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QUANTITATIVE APPROXIMATION BY MULTIPLE SIGMOIDS KANTOROVICH-SHILKRET QUASI-INTERPOLATION NEURAL NETWORK OPERATORS

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Abstract. In this article, we derive multivariate quantitative approximation by Kantorovich-Shilkret type quasi-interpolation neural network operators with respect to supremum and L_p norms. This is done with rates using the multivariate modulus of continuity. We approximate continuous and bounded functions on $\mathbb{R}^N$, $N \in \mathbb{N}$. When they are also uniformly continuous, we have pointwise and uniform convergences, plus L_p estimates. We include also the related complex approximation. Our activation functions are induced by multiple general sigmoid functions.

1. Introduction

The author in [**1**] and [**2**], see Chapters 2–5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treated there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be compact support. Also in [**2**], he gave the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see Chaptes 4–5 there.

The author inspired by [**16**], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [**3, 4, 5, 6, 7**], by treating both the univariate and multivariate cases. He did also the corresponding fractional case [**8**]. For recent works, see [**9, 10, 11, 12, 13, 14, 15**].

The author here performs multivariate multiple general sigmoid activation functions based neural network approximation to continuous functions over the whole

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$\mathbb{R}^N$, $N \in \mathbb{N}$, then he extends his results to complex valued functions. L_p approximations are included. All convergences here are with rates expressed via the modulus of continuity of the involved function and given by very tight Jackson type inequalities.

The author comes up with the "right" precisely defined flexible quasi-interpolation, Kantorovich-Shilkret type integral coefficient neural networks operators associated with multiple general sigmoid activation functions. In preparation to prove our results, we present important properties of the general density functions defining our operators.

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma(\langle a_j \cdot x \rangle + b_j), \quad x \in \mathbb{R}^s, \ s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental neural network models, the activation functions are based on multiple general sigmoid activation functions. About neural networks in general, read [**17, 18, 19**].

In recent years, non-additive integrals, like the N. Shilkret one [**20**], have become fashionable and more useful in Economics, etc.

2. Background

2.1. About Shilkret integral

Here we follow [**20**].

Let $\mathcal{F}$ be a σ-field of subsets of an arbitrary set Ω. An extended non-negative real valued function μ on $\mathcal{F}$ is called maxitive if $\mu(\emptyset) = 0$ and

$$\mu(\cup_{i \in I} E_i) = \sup_{i \in I} \mu(E_i), \tag{1}$$

where the set I is of cardinality at most countable. We also call μ a maxitive measure. Here f stands for a non-negative measurable function on Ω. In [**20**], Niel Shilkret developed his non-additive integral defined as follows:

$$(N^*) \int_D f \mathrm{d}\mu := \sup_{y \in Y} \{ y \cdot \mu(D \cap \{f \geq y\}) \}, \tag{2}$$

where $Y = [0, m]$ or $Y = [0, m)$ with $0 < m \leq \infty$, and $D \in \mathcal{F}$. Here we take $Y = [0, \infty)$.

It is easily proved that

$$(N^*) \int_D f \mathrm{d}\mu = \sup_{y > 0} \{ y \cdot \mu(D \cap \{f > y\}) \}. \tag{3}$$

The Shilkret integral takes values in $[0, \infty]$.

The Shilkret integral ([**20**]) has the following properties:

$$(N^*)\int_{\Omega}\chi_E \mathrm{d}\mu=\mu(E), \tag{4}$$

where χ_E is the indicator function on $E\in\mathcal{F}$,

$$(N*)\int_D cf\mathrm{d}\mu=c(N^*)\int_D f\mathrm{d}\mu,\quad c\geq 0, \tag{5}$$

$$(N^*)\int_D \sup_{n\in\mathbb{N}} f_n\mathrm{d}\mu=\sup_{n\in\mathbb{N}}(N^*)\int_D f_n\mathrm{d}\mu, \tag{6}$$

where f_n, $n\in\mathbb{N}$, is an increasing sequence of elementary (countably valued) functions converging uniformly to f. Furthermore, we have

$$(N^*)\int_D f\mathrm{d}\mu\geq 0, \tag{7}$$

$$f\geq g\ \text{ implies }\ (N^*)\int_D f\mathrm{d}\mu\geq(N^*)\int_D g\mathrm{d}\mu, \tag{8}$$

where $f,g\colon\Omega\to[0,\infty]$ are measurable.

Let $a\leq f(\omega)\leq b$ for almost every $\omega\in E$, then

$$a\mu(E)\leq(N^*)\int_E f\mathrm{d}\mu\leq b\mu(E),$$

$$(N^*)\int_E 1\mathrm{d}\mu=\mu(E),$$

$f>0$ almost everywhere and $(N^*)\int_E f\mathrm{d}\mu=0$ imply $\mu(E)=0$,

$(N^*)\int_\Omega f\mathrm{d}\mu=0$ if and only if $f=0$ almost everywhere,

$(N^*)\int_\Omega f\mathrm{d}\mu<\infty$ implies

$$\overline{N}(f):=\{\omega\in\Omega|f(\omega)\neq 0\}\ \text{ has }\sigma\text{-finite measure}, \tag{9}$$

$$(N^*)\int_D(f+g)\mathrm{d}\mu\leq(N^*)\int_D f\mathrm{d}\mu+(N^*)\int_D g\mathrm{d}\mu,$$

and

$$\left|(N^*)\int_D f\mathrm{d}\mu-(N^*)\int_D g\mathrm{d}\mu\right|\leq(N^*)\int_D|f-g|\mathrm{d}\mu. \tag{10}$$

From now on, in this article, we assume $\mu\colon\mathcal{F}\to[0,+\infty)$.

2.2. On activation functions

Let $i=1,\dots,N\in\mathbb{N}$ and $h_i\colon\mathbb{R}\to[-1,1]$ be a general sigmoid function such that it is strictly increasing, $h_i(0)=0$, $h_i(-x)=-h_i(x)$, $h_i(+\infty)=1$, $h_i(-\infty)=-1$. Also h_i is strictly convex over $(-\infty,0]$ and strictly concave over $[0,+\infty)$, with $h_i^{(2)}\in C(\mathbb{R},[-1,1])$.

Some examples of related sigmoid functions follow:

$$\frac{1}{1+e^{-x}}; \quad \tanh x; \quad \frac{2}{\pi}\arctan\left(\frac{\pi}{2}x\right); \quad \frac{x}{\sqrt[2m]{1+x^{2m}}},\ m\in\mathbb{N};$$

$$\frac{4}{\pi}gd\left(x\right); \quad \frac{x}{\left(1+\left|x\right|\right)^{\frac{1}{\lambda}}},\ \lambda \text{ is odd}; \quad erf\left(\frac{\sqrt{\pi}}{2}x\right);$$

$$\frac{1}{1+e^{-\mu x}}; \quad \tanh \mu x,\ \mu>0; \qquad \text{for all } x\in\mathbb{R}.$$

We consider the activation function

$$\psi_i\left(x\right) := \frac{1}{4}\left(h_i\left(x+1\right)-h_i\left(x-1\right)\right), \quad x\in\mathbb{R},\ i=1,\dots,N. \tag{11}$$

As in [**11**, p. 285], we get $\psi_i\left(-x\right)=\psi_i\left(x\right)$, thus ψ_i is an even function. Since $x+1>x-1$, then $h_i\left(x+1\right)>h_i\left(x-1\right)$, and $\psi_i\left(x\right)>0$, all $x\in\mathbb{R}$.

We see that

$$\psi_i\left(0\right)=\frac{h_i\left(1\right)}{2}, \quad i=1,\dots,N. \tag{12}$$

Let $x>1$, we have that

$$\psi_i'\left(x\right)=\frac{1}{4}\left(h_i'\left(x+1\right)-h_i'\left(x-1\right)\right)<0$$

by h_i' being strictly decreasing over $[0,+\infty)$.

Let now $0<x<1$, then $1-x>0$ and $0<1-x<1+x$. It holds $h_i'\left(x-1\right)=h_i'\left(1-x\right)>h_i'\left(x+1\right)$, so that again $\psi_i'\left(x\right)<0$. Consequently, ψ_i is stritly decreasing on $\left(0,+\infty\right)$.

Clearly, ψ_i is strictly increasing on $(-\infty,0)$, and $\psi_i'\left(0\right)=0$.

See that

$$\lim_{x\to+\infty}\psi_i\left(x\right)=\frac{1}{4}\left(h_i\left(+\infty\right)-h_i\left(+\infty\right)\right)=0 \tag{13}$$

and

$$\lim_{x\to-\infty}\psi_i\left(x\right)=\frac{1}{4}\left(h_i\left(-\infty\right)-h_i\left(-\infty\right)\right)=0. \tag{14}$$

That is, the x-axis is the horizontal asymptote on ψ_i.

Conclusion: ψ is a bell symmetric function with maximum

$$\psi_i\left(0\right)=\frac{h_i\left(1\right)}{2}.$$

We need the following theorems.

Theorem 1. *We have*

$$\sum_{i=-\infty}^{\infty}\psi_i\left(x-i\right)=1 \quad \textit{for all } x\in\mathbb{R},\ i=1,\dots,N. \tag{15}$$

Proof. Exactly the same as in [**11**, p. 286], is omitted. □

Theorem 2. *It holds*

$$\int_{-\infty}^{\infty} \psi_i(x)\,\mathrm{d}x = 1, \quad i = 1, \dots, N. \tag{16}$$

Proof. Similar to [**11**, p. 287]. It is omitted. □

Thus $\psi_i(x)$ is a density function on $\mathbb{R}$, $i = 1, \dots, N$.
We also need the following theorem.

Theorem 3. *Let* $0 < \alpha < 1$, *and* $n \in \mathbb{N}$ *with* $n^{1-\alpha} > 2$. *It holds*

$$\sum_{\substack{k=-\infty \\ |nx-k| \geq n^{1-\alpha}}}^{\infty} \psi_i(nx-k) < \frac{\left(1 - h_i\left(n^{1-\alpha} - 2\right)\right)}{2}, \quad i = 1, \dots, N. \tag{17}$$

Notice that

$$\lim_{n \to +\infty} \frac{\left(1 - h_i\left(n^{1-\alpha} - 2\right)\right)}{2} = 0, \quad i = 1, \dots, N.$$

Proof. Similar to [**13**], as such is omitted. □

We make the following remark.

Remark 4. We define

$$Z(x_1, \dots, x_N) := Z(x) := \prod_{i=1}^{N} \psi_i(x_i), \quad x = (x_1, \dots, x_N) \in \mathbb{R}^N,\ N \in \mathbb{N}. \tag{18}$$

It has the properties:

(i)

$$Z(x) > 0 \quad \text{for all } x \in \mathbb{R}^N, \tag{19}$$

(ii)

$$\begin{aligned} \sum_{k=-\infty}^{\infty} Z(x-k) &:= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} Z(x_1 - k_1, \dots, x_N - k_N) \\ &= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} \prod_{i=1}^{N} \psi_i(x_i - k_i) \\ &= \prod_{i=1}^{N} \left(\sum_{k_i=-\infty}^{\infty} \psi_i(x_i - k_i) \right) \overset{(5)}{=} 1. \end{aligned}$$

Hence

$$\sum_{k=-\infty}^{\infty} Z(x-k) = 1. \tag{20}$$

(iii)

$$\sum_{k=-\infty}^{\infty} Z(nx-k) = 1 \quad \text{for all } x \in \mathbb{R}^N,\ n \in \mathbb{N}, \tag{21}$$

(iv)
(22)
$$\int_{\mathbb{R}^N} Z(x)\,\mathrm{d}x = \int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}\psi_i(x_i)\right)\mathrm{d}x_1\ldots\mathrm{d}x_N = \prod_{i=1}^{N}\left(\int_{-\infty}^{\infty}\psi_i(x_i)\,\mathrm{d}x_i\right) \overset{(16)}{=} 1.$$

Thus,

$$\int_{\mathbb{R}^N} Z(x)\,\mathrm{d}x = 1, \tag{23}$$

that is, Z is a multivariate density function.

Here denote $x = (x_1, \ldots, x_N)$, $\|x\|_\infty := \max\{|x_1|, \ldots, |x_N|\}$, $x \in \mathbb{R}^N$, also set $\infty := (\infty, \ldots, \infty)$, $-\infty := (-\infty, \ldots, -\infty)$ upon the multivariate context, $0 < \beta < 1$.

(v) We have

$$\sum_{\substack{k=-\infty \\ \|\frac{k}{n}-x\|_\infty > \frac{1}{n^\beta}}}^{\infty} Z(nx-k) = \sum_{\substack{k_1=-\infty \\ \|\frac{k}{n}-x\|_\infty > \frac{1}{n^\beta}}}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty}\left(\prod_{i=1}^{N}\psi_i(nx_i-k_i)\right)$$

$$= \prod_{i=1}^{N}\left(\sum_{\substack{k_i=-\infty \\ \|\frac{k}{n}-x\|_\infty > \frac{1}{n^\beta}}}^{\infty}\psi_i(nx_i-k_i)\right) \quad (\text{for some } r \in \{1, \ldots, N\})$$

$$\le \left(\prod_{\substack{i=1 \\ i\neq r}}^{N}\left(\sum_{k_i=-\infty}^{\infty}\psi_i(nx_i-k_i)\right)\right)\left(\sum_{\substack{k_r=-\infty \\ |\frac{k_r}{n}-x_r|_\infty > \frac{1}{n^\beta}}}^{\infty}\psi_r(nx_r-k_r)\right) \tag{24}$$

$$= \sum_{\substack{k_r=-\infty \\ |\frac{k_r}{n}-x_r|_\infty > \frac{1}{n^\beta}}}^{\infty}\psi_r(nx_r-k_r) = \sum_{\substack{k_r=-\infty \\ |nx_r-k_r| > n^{1-\beta}}}^{\infty}\psi_r(nx_r-k_r)$$

$$\overset{(17)}{<} \frac{1-h_r\left(n^{1-\beta}-2\right)}{2} \le \max_{i\in\{1,\ldots,N\}}\left(\frac{1-h_i\left(n^{1-\beta}-2\right)}{2}\right).$$

That is,

$$\sum_{\substack{k=-\infty \\ \|\frac{k}{n}-x\|_\infty > \frac{1}{n^\beta}}}^{\infty} Z(nx-k) < \max_{i\in\{1,\ldots,N\}}\left(\frac{1-h_i\left(n^{1-\beta}-2\right)}{2}\right), \tag{25}$$

$0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, for all $x \in \mathbb{R}^N$.

Denote by

$$\delta_N(\beta, n) := \max_{i\in\{1,\ldots,N\}}\left(\frac{1-h_i\left(n^{1-\beta}-2\right)}{2}\right), \quad 0 < \beta < 1. \tag{26}$$

For $f \in C_B^+\left(\mathbb{R}^N\right)$ (continuous and bounded functions from $\mathbb{R}^N$ into $\mathbb{R}_+$), we define the first modulus of continuity

$$\omega_1(f,\delta) := \sup_{\substack{x,y\in\mathbb{R}^N \\ \|x-y\|_\infty \leq h}} |f(x) - f(y)|, \quad h > 0. \tag{27}$$

Given that $f \in C_U^+\left(\mathbb{R}^N\right)$ (uniformly continuous from $\mathbb{R}^N$ into $\mathbb{R}_+$, same definition for ω_1), we have

$$\lim_{h\to 0} \omega_1(f,h) = 0. \tag{28}$$

When $N = 1$, ω_1 is defined as in (27), with $\|\cdot\|_\infty$ collapsing to $|\cdot|$ and having the property (28).

3. Main Results

We need the following definition.

Definition 5. Let $\mathcal{L}$ be the Lebesgue σ-algebra on $\mathbb{R}^N$, $N \in \mathbb{N}$, and the maxitive measure $\mu\colon \mathcal{L} \to [0,+\infty)$ such that for any $A \in \mathcal{L}$ with $A \neq \emptyset$, we get $\mu(A) > 0$.

For $f \in C_B^+\left(\mathbb{R}^N\right)$, we define the multivariate Kantorovich-Shilkret type neural network operators for any $x \in \mathbb{R}^N$:

$$\begin{aligned} T_n^\mu(f,x) =\ & T_n^\mu(f,x_1,\ldots,x_N) \\ := & \sum_{k=-\infty}^{\infty} \left(\frac{(N^*)\int_{\left[0,\frac{1}{N}\right]^N} f\left(t+\frac{k}{n}\right) \mathrm{d}\mu(t)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)} \right) Z(nx-k) \\ = & \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} \left(\frac{(N^*)\int_0^{\frac{1}{n}} \ldots \int_0^{\frac{1}{n}} f\left(t_1+\frac{k_1}{n}, t_2+\frac{k_2}{n}, \ldots, t_N+\frac{k_N}{n}\right) \mathrm{d}\mu(t_1,\ldots,t_N)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)} \right) \\ & \times \left(\prod_{i=1}^N \psi_i(nx_i - k_i) \right), \end{aligned} \tag{29}$$

where $x = (x_1,\ldots,x_N) \in \mathbb{R}^N$, $k = (k_1,\ldots,k_N)$, $t = (t_1,\ldots,t_N)$, $n \in \mathbb{N}$.

Clearly here $\mu\left(\left[0,\frac{1}{n}\right]^N\right) > 0$ for all $n \in \mathbb{N}$.

Above we notice that

$$\|T_n^\mu(f)\|_\infty \leq \|f\|_\infty, \tag{30}$$

so that $T_n^\mu(f,x)$ is well-defined.

We make the following remark.

Remark 6. Let $t \in \left[0,\frac{1}{n}\right]^N$ and $x \in \mathbb{R}^N$, then

$$f\left(t+\frac{k}{n}\right) = f\left(t+\frac{k}{n}\right) - f(x) + f(x) \leq \left| f\left(t+\frac{k}{n}\right) - f(x) \right| + f(x), \tag{31}$$

hence

$$
\begin{aligned}
&(N^*)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right) \mathrm{d}\mu(t) \\
&\leq (N^*)\int_{\left[0,\frac{1}{n}\right]^N} \left| f\left(t+\frac{k}{n}\right) - f(x)\right| \mathrm{d}\mu(t) + f(x)\,\mu\left(\left[0,\frac{1}{n}\right]^N\right).
\end{aligned} \tag{32}
$$

That is,

$$
\begin{aligned}
&(N^*)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right) \mathrm{d}\mu(t) - f(x)\,\mu\left(\left[0,\frac{1}{n}\right]^N\right) \\
&\leq (N^*)\int_{\left[0,\frac{1}{n}\right]^N} \left| f\left(t+\frac{k}{n}\right) - f(x)\right| \mathrm{d}\mu(t).
\end{aligned} \tag{33}
$$

Similarly, we have

$$
f(x) = f(x) - f\left(t+\frac{k}{n}\right) + f\left(t+\frac{k}{n}\right) \leq \left| f\left(t+\frac{k}{n}\right) - f(x)\right| + f\left(t+\frac{k}{n}\right),
$$

hence

$$
\begin{aligned}
&(N^*)\int_{\left[0,\frac{1}{n}\right]^N} f(x)\,\mathrm{d}\mu(t) \\
&\leq (N^*)\int_{\left[0,\frac{1}{n}\right]^N} \left| f\left(t+\frac{k}{n}\right) - f(x)\right| \mathrm{d}\mu(t) + (N^*)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right) \mathrm{d}\mu(t).
\end{aligned}
$$

That is,

$$
\begin{aligned}
&f(x)\,\mu\left(\left[0,\frac{1}{n}\right]^N\right) - (N^*)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right) \mathrm{d}\mu(t) \\
&\leq (N^*)\int_{\left[0,\frac{1}{n}\right]^N} \left| f\left(t+\frac{k}{n}\right) - f(x)\right| \mathrm{d}\mu(t).
\end{aligned} \tag{34}
$$

By (33) and (34), we derive

$$
\begin{aligned}
&\left|(N^*)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right) \mathrm{d}\mu(t) - f(x)\,\mu\left(\left[0,\frac{1}{n}\right]^N\right)\right| \\
&\leq (N^*)\int_{\left[0,\frac{1}{n}\right]^N} \left| f\left(t+\frac{k}{n}\right) - f(x)\right| \mathrm{d}\mu(t).
\end{aligned} \tag{35}
$$

In particular, it holds

$$
\left| \frac{(N^*)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{k}{n}\right) \mathrm{d}\mu(t)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)} - f(x)\right| \leq \frac{(N^*)\int_{\left[0,\frac{1}{n}\right]^N} \left| f\left(t+\frac{k}{n}\right) - f(x)\right| \mathrm{d}\mu(t)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)}. \tag{36}
$$

We present the following approximation result.

Theorem 7. *Let* $f \in C_B^+\left(\mathbb{R}^N\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$ *with* $n^{1-\beta} > 2$. *Then*

i)

$$\sup_{\mu} \left| T_n^{\mu}(f, x) - f(x) \right| \le \omega_1\left(f, \frac{1}{n} + \frac{1}{n^{\beta}}\right) + 2\|f\|_{\infty} \delta_N(\beta, n) =: \lambda_n, \tag{37}$$

ii)

$$\sup_{\mu} \left\| T_n^{\mu}(f) - f \right\|_{\infty} \le \lambda_n. \tag{38}$$

Given that $f \in \left(C_U^+\left(\mathbb{R}^N\right) \cap C_B^+\left(\mathbb{R}^N\right)\right)$, *we obtain* $\lim_{n\to\infty} T_n^{\mu}(f) = f$ *uniformly. Above* $\delta_N(\beta, n)$ *is as in* (26).

Proof. We observe

$$\begin{aligned}
&\left| T_n^{\mu}(f, x) - f(x) \right| \\
&= \left| \sum_{k=-\infty}^{\infty} \left(\frac{(N^*)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t + \frac{k}{n}\right) d\mu(t)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)} \right) Z(nx - k) - \sum_{k=-\infty}^{\infty} f(x) Z(nx - k) \right| \\
&= \left| \sum_{k=-\infty}^{\infty} \left(\left(\frac{(N^*)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t + \frac{k}{n}\right) d\mu(t)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)} \right) - f(x) \right) Z(nx - k) \right| \\
&\le \sum_{k=-\infty}^{\infty} \left| \left(\frac{(N^*)\int_{\left[0,\frac{1}{n}\right]^N} f\left(t + \frac{k}{n}\right) d\mu(t)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)} \right) - f(x) \right| Z(nx - k) \\
&\overset{(36)}{\le} \sum_{k=-\infty}^{\infty} \left(\frac{(N^*)\int_{\left[0,\frac{1}{n}\right]^N} \left| f\left(t + \frac{k}{n}\right) - f(x) \right| d\mu(t)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)} \right) Z(nx - k) \\
&= \sum_{\substack{k=-\infty \\ \left\|\frac{k}{n} - x\right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\infty} \left(\frac{(N^*)\int_{\left[0,\frac{1}{n}\right]^N} \left| f\left(t + \frac{k}{n}\right) - f(x) \right| d\mu(t)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)} \right) Z(nx - k) \qquad (39) \\
&+ \sum_{\substack{k=-\infty \\ \left\|\frac{k}{n} - x\right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\infty} \left(\frac{(N^*)\int_{\left[0,\frac{1}{n}\right]^N} \left| f\left(t + \frac{k}{n}\right) - f(x) \right| d\mu(t)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)} \right) Z(nx - k) \\
&\le \sum_{\substack{k=-\infty \\ \left\|\frac{k}{n} - x\right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\infty} \left(\frac{(N^*)\int_{\left[0,\frac{1}{n}\right]^N} \omega_1\left(f, \|t\|_{\infty} + \left\|\frac{k}{n} - x\right\|_{\infty}\right) d\mu(t)}{\mu\left(\left[0,\frac{1}{n}\right]^N\right)} \right) Z(nx - k) \\
&+ 2\|f\|_{\infty} \left(\sum_{\substack{k=-\infty \\ \left\|\frac{k}{n} - x\right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\infty} Z(nx - k) \right) \text{ (by (25))}
\end{aligned}$$

$$\leq \omega_1\left(f,\frac{1}{n}+\frac{1}{n^{\beta}}\right)+2\left\|f\right\|_{\infty}\delta_N\left(\beta,n\right), \tag{40}$$

proving the claim. □

Additionally we give the following definition.

Definition 8. Denote by $C_B^+\left(\mathbb{R}^N,\mathbb{C}\right)=\{f\colon\mathbb{R}^N\to\mathbb{C}|f=f_1+\mathrm{i}f_2,$ where $f_1,f_2\in C_B^+\left(\mathbb{R}^N\right)$, $N\in\mathbb{N}\}$. For $f\in C_B^+\left(\mathbb{R}^N,\mathbb{C}\right)$, we set

$$T_n^{\mu}\left(f,x\right):=\ T_n^{\mu}\left(f_1,x\right)+\mathrm{i}\,T_n^{\mu}\left(f_2,x\right) \tag{41}$$

for all $n\in\mathbb{N}$, $x\in\mathbb{R}^N$, $\mathrm{i}=\sqrt{-1}$.

Theorem 9. *Let $f\in C_B^+\left(\mathbb{R}^N,\mathbb{C}\right)$, $f=f_1+\mathrm{i}f_2$, $N\in\mathbb{N}$, $0<\beta<1$, $x\in\mathbb{R}^N$, $n\in\mathbb{N}$ with $n^{1-\beta}>2$. Then*

i)

$$\begin{aligned}\sup_{\mu}\left|T_n^{\mu}\left(f,x\right)-f\left(x\right)\right|\leq&\left[\omega_1\left(f_1,\frac{1}{n}+\frac{1}{n^{\beta}}\right)+\omega_1\left(f_2,\frac{1}{n}+\frac{1}{n^{\beta}}\right)\right]\\&+2\left(\left\|f_1\right\|_{\infty}+\left\|f_2\right\|_{\infty}\right)\delta_N\left(\beta,n\right)=:l_n,\end{aligned} \tag{42}$$

ii)

$$\sup_{\mu}\left\|T_n^{\mu}\left(f\right)-f\right\|\leq l_n. \tag{43}$$

Proof.

$$\begin{aligned}&\left|T_n^{\mu}\left(f,x\right)-f\left(x\right)\right|\\&=\left|T_n^{\mu}\left(f_1,x\right)+\mathrm{i}\,T_n^{\mu}\left(f_2,x\right)-f_1\left(x\right)-\mathrm{i}f_2\left(x\right)\right|\\&=\left|\left(T_n^{\mu}\left(f_1,x\right)-f_1\left(x\right)\right)+\mathrm{i}\left(T_n^{\mu}\left(f_2,x\right)-f_2\left(x\right)\right)\right|\\&\leq\left|T_n^{\mu}\left(f_1,x\right)-f_1\left(x\right)\right|+\left|T_n^{\mu}\left(f_2,x\right)-f_2\left(x\right)\right|\\&\overset{(37)}{\leq}\left(\omega_1\left(f_1,\frac{1}{n}+\frac{1}{n^{\beta}}\right)+2\left\|f_1\right\|_{\infty}\delta_N\left(\beta,n\right)\right)\\&\quad+\left(\omega_1\left(f_2,\frac{1}{n}+\frac{1}{n^{\beta}}\right)+2\left\|f_2\right\|_{\infty}\delta_N\left(\beta,n\right)\right)\end{aligned} \tag{44}$$

$$=\left[\omega_1\left(f_1,\frac{1}{n}+\frac{1}{n^{\beta}}\right)+\omega_1\left(f_2,\frac{1}{n}+\frac{1}{n^{\beta}}\right)\right]+2\left(\left\|f_1\right\|_{\infty}+\left\|f_2\right\|_{\infty}\right)\delta_N\left(\beta,n\right), \tag{45}$$

proving the claim. □

We finish with an L_{p_1}, $p_1\geq1$, estimate.

Theorem 10. *Let $f\in C_B^+\left(\mathbb{R}^N,\mathbb{C}\right)$, $f=f_1+\mathrm{i}f_2$, $N\in\mathbb{N}$, $0<\beta<1$, $n\in\mathbb{N}$ with $n^{1-\beta}>2$, and $p_1\geq1$. Then*

$$\left\|T_n^{\mu}\left(f\right)-f\right\|_{p_1,\Lambda}\leq l_n\left|\Lambda\right|^{\frac{1}{p_1}}, \tag{46}$$

where $|\Lambda|<\infty$ is the Lebesgue measure of compact $\Lambda\subset\mathbb{R}^N$, and l_n as in (42).

Proof. By integrating (42), etc. □

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