}^{i\alpha} f(x_0 + u(x_1 - x_0))) (1) \\ &+ \frac{1}{\Gamma((n+1)\alpha)} \int_u^1 (t-u)^{(n+1)\alpha-1} \left(D_{1-}^{(n+1)\alpha} f(x_0 + u(x_1 - x_0)) \right) (t) dt \end{aligned}$$

for all $u \in [0, 1]$.

Proof. By Theorem 7. □

3. MAIN RESULTS

3.1. About general alternative mixed fractional Ostrowski type inequalities

We present the following theorem.

Theorem 10. Let X, Y be Banach spaces, $0 < \alpha \leq 1$, $n \in \mathbb{N}$. Here $x_0, x^*, x_1 \in X$ and are not necessarily colinear; $0 \leq u \leq 1$.

Let $f: X \rightarrow Y$ and $g: X \rightarrow \mathbb{R}$ such that $f(x^* + u(x_1 - x^*)) \in C^1([0, 1], Y)$,
 $g(x^* + u(x_1 - x^*)) \in C^1([0, 1], \mathbb{R})$.

For $k = 1, \dots, n$, we assume that $D_{*0}^{k\alpha} f(x^* + u(x_1 - x^*)) \in C^1([0, 1], Y)$,
 $D_{*0}^{(n+1)\alpha} f(x^* + u(x_1 - x^*)) \in C([0, 1], Y)$, and

$D_{*0}^{k\alpha} g(x^* + u(x_1 - x^*)) \in C^1([0, 1], \mathbb{R})$, $D_{*0}^{(n+1)\alpha} g(x^* + u(x_1 - x^*)) \in C([0, 1], \mathbb{R})$.

Furthermore, we suppose that $f(x_0 + u(x^* - x_0)) \in C^1([0, 1], Y)$,
 $g(x_0 + u(x^* - x_0)) \in C^1([0, 1], \mathbb{R})$;

and for $k = 1, \dots, n$, we assume that $D_{1-}^{k\alpha} f(x_0 + u(x^* - x_0)) \in C^1([0, 1], Y)$,
 $D_{1-}^{(n+1)\alpha} f(x_0 + u(x^* - x_0)) \in C([0, 1], Y)$, and

$D_{1-}^{k\alpha} g(x_0 + u(x^* - x_0)) \in C^1([0, 1], \mathbb{R})$, $D_{1-}^{(n+1)\alpha} g(x_0 + u(x^* - x_0)) \in C([0, 1], \mathbb{R})$.

Finally, we assume that $D_{*0}^{i\alpha} (f(x^* + u(x_1 - x^*))) (0) = 0$,

$D_{*0}^{i\alpha} (g(x^* + u(x_1 - x^*))) (0) = 0$ for all $i = 1, \dots, n$, along with

$D_{1-}^{i\alpha} (f(x_0 + u(x^* - x_0))) (1) = 0$, $D_{1-}^{i\alpha} (g(x_0 + u(x^* - x_0))) (1) = 0$

for all $i = 1, \dots, n$.

Then

$$K(f, g)(x^*)$$

$$\begin{aligned} &:= 2 \int_{\langle x_0 x^* x_1 \rangle} f(x) g(x) dx - f(x^*) \int_{\langle x_0 x^* x_1 \rangle} g(x) dx - g(x^*) \int_{\langle x_0 x^* x_1 \rangle} f(x) dx \\ &= \frac{1}{\Gamma((n+1)\alpha)} \\ &\times \left\{ \int_0^1 \left[g(x^* + u(x_1 - x^*)) \int_0^u (u-t)^{(n+1)\alpha-1} \left(D_{*0}^{(n+1)\alpha} (f(x^* + t(x_1 - x^*))) \right) (t) dt \right. \right. \end{aligned}$$

$$\begin{aligned}
 & + f(x^* + u(x_1 - x^*)) \int_0^u (u-t)^{(n+1)\alpha-1} \left(D_{*0}^{(n+1)\alpha} (g(x^* + t(x_1 - x^*))) \right) (t) dt \Big] du \Big] \\
 (21) \quad & + \left[\int_0^1 \left[g(x_0 + u(x^* - x_0)) \int_u^1 (t-u)^{(n+1)\alpha-1} \left(D_{1-}^{(n+1)\alpha} f(x_0 + t(x^* - x_0)) \right) (t) dt \right. \right. \\
 & \left. \left. + f(x_0 + u(x^* - x_0)) \int_u^1 (t-u)^{(n+1)\alpha-1} \left(D_{1-}^{(n+1)\alpha} g(x_0 + t(x^* - x_0)) \right) (t) dt \right] du \right] \Big\}.
 \end{aligned}$$

Proof. Along with $f: X \rightarrow Y$, we consider $g: X \rightarrow \mathbb{R}$ with the same properties as f . We also have the line segments $\overline{x_0 x^*}$ and $\overline{x^* x_1}$ with not necessarily colinear $x_0, x^*, x_1 \in M$. Assume that $D_{*0}^{i\alpha} (f(x^* + u(x_1 - x^*))) (0) = 0$, and $D_{*0}^{i\alpha} (g(x^* + u(x_1 - x^*))) (0) = 0$, $i = 1, \dots, n$.

We apply Theorem 8.

Then (on $\overline{x^* x_1}$)

$$\begin{aligned}
 & f(x^* + u(x_1 - x^*)) - f(x^*) \\
 (22) \quad & = \frac{1}{\Gamma((n+1)\alpha)} \int_0^u (u-t)^{(n+1)\alpha-1} \left(D_{*0}^{(n+1)\alpha} (f(x^* + t(x_1 - x^*))) \right) (t) dt
 \end{aligned}$$

for all $u \in [0, 1]$.

Similarly, it holds

$$\begin{aligned}
 & g(x^* + u(x_1 - x^*)) - g(x^*) \\
 (23) \quad & = \frac{1}{\Gamma((n+1)\alpha)} \int_0^u (u-t)^{(n+1)\alpha-1} \left(D_{*0}^{(n+1)\alpha} (g(x^* + t(x_1 - x^*))) \right) (t) dt
 \end{aligned}$$

for all $u \in [0, 1]$.

Assume also (on $\overline{x_0 x^*}$) $(D_{1-}^{i\alpha} (f(x_0 + u(x^* - x_0)))) (1) = 0$,

$D_{1-}^{i\alpha} (g(x_0 + u(x^* - x_0))) (1) = 0$, $i = 1, \dots, n$.

We apply Theorem 9.

Then

$$\begin{aligned}
 & f(x_0 + u(x^* - x_0)) - f(x^*) \\
 (24) \quad & = \frac{1}{\Gamma((n+1)\alpha)} \int_u^1 (t-u)^{(n+1)\alpha-1} \left(D_{1-}^{(n+1)\alpha} (f(x_0 + u(x^* - x_0))) \right) (t) dt
 \end{aligned}$$

for all $u \in [0, 1]$.

Similarly, it holds

$$\begin{aligned}
 & g(x_0 + u(x^* - x_0)) - g(x^*) \\
 (25) \quad & = \frac{1}{\Gamma((n+1)\alpha)} \int_u^1 (t-u)^{(n+1)\alpha-1} \left(D_{1-}^{(n+1)\alpha} g(x_0 + u(x^* - x_0)) \right) (t) dt
 \end{aligned}$$

for all $u \in [0, 1]$.

We have

$$\begin{aligned}
 & g(x^* + u(x_1 - x^*)) f(x^* + u(x_1 - x^*)) - g(x^* + u(x_1 - x^*)) f(x^*) \\
 (26) \quad & = \frac{g(x^* + u(x_1 - x^*))}{\Gamma((n+1)\alpha)} \int_0^u (u-t)^{(n+1)\alpha-1} \left(D_{*0}^{(n+1)\alpha} (f(x^* + t(x_1 - x^*))) \right) (t) dt
 \end{aligned}$$

for all $u \in [0, 1]$.

Similarly, we get

$$(27) \quad \begin{aligned} & f(x^* + u(x_1 - x^*))g(x^* + u(x_1 - x^*)) - f(x^* + u(x_1 - x^*))g(x^*) \\ &= \frac{f(x^* + u(x_1 - x^*))}{\Gamma((n+1)\alpha)} \int_0^u (u-t)^{(n+1)\alpha-1} \left(D_{*0}^{(n+1)\alpha} (g(x^* + t(x_1 - x^*))) \right) (t) dt \end{aligned}$$

for all $u \in [0, 1]$.

Furthermore, we have

$$(28) \quad \begin{aligned} & g(x_0 + u(x^* - x_0))f(x_0 + u(x^* - x_0)) - g(x_0 + u(x^* - x_0))f(x^*) \\ &= \frac{g(x_0 + u(x^* - x_0))}{\Gamma((n+1)\alpha)} \int_u^1 (t-u)^{(n+1)\alpha-1} \left(D_{1-}^{(n+1)\alpha} f(x_0 + t(x^* - x_0)) \right) (t) dt \end{aligned}$$

for all $u \in [0, 1]$.

Next, we have

$$(29) \quad \begin{aligned} & f(x_0 + u(x^* - x_0))g(x_0 + u(x^* - x_0)) - f(x_0 + u(x^* - x_0))g(x^*) \\ &= \frac{f(x_0 + u(x^* - x_0))}{\Gamma((n+1)\alpha)} \int_u^1 (t-u)^{(n+1)\alpha-1} \left(D_{1-}^{(n+1)\alpha} g(x_0 + t(x^* - x_0)) \right) (t) dt \end{aligned}$$

for all $u \in [0, 1]$.

Adding (26) and (27), we get

$$(30) \quad \begin{aligned} & 2f(x^* + u(x_1 - x^*))g(x^* + u(x_1 - x^*)) \\ & - g(x^* + u(x_1 - x^*))f(x^*) - f(x^* + u(x_1 - x^*))g(x^*) \\ &= \frac{1}{\Gamma((n+1)\alpha)} \\ & \times \left[g(x^* + u(x_1 - x^*)) \int_0^u (u-t)^{(n+1)\alpha-1} \left(D_{*0}^{(n+1)\alpha} (f(x^* + t(x_1 - x^*))) \right) (t) dt \right. \\ & \quad \left. + f(x^* + u(x_1 - x^*)) \int_0^u (u-t)^{(n+1)\alpha-1} \left(D_{*0}^{(n+1)\alpha} (g(x^* + t(x_1 - x^*))) \right) (t) dt \right] \end{aligned}$$

for all $u \in [0, 1]$.

Adding (28) and (29), we obtain

$$(31) \quad \begin{aligned} & 2g(x_0 + u(x^* - x_0))f(x_0 + u(x^* - x_0)) \\ & - g(x_0 + u(x^* - x_0))f(x^*) - f(x_0 + u(x^* - x_0))g(x^*) \\ &= \frac{1}{\Gamma((n+1)\alpha)} \\ & \times \left[g(x_0 + u(x^* - x_0)) \int_u^1 (t-u)^{(n+1)\alpha-1} \left(D_{1-}^{(n+1)\alpha} f(x_0 + t(x^* - x_0)) \right) (t) dt \right. \\ & \quad \left. + f(x_0 + u(x^* - x_0)) \int_u^1 (t-u)^{(n+1)\alpha-1} \left(D_{1-}^{(n+1)\alpha} g(x_0 + t(x^* - x_0)) \right) (t) dt \right] \end{aligned}$$

for all $u \in [0, 1]$.

Next, we integrate (30) and (31) to obtain

$$\begin{aligned}
 (32) \quad & 2 \int_{x^*}^{x_1} f(x) g(x) dx - f(x^*) \int_{x^*}^{x_1} g(x) dx - g(x^*) \int_{x^*}^{x_1} f(x) dx \\
 &= \frac{1}{\Gamma((n+1)\alpha)} \\
 &\times \left[\int_0^1 \left[g(x^* + u(x_1 - x^*)) \int_0^u (u-t)^{(n+1)\alpha-1} \left(D_{*0}^{(n+1)\alpha} (f(x^* + t(x_1 - x^*))) \right) (t) dt \right. \right. \\
 &\quad \left. \left. + f(x^* + u(x_1 - x^*)) \int_0^u (u-t)^{(n+1)\alpha-1} \left(D_{*0}^{(n+1)\alpha} (g(x^* + t(x_1 - x^*))) \right) (t) dt \right] du \right]
 \end{aligned}$$

and

$$\begin{aligned}
 (33) \quad & 2 \int_{x_0}^{x^*} f(x) g(x) dx - f(x^*) \int_{x_0}^{x^*} g(x) dx - g(x^*) \int_{x_0}^{x^*} f(x) dx \\
 &= \frac{1}{\Gamma((n+1)\alpha)} \\
 &\times \left[\int_0^1 \left[g(x_0 + u(x^* - x_0)) \int_u^1 (t-u)^{(n+1)\alpha-1} \left(D_{1-}^{(n+1)\alpha} f(x_0 + t(x^* - x_0)) \right) (t) dt \right. \right. \\
 &\quad \left. \left. + f(x_0 + u(x^* - x_0)) \int_u^1 (t-u)^{(n+1)\alpha-1} \left(D_{1-}^{(n+1)\alpha} g(x_0 + t(x^* - x_0)) \right) (t) dt \right] du \right].
 \end{aligned}$$

Adding (32) and (33), we derive

$$\begin{aligned}
 (34) \quad & K(f, g)(x^*) \\
 &:= 2 \int_{\langle x_0 x^* x_1 \rangle} f(x) g(x) dx - f(x^*) \int_{\langle x_0 x^* x_1 \rangle} g(x) dx - g(x^*) \int_{\langle x_0 x^* x_1 \rangle} f(x) dx \\
 &= \frac{1}{\Gamma((n+1)\alpha)} \\
 &\times \left\{ \left[\int_0^1 \left[g(x^* + u(x_1 - x^*)) \int_0^u (u-t)^{(n+1)\alpha-1} \left(D_{*0}^{(n+1)\alpha} (f(x^* + t(x_1 - x^*))) \right) (t) dt \right. \right. \right. \\
 &\quad \left. \left. + f(x^* + u(x_1 - x^*)) \int_0^u (u-t)^{(n+1)\alpha-1} \left(D_{*0}^{(n+1)\alpha} (g(x^* + t(x_1 - x^*))) \right) (t) dt \right] du \right. \\
 &\quad \left. + \left[\int_0^1 \left[g(x_0 + u(x^* - x_0)) \int_u^1 (t-u)^{(n+1)\alpha-1} \left(D_{1-}^{(n+1)\alpha} f(x_0 + t(x^* - x_0)) \right) (t) dt \right. \right. \right. \\
 &\quad \left. \left. + f(x_0 + u(x^* - x_0)) \int_u^1 (t-u)^{(n+1)\alpha-1} \left(D_{1-}^{(n+1)\alpha} g(x_0 + t(x^* - x_0)) \right) (t) dt \right] du \right] \right\},
 \end{aligned}$$

proving the claim. $\square$

We present the following alternative general fractional Ostrowski type inequalities:

Theorem 11. *Same as in Theorem 10. Then:*

i)

$$\begin{aligned}
& \|K(f, g)(x^*)\|_2 \leq \frac{1}{\Gamma((n+1)\alpha + 2)} \\
& \times \left\{ \|g(x^* + u(x_1 - x^*))\|_{u, \infty, [0, 1]} \left\| \left\| D_{*0}^{(n+1)\alpha} (f(x^* + t(x_1 - x^*))) (t) \right\|_2 \right\|_{t, \infty, [0, 1]} \right. \\
(35) \quad & + \left\| \|f(x^* + u(x_1 - x^*))\|_2 \right\|_{u, \infty, [0, 1]} \left\| \left\| D_{*0}^{(n+1)\alpha} (g(x^* + t(x_1 - x^*))) (t) \right\|_2 \right\|_{t, \infty, [0, 1]} \\
& + \|g(x_0 + u(x^* - x_0))\|_{u, \infty, [0, 1]} \left\| \left\| D_{1-}^{(n+1)\alpha} f(x_0 + t(x^* - x_0)) (t) \right\|_2 \right\|_{t, \infty, [0, 1]} \\
& + \left\| \|f(x_0 + u(x^* - x_0))\|_2 \right\|_{u, \infty, [0, 1]} \left\| \left\| D_{1-}^{(n+1)\alpha} g(x_0 + t(x^* - x_0)) (t) \right\|_2 \right\|_{t, \infty, [0, 1]} \Big\},
\end{aligned}$$

ii) if $\alpha \geq \frac{1}{n+1}$, we obtain

$$\begin{aligned}
& \|K(f, g)(x^*)\|_2 \leq \frac{1}{\Gamma((n+1)\alpha + 1)} \\
& \times \left\{ \|g(x^* + u(x_1 - x^*))\|_{u, \infty, [0, 1]} \left\| \left\| D_{*0}^{(n+1)\alpha} (f(x^* + t(x_1 - x^*))) (t) \right\|_2 \right\|_{t, L_1([0, 1])} \right. \\
(36) \quad & + \left\| \|f(x^* + u(x_1 - x^*))\|_2 \right\|_{u, \infty, [0, 1]} \left\| \left\| D_{*0}^{(n+1)\alpha} (g(x^* + t(x_1 - x^*))) (t) \right\|_2 \right\|_{t, L_1([0, 1])} \\
& + \|g(x_0 + u(x^* - x_0))\|_{u, \infty, [0, 1]} \left\| \left\| D_{1-}^{(n+1)\alpha} (f(x_0 + t(x^* - x_0))) (t) \right\|_2 \right\|_{t, L_1([0, 1])} \\
& + \left\| \|f(x_0 + u(x^* - x_0))\|_2 \right\|_{u, \infty, [0, 1]} \left\| \left\| D_{1-}^{(n+1)\alpha} (g(x_0 + t(x^* - x_0))) (t) \right\|_2 \right\|_{t, L_1([0, 1])} \Big\},
\end{aligned}$$

iii) if $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1; \alpha > \frac{1}{(n+1)q}$, we derive

$$\begin{aligned}
& \|K(f, g)(x^*)\|_2 \leq \frac{1}{(p((n+1)\alpha - 1) + 1)^{\frac{1}{p}} \Gamma((n+1)\alpha) \left((n+1)\alpha + \frac{1}{p}\right)} \\
& \times \left\{ \|g(x^* + u(x_1 - x^*))\|_{u, \infty, [0, 1]} \left\| \left\| D_{*0}^{(n+1)\alpha} f(x^* + t(x_1 - x^*)) (t) \right\|_2 \right\|_{t, L_q([0, 1])} \right. \\
(37) \quad & + \left\| \|f(x^* + u(x_1 - x^*))\|_2 \right\|_{u, \infty, [0, 1]} \left\| \left\| D_{*0}^{(n+1)\alpha} g(x^* + t(x_1 - x^*)) (t) \right\|_2 \right\|_{t, L_q([0, 1])} \\
& + \|g(x_0 + u(x^* - x_0))\|_{u, \infty, [0, 1]} \left\| \left\| D_{1-}^{(n+1)\alpha} f(x_0 + t(x^* - x_0)) (t) \right\|_2 \right\|_{t, L_q([0, 1])} \\
& + \left\| \|f(x_0 + u(x^* - x_0))\|_2 \right\|_{u, \infty, [0, 1]} \left\| \left\| D_{1-}^{(n+1)\alpha} g(x_0 + t(x^* - x_0)) (t) \right\|_2 \right\|_{t, L_q([0, 1])} \Big\}.
\end{aligned}$$

Proof. We have that

$$\begin{aligned}
& \|K(f, g)(x^*)\|_2 \\
& = \left\| 2 \int_{\langle x_0 x^* x_1 \rangle} f(x) g(x) dx - f(x^*) \int_{\langle x_0 x^* x_1 \rangle} g(x) dx - g(x^*) \int_{\langle x_0 x^* x_1 \rangle} f(x) dx \right\|_2 \\
& \stackrel{(21)}{\leq} \frac{1}{\Gamma((n+1)\alpha)} \\
(38) \quad & \times \left\{ \left\| \int_0^1 \left[g(x^* + u(x_1 - x^*)) \int_0^u (u-t)^{(n+1)\alpha-1} \left(D_{*0}^{(n+1)\alpha} (f(x^* + t(x_1 - x^*))) (t) \right) dt \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
 & + f(x^* + u(x_1 - x^*)) \int_0^u (u-t)^{(n+1)\alpha-1} \left(D_{*0}^{(n+1)\alpha} (g(x^* + t(x_1 - x^*))) \right) (t) dt \Big] du \Big\|_2 \\
 & + \left\| \int_0^1 \left[g(x_0 + u(x^* - x_0)) \int_u^1 (t-u)^{(n+1)\alpha-1} \left(D_{1-}^{(n+1)\alpha} f(x_0 + t(x^* - x_0)) \right) (t) dt \right. \right. \\
 & \quad \left. \left. + f(x_0 + u(x^* - x_0)) \int_u^1 (t-u)^{(n+1)\alpha-1} \left(D_{1-}^{(n+1)\alpha} g(x_0 + t(x^* - x_0)) \right) (t) dt \right] du \right\|_2 \Big\} \\
 & \leq \frac{1}{\Gamma((n+1)\alpha)} \\
 & \times \left\{ \left\| \int_0^1 \left[f(x^* + u(x_1 - x^*)) \int_0^u (u-t)^{(n+1)\alpha-1} \left(D_{*0}^{(n+1)\alpha} (f(x^* + t(x_1 - x^*))) \right) (t) dt \right] du \right\|_2 \right. \\
 & \quad (39) \\
 & \quad \left. + \left\| \int_0^1 \left[f(x^* + u(x_1 - x^*)) \int_0^u (u-t)^{(n+1)\alpha-1} \left(D_{*0}^{(n+1)\alpha} (g(x^* + t(x_1 - x^*))) \right) (t) dt \right] du \right\|_2 \right. \\
 & \quad + \left\| \int_0^1 \left[g(x_0 + u(x^* - x_0)) \int_u^1 (t-u)^{(n+1)\alpha-1} \left(D_{1-}^{(n+1)\alpha} f(x_0 + t(x^* - x_0)) \right) (t) dt \right] du \right\|_2 \\
 & \quad \left. + \left\| \int_0^1 \left[f(x_0 + u(x^* - x_0)) \int_u^1 (t-u)^{(n+1)\alpha-1} \left(D_{1-}^{(n+1)\alpha} (g(x_0 + t(x^* - x_0))) \right) (t) dt \right] du \right\|_2 \right\} \\
 & \leq \frac{1}{\Gamma((n+1)\alpha)} \\
 & \times \left\{ \int_0^1 \left[|g(x^* + u(x_1 - x^*))| \int_0^u (u-t)^{(n+1)\alpha-1} \left\| D_{*0}^{(n+1)\alpha} (f(x^* + t(x_1 - x^*))) (t) \right\|_2 dt \right] du \right. \\
 & \quad + \int_0^1 \left[\|f(x^* + u(x_1 - x^*))\|_2 \int_0^u (u-t)^{(n+1)\alpha-1} \left\| D_{*0}^{(n+1)\alpha} (g(x^* + t(x_1 - x^*))) (t) \right\|_2 dt \right] du \\
 & \quad + \int_0^1 \left[|g(x_0 + u(x^* - x_0))| \int_u^1 (t-u)^{(n+1)\alpha-1} \left\| D_{1-}^{(n+1)\alpha} f(x_0 + t(x^* - x_0)) (t) \right\|_2 dt \right] du \\
 & \quad (40) \\
 & \quad \left. + \int_0^1 \left[\|f(x_0 + u(x^* - x_0))\|_2 \int_u^1 (t-u)^{(n+1)\alpha-1} \left\| D_{1-}^{(n+1)\alpha} g(x_0 + t(x^* - x_0)) (t) \right\|_2 dt \right] du \right\} \\
 & =: (\psi).
 \end{aligned}$$

i) We observe that

$$\begin{aligned}
 & (\psi) \leq \frac{1}{\Gamma((n+1)\alpha)} \\
 & \times \left\{ \int_0^1 |g(x^* + u(x_1 - x^*))| \frac{u^{(n+1)\alpha}}{(n+1)\alpha} \left\| \left\| D_{*0}^{(n+1)\alpha} (f(x^* + t(x_1 - x^*))) (t) \right\|_2 \right\|_{t, \infty, [0,1]} du \right. \\
 & \quad (41) \\
 & \quad + \int_0^1 \|f(x^* + u(x_1 - x^*))\|_2 \frac{u^{(n+1)\alpha}}{(n+1)\alpha} \left\| \left\| D_{*0}^{(n+1)\alpha} (g(x^* + t(x_1 - x^*))) (t) \right\|_2 \right\|_{t, \infty, [0,1]} du \\
 & \quad + \int_0^1 |g(x_0 + u(x^* - x_0))| \frac{(1-u)^{(n+1)\alpha}}{(n+1)\alpha} \left\| \left\| D_{1-}^{(n+1)\alpha} f(x_0 + t(x^* - x_0)) (t) \right\|_2 \right\|_{t, \infty, [0,1]} du \\
 & \quad \left. + \int_0^1 \|f(x_0 + u(x^* - x_0))\|_2 \frac{(1-u)^{(n+1)\alpha}}{(n+1)\alpha} \left\| \left\| D_{1-}^{(n+1)\alpha} g(x_0 + t(x^* - x_0)) (t) \right\|_2 \right\|_{t, \infty, [0,1]} du \right\}
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma((n+1)\alpha+2)} \\
&\times \left\{ \|g(x^* + u(x_1 - x^*))\|_{u,\infty,[0,1]} \left\| \left\| D_{*0}^{(n+1)\alpha} (f(x^* + t(x_1 - x^*))) (t) \right\|_2 \right\|_{t,\infty,[0,1]} \right. \\
&\quad + \left\| f(x^* + u(x_1 - x^*)) \right\|_2 \|D_{*0}^{(n+1)\alpha} (g(x^* + t(x_1 - x^*))) (t)\|_{t,\infty,[0,1]} \\
(42) \quad &\quad + \|g(x_0 + u(x^* - x_0))\|_{u,\infty,[0,1]} \left\| \left\| D_{1-}^{(n+1)\alpha} f(x_0 + t(x^* - x_0)) (t) \right\|_2 \right\|_{t,\infty,[0,1]} \\
&\quad + \left\| f(x_0 + u(x^* - x_0)) \right\|_2 \|D_{1-}^{(n+1)\alpha} g(x_0 + t(x^* - x_0)) (t)\|_{t,\infty,[0,1]} \Big\},
\end{aligned}$$

proving (i).

ii) Let $(n+1)\alpha \geq 1$ or $\alpha \geq \frac{1}{n+1}$, then

$$\begin{aligned}
(\psi) &\leq \frac{1}{\Gamma((n+1)\alpha)} \\
&\times \left\{ \int_0^1 |g(x^* + u(x_1 - x^*))| u^{(n+1)\alpha-1} \left\| \left\| D_{*0}^{(n+1)\alpha} (f(x^* + t(x_1 - x^*))) (t) \right\|_2 \right\|_{t,L_1([0,1])} du \\
(43) \quad &\quad + \int_0^1 \|f(x^* + u(x_1 - x^*))\|_2 u^{(n+1)\alpha-1} \left\| \left\| D_{*0}^{(n+1)\alpha} (g(x^* + t(x_1 - x^*))) (t) \right\|_2 \right\|_{t,L_1([0,1])} du \\
&\quad + \int_0^1 |g(x_0 + u(x^* - x_0))| (1-u)^{(n+1)\alpha-1} \left\| \left\| D_{1-}^{(n+1)\alpha} (f(x_0 + t(x^* - x_0))) (t) \right\|_2 \right\|_{t,L_1([0,1])} du \\
&\quad + \int_0^1 \|f(x_0 + u(x^* - x_0))\|_2 (1-u)^{(n+1)\alpha-1} \left\| \left\| D_{1-}^{(n+1)\alpha} (g(x_0 + t(x^* - x_0))) (t) \right\|_2 \right\|_{t,L_1([0,1])} du \Big\} \\
&\leq \frac{1}{\Gamma((n+1)\alpha+1)} \\
&\times \left\{ \|g(x^* + u(x_1 - x^*))\|_{u,\infty,[0,1]} \left\| \left\| D_{*0}^{(n+1)\alpha} (f(x^* + t(x_1 - x^*))) (t) \right\|_2 \right\|_{t,L_1([0,1])} \right. \\
(44) \quad &\quad + \left\| f(x^* + u(x_1 - x^*)) \right\|_2 \|D_{*0}^{(n+1)\alpha} (g(x^* + t(x_1 - x^*))) (t)\|_{t,L_1([0,1])} \\
&\quad + \|g(x_0 + u(x^* - x_0))\|_{u,\infty,[0,1]} \left\| \left\| D_{1-}^{(n+1)\alpha} (f(x_0 + t(x^* - x_0))) (t) \right\|_2 \right\|_{t,L_1([0,1])} \\
&\quad + \left\| f(x_0 + u(x^* - x_0)) \right\|_2 \|D_{1-}^{(n+1)\alpha} (g(x_0 + t(x^* - x_0))) (t)\|_{t,L_1([0,1])} \Big\},
\end{aligned}$$

proving (ii).

iii) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $(n+1)\alpha > \frac{1}{q}$ or $\alpha > \frac{1}{(n+1)q}$. Then (by applying Hölder's inequality), we obtain

$$\begin{aligned}
(\psi) &\leq \frac{1}{\Gamma((n+1)\alpha)} \\
&\times \left\{ \int_0^1 |g(x^* + u(x_1 - x^*))| \frac{u^{(n+1)\alpha-1+\frac{1}{p}}}{(p((n+1)\alpha-1)+1)^{\frac{1}{p}}} \right.
\end{aligned}$$

$$\begin{aligned}
 & \times \left\| \left\| \left(D_{*0}^{(n+1)\alpha} f(x^* + t(x_1 - x^*)) \right) (t) \right\|_2 \right\|_{t, L_q([0,1])} du \\
 (45) \quad & + \int_0^1 \|f(x^* + u(x_1 - x^*))\|_2 \frac{u^{(n+1)\alpha-1+\frac{1}{p}}}{(p((n+1)\alpha-1)+1)^{\frac{1}{p}}} \\
 & \times \left\| \left\| \left(D_{*0}^{(n+1)\alpha} g(x^* + t(x_1 - x^*)) \right) (t) \right\|_2 \right\|_{t, L_q([0,1])} du \\
 & + \int_0^1 |g(x_0 + u(x^* - x_0))| \frac{(1-u)^{(n+1)\alpha-1+\frac{1}{p}}}{(p((n+1)\alpha-1)+1)^{\frac{1}{p}}} \\
 & \times \left\| \left\| \left(D_{1-}^{(n+1)\alpha} f(x_0 + t(x^* - x_0)) \right) (t) \right\|_2 \right\|_{t, L_q([0,1])} du \\
 & + \int_0^1 \|f(x_0 + u(x^* - x_0))\|_2 \frac{(1-u)^{(n+1)\alpha-1+\frac{1}{p}}}{(p((n+1)\alpha-1)+1)^{\frac{1}{p}}} \\
 & \times \left\| \left\| \left(D_{1-}^{(n+1)\alpha} g(x_0 + t(x^* - x_0)) \right) (t) \right\|_2 \right\|_{t, L_q([0,1])} du \Big\} \\
 & \leq \frac{1}{(p((n+1)\alpha-1)+1)^{\frac{1}{p}} \Gamma((n+1)\alpha) \left((n+1)\alpha + \frac{1}{p} \right)} \\
 & \times \left\{ \|g(x^* + u(x_1 - x^*))\|_{u, \infty, [0,1]} \left\| \left\| \left(D_{*0}^{(n+1)\alpha} f(x^* + t(x_1 - x^*)) \right) (t) \right\|_2 \right\|_{t, L_q([0,1])} \right. \\
 & + \|f(x^* + u(x_1 - x^*))\|_{u, \infty, [0,1]} \left\| \left\| \left(D_{*0}^{(n+1)\alpha} g(x^* + t(x_1 - x^*)) \right) (t) \right\|_2 \right\|_{t, L_q([0,1])} \\
 & + \|g(x_0 + u(x^* - x_0))\|_{u, \infty, [0,1]} \left\| \left\| \left(D_{1-}^{(n+1)\alpha} f(x_0 + t(x^* - x_0)) \right) (t) \right\|_2 \right\|_{t, L_q([0,1])} \\
 (46) \quad & \left. + \|f(x_0 + u(x^* - x_0))\|_{u, \infty, [0,1]} \left\| \left\| \left(D_{1-}^{(n+1)\alpha} g(x_0 + t(x^* - x_0)) \right) (t) \right\|_2 \right\|_{t, L_q([0,1])} \right\},
 \end{aligned}$$

proving (iii). $\square$

We make the following remark.

Remark 12. We notice that $(0 < \alpha < 1)$

$$(47) \quad D_{*0}^\alpha (f(x^* + u(x_1 - x^*))) \stackrel{(8)}{=} \frac{1}{\Gamma(1-\alpha)} \int_0^u (u-t)^{-\alpha} (f'(x^* + u(x_1 - x^*))) (x_1 - x^*) dt$$

and

$$\begin{aligned}
 & \|D_{*0}^\alpha (f(x^* + u(x_1 - x^*)))\|_2 \\
 (48) \quad & \leq \frac{1}{\Gamma(1-\alpha)} \int_0^u (u-t)^{-\alpha} \|f'(x^* + u(x_1 - x^*)) (x_1 - x^*)\|_2 dt \\
 & \leq \frac{u^{1-\alpha}}{\Gamma(2-\alpha)} \| \|f'(x^* + u(x_1 - x^*)) (x_1 - x^*)\|_2 \|_{t, \infty, [0,1]}.
 \end{aligned}$$

Clearly, we have

$$(49) \quad D_{*0}^\alpha (f(x^* + u(x_1 - x^*))) (0) = 0$$

and similarly, it holds

$$(50) \quad D_{*0}^\alpha (g(x^* + u(x_1 - x^*))) (0) = 0.$$

Next, we have ($0 < \alpha < 1$)

$$(51) \quad \begin{aligned} & D_{1-}^\alpha (f(x_0 + u(x^* - x_0))) \\ & \stackrel{(13)}{=} \frac{-1}{\Gamma(1-\alpha)} \int_u^1 (t-u)^{-\alpha} f'(x_0 + u(x^* - x_0)) (x^* - x_0) dt \end{aligned}$$

and

$$(52) \quad \begin{aligned} & \|D_{1-}^\alpha (f(x_0 + u(x^* - x_0)))\|_2 \\ & \leq \frac{1}{\Gamma(1-\alpha)} \int_u^1 (t-u)^{-\alpha} \|f'(x_0 + u(x^* - x_0)) (x^* - x_0)\|_2 dt \\ & \leq \frac{(1-u)^{1-\alpha}}{\Gamma(2-\alpha)} \| \|f'(x_0 + u(x^* - x_0)) (x^* - x_0)\|_2 \|_{t,\infty,[0,1]}. \end{aligned}$$

Hence we get

$$(53) \quad D_{1-}^\alpha (f(x_0 + u(x^* - x_0))) (1) = 0$$

and similarly,

$$(54) \quad D_{1-}^\alpha (g(x_0 + u(x^* - x_0))) (1) = 0.$$

The above (49), (50), (53), (54) are valid for any $x_0, x^*, x_1 \in M$.

When $n = 1$, by Theorems 8 and 9, we obtain directly and without any initial conditions the following ($0 < \alpha < 1$)

$$\begin{aligned} & f(x^* + u(x_1 - x^*)) - f(x^*) \\ & = \frac{1}{\Gamma(2\alpha)} \int_0^u (u-t)^{2\alpha-1} (D_{*0}^{2\alpha} (f(x^* + u(x_1 - x^*)))) (t) dt \end{aligned}$$

and

$$(55) \quad \begin{aligned} & g(x^* + u(x_1 - x^*)) - g(x^*) \\ & = \frac{1}{\Gamma(2\alpha)} \int_0^u (u-t)^{2\alpha-1} (D_{*0}^{2\alpha} (g(x^* + u(x_1 - x^*)))) (t) dt \end{aligned}$$

for all $u \in [0, 1]$.

Similarly, we derive that

$$\begin{aligned} & f(x_0 + u(x^* - x_0)) - f(x^*) \\ & = \frac{1}{\Gamma(2\alpha)} \int_u^1 (t-u)^{2\alpha-1} (D_{1-}^{2\alpha} f(x_0 + u(x^* - x_0))) (t) dt \end{aligned}$$

and

$$(56) \quad \begin{aligned} & g(x_0 + u(x^* - x_0)) - g(x^*) \\ & = \frac{1}{\Gamma(2\alpha)} \int_u^1 (t-u)^{2\alpha-1} (D_{1-}^{2\alpha} g(x_0 + u(x^* - x_0))) (t) dt \end{aligned}$$

for all $u \in [0, 1]$.

So Theorems 10 and 11 are valid, without any initial conditions, when $n = 1$, $0 < \alpha < 1$.

3.2. About alternative general mixed fractional Grüss type inequalities

We make the following remark.

Remark 13 (to Theorem 10). Case of $n = 1$, $0 < \alpha < 1$, and from now on $x^* \in \overline{x_0 x_1}$. We have

$$K(f, g)(x^*) = K(f, g)(x_0 + u(x_1 - x_0)) \quad \text{for some } u \in [0, 1].$$

Furthermore, it holds

$$(57) \quad \begin{aligned} & \int_0^1 K(f, g)(x_0 + u(x_1 - x_0)) \, du \\ &= 2 \left[\int_{x_0}^{x_1} f(x) g(x) \, dx - \left(\int_{x_0}^{x_1} f(x) \, dx \right) \left(\int_{x_0}^{x_1} g(x) \, dx \right) \right] \end{aligned}$$

by (21) being valid for any $x^* \in \overline{x_0 x_1}$. That is,

$$(58) \quad \begin{aligned} \Delta(f, g; x_0, x_1) &:= \left\| \int_{x_0}^{x_1} f(x) g(x) \, dx - \left(\int_{x_0}^{x_1} f(x) \, dx \right) \left(\int_{x_0}^{x_1} g(x) \, dx \right) \right\|_2 \\ &= \frac{1}{2} \left\| \int_0^1 K(f, g)(x_0 + u(x_1 - x_0)) \, du \right\|_2 \\ &\leq \frac{1}{2} \int_0^1 \|K(f, g)(x_0 + u(x_1 - x_0))\|_2 \, du. \end{aligned}$$

We denote by

$$(59) \quad \begin{aligned} M(f, g) &:= \max \left\{ \sup_{x^* \in \overline{x_0 x_1}} \|g(x^* + u(x_1 - x^*))\|_{u, \infty, [0, 1]}, \right. \\ &\quad \sup_{x^* \in \overline{x_0 x_1}} \| \|f(x^* + u(x_1 - x^*))\|_2 \|_{u, \infty, [0, 1]}, \\ &\quad \sup_{x^* \in \overline{x_0 x_1}} \|g(x_0 + u(x^* - x_0))\|_{u, \infty, [0, 1]}, \\ &\quad \left. \sup_{x^* \in \overline{x_0 x_1}} \| \|f(x_0 + u(x^* - x_0))\|_2 \|_{u, \infty, [0, 1]} \right\} < \infty, \end{aligned}$$

and call

$$(60) \quad \begin{aligned} N_1(f, g) &:= \max \left\{ \sup_{x^* \in \overline{x_0 x_1}} \| \|D_{*0}^{2\alpha}(f(x^* + t(x_1 - x^*))) (t)\|_2 \|_{t, \infty, [0, 1]}, \right. \\ &\quad \sup_{x^* \in \overline{x_0 x_1}} \| \|D_{*0}^{2\alpha}(g(x^* + t(x_1 - x^*))) (t)\|_2 \|_{t, \infty, [0, 1]}, \\ &\quad \sup_{x^* \in \overline{x_0 x_1}} \| \|D_{1-}^{2\alpha}f(x_0 + t(x^* - x_0)) (t)\|_2 \|_{t, \infty, [0, 1]}, \\ &\quad \left. \sup_{x^* \in \overline{x_0 x_1}} \| \|D_{1-}^{2\alpha}g(x_0 + t(x^* - x_0)) (t)\|_2 \|_{t, \infty, [0, 1]} \right\} \end{aligned}$$

assumed to be finite,

and call

$$(61) \quad N_2(f, g) := \max \left\{ \begin{aligned} &\sup_{x^* \in \overline{x_0 x_1}} \left\| \left\| D_{*0}^{2\alpha} (f(x^* + t(x_1 - x^*))) (t) \right\|_2 \right\|_{t, L_1([0,1])}, \\ &\sup_{x^* \in \overline{x_0 x_1}} \left\| \left\| D_{*0}^{2\alpha} (g(x^* + t(x_1 - x^*))) (t) \right\|_2 \right\|_{t, L_1([0,1])}, \\ &\sup_{x^* \in \overline{x_0 x_1}} \left\| \left\| D_{1-}^{2\alpha} (f(x_0 + t(x^* - x_0))) (t) \right\|_2 \right\|_{t, L_1([0,1])}, \\ &\sup_{x^* \in \overline{x_0 x_1}} \left\| \left\| D_{1-}^{2\alpha} (g(x_0 + t(x^* - x_0))) (t) \right\|_2 \right\|_{t, L_1([0,1])} \end{aligned} \right\}$$

assumed to be finite,
and call

$$(62) \quad N_3(f, g) := \max \left\{ \begin{aligned} &\sup_{x^* \in \overline{x_0 x_1}} \left\| \left\| D_{*0}^{2\alpha} f(x^* + t(x_1 - x^*)) (t) \right\|_2 \right\|_{t, L_q([0,1])}, \\ &\sup_{x^* \in \overline{x_0 x_1}} \left\| \left\| D_{*0}^{2\alpha} g(x^* + t(x_1 - x^*)) (t) \right\|_2 \right\|_{t, L_q([0,1])}, \\ &\sup_{x^* \in \overline{x_0 x_1}} \left\| \left\| D_{1-}^{2\alpha} f(x_0 + t(x^* - x_0)) (t) \right\|_2 \right\|_{t, L_q([0,1])}, \\ &\sup_{x^* \in \overline{x_0 x_1}} \left\| \left\| D_{1-}^{2\alpha} g(x_0 + t(x^* - x_0)) (t) \right\|_2 \right\|_{t, L_q([0,1])} \end{aligned} \right\}$$

assumed to be finite.

We present the following Grüss type inequalities:

Theorem 14. *Following the assumptions of Theorem 10 for $n = 1$, $0 < \alpha < 1$ and $x^* \in \overline{x_0 x_1}$; $x_0, x^*, x_1 \in M$, but without any initial conditions. All the rest as in Remark 13. Then:*

i)

$$(63) \quad \Delta(f, g; x_0, x_1) \leq \frac{2M(f, g) N_1(f, g)}{\Gamma(2(\alpha + 1))},$$

ii) when $\alpha \geq \frac{1}{2}$, we have

$$(64) \quad \Delta(f, g; x_0, x_1) \leq \frac{2M(f, g) N_2(f, g)}{\Gamma(2\alpha + 1)},$$

iii) when $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$; $\alpha > \frac{1}{2q}$, we have

$$(65) \quad \Delta(f, g; x_0, x_1) \leq \frac{2M(f, g) N_3(f, g)}{(p(2\alpha - 1) + 1)^{\frac{1}{p}} \Gamma(2\alpha) \left(2\alpha + \frac{1}{p}\right)}.$$

Proof. By Theorem 11, see also Remark 12. □

Since M is compact, an interesting application can exist when x_0, x_1 are the endpoints of its diameter.

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