Acta Math. Univ. Comenianae
Vol. XCIII, 1 (2024), pp. 41–58

ON NÖRLUND $\mathcal{I}_2$-LACUNARY STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES IN INTUITIONISTIC FUZZY NORMED SPACES

Ö. KIŞI, M. GÜRDAL and C. CHOUDHURY

Abstract. Investigating the concepts of Nörlund $\mathcal{I}_2$-lacunary statistical convergence and Nörlund $\mathcal{I}_2$-lacunary statistical convergence of double sequence in intuitionistic fuzzy normed spaces (IFNS) is the goal of this article. Then, we introduce the concepts of Nörlund strongly $\mathcal{I}_2^*$-lacunary convergence, Nörlund strongly $\mathcal{I}_2$-lacunary Cauchy, and Nörlund strongly $\mathcal{I}_2^*$-lacunary Cauchy double sequences in IFNS and get astounding results.

1. Introduction and preliminaries

Statistical convergence and ideal convergence of a sequence were introduced by Fast [**8**] and Kostyrko et al. [**16**], respectively. These concepts were rethought for double sequences by Mursaleen and Edely [**24**] and Das et al. [**6**], respectively. Later, these concepts have been generalized in many directions. More details on statistical convergence and ideal convergence, and on applications of these concepts can be found in [**7, 11, 21, 23, 25, 26, 27, 28, 31, 32**]. In another direction, a new type of convergence called lacunary statistical convergence was introduced in [**9**]. The relation between lacunary statistical convergence and statistical convergence was established among other related things in [**9**]. Some works in lacunary statistical convergence can be found in [**5, 18, 35**].

In many areas of mathematics and engineering, the fuzzy theory has recently become the subject of the most active research. The fuzzy set theory was initially developed by Zadeh [**38**] in 1965. After that, several research publications have used the idea of fuzzy sets (numbers) and many conventional theories have also been fuzzified. Later on, Atanassov [**1**] put forward intuitionistic fuzzy set (IFS), which is the extension of fuzzy set. Many complicated issues relating to many fields have been solved using fuzzy sets (FSs) and intuitionistic fuzzy sets, particularly

Received November 8, 2022; revised March 14, 2024.
2020 *Mathematics Subject Classification.* Primary 40F05, 46A45, 40A05.
Key words and phrases. Intuitionistic normed spaces; ideal convergence; Nörlund convergent space.
The third author is grateful to the University Grants Commission, India for their fellowships funding under the UGC-SRF scheme (F. No. 16-6(DEC. 2018)/2019(NET/CSIR)) during the preparation of this paper.

in decision-making. Fuzzy and probabilistic metric spaces were combined to create the fuzzy metric space (FMS), which was invented by Kramosil and Michalek [**17**]. Park [**30**] looked upon intuitionistic fuzzy metric space and also FMSs (IFMS). Lael and Nourouzi began their investigation of an IF-normed space in [**19**], which was motivated by Park's description of an IFM. By Karakuş et al. [**13**], statistical convergence in IFNS was shown. One can consult [**12, 23, 25, 26, 27, 33, 34**]'s writings for in-depth research on this subject. We refer the reader also to the recent monographs [**2**] and [**3**], and references therein, devoted to summability theory and the spaces of double sequences generated by some four dimensional triangle matrices with a new approach.

Nörlund sequence space was investigated by Wang [**37**] as follows:

$$\mathcal{N}^f = \left\{ \varpi = (\varpi_u) \in l_\infty : \sum_{k=0}^{\infty} \left| \frac{1}{A_k} \sum_{u=0}^{k} f_{k-u} \varpi_u \right|^p < \infty, \, 1 \leq p < \infty \right\},$$

where $A_k = \sum_{u=0}^{k} f_u$. The spaces $l_\infty \left(\mathcal{N}^f \right)$ and $l_p \left(\mathcal{N}^f \right)$ consist of all sequences whose Nörlund transforms are in the space l_∞ and l_p, where $1 \leq p < \infty$.

Wang [**37**] utilized the Nörlund matrix $\mathcal{N}^f$ in the theory of sequence space for the first time. Recall that in [**10**], $f = (f_u)$ was presumed to be a non-negative sequence of real numbers and $T_j = \sum_{u=0}^{j} f_u$, for all $j \in \mathbb{N}$ with $f_0 > 0$, where $\mathbb{N}$ denotes the set of non-negative integers. Afterwards, Nörlund matrix $\mathcal{N}^f = \left(a_{ju}^f \right)$ w.r.t. the sequence $f = (f_u)$ is identified as follows:

$$a_{ju}^f = \begin{cases} \frac{f_{j-u}}{T_j} & \text{if } 0 \leq u \leq j, \\ 0 & \text{if } u > j, \end{cases}$$

for each $j, u \in \mathbb{N}$.

Wang [**37**] used the Nörlund matrix to determine the sequence space $l_\infty \left(\mathcal{N}^f \right)$ as the domain of Nörlund mean $\mathcal{N}^f$-transform are in the space l_∞. Tuğ and Başar [**36**] worked the sequence spaces $c_0 \left(\mathcal{N}^f \right)$ and $c \left(\mathcal{N}^f \right)$ as the set of all sequences with $\mathcal{N}^f$ in the spaces c_0 and c, respectively. Also, Tuğ and Başar [**36**] put forward the sequence $\mathcal{N}_p^f (\varpi)$ to indicate the $\mathcal{N}^f$-transform of the sequence $\varpi = (\varpi_u) \in w$, where the sequence $\mathcal{N}_p^f (\varpi)$ is given by

$$\mathcal{N}_p^f (\varpi) := \frac{1}{T_p} \sum_{u=0}^{p} f_{p-u} \varpi_u$$

for all $p \in \mathbb{N}$.

Khan et al. [**14**] recently investigated the space of Nörlund $\mathcal{I}$-convergent sequences using the concepts of domain of Nörlund matrix $\mathcal{N}^f$ and $\mathcal{I}$-convergence. The intuitionistic fuzzy Nörlund $\mathcal{I}$-convergent sequence spaces were studied by Khan and Khan [**15**].

In this study, we offer a novel notion of convergence for double sequences in IFNS utilizing the lacunary sequence, ideal convergence, and Nörlund convergent space. We specifically want to explore the intuitionistic Nörlund $\mathcal{I}_2$-lacunary statistically convergent sequence space.

Now, we recall the following basic concepts. They will be needed in the course of the paper.

Menger [**20**] explored triangular norms, or t-norms. Menger proposed using probability distributions rather than using numbers to represent distance in the problem of calculating the separation between two components in space. In metric space circumstances, T-norms are used to generalize with the probability distribution of triangle inequality. Triangular conorms (t-conorms) are also known as dual t-norm operations.

Lael and Nourouzi [**19**] introduced the idea of IFNS.

Definition 1.1. The five-tuple $(X, \Phi, \Psi, \triangle, \diamondsuit)$ is named to be an IF-normed space when X is a real vector space, and Φ, Ψ are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions for all $\varpi, q \in X$ and $t, s > 0$:

(a) $\Phi(\varpi, t) + \Psi(\varpi, t) \leq 1$,
(b) $\Phi(\varpi, t) > 0$,
(c) $\Phi(\varpi, t) = 1$ for all $t \in \mathbb{R}^+$ iff $\varpi = 0$,
(d) $\Phi(c\varpi, t) = \Phi\left(\varpi, \frac{t}{|c|}\right)$ for all $t \in \mathbb{R}^+$ and $c \neq 0$,
(e) $\Phi(\varpi, t) \triangle \Phi(q, s) \leq \Phi(\varpi + q, t + s)$,
(f) $\Phi(\varpi, t) : (0, \infty) \to [0, 1]$ is continuous in t,
(g) $\lim_{t\to\infty} \Phi(\varpi, t) = 1$ and $\lim_{t\to 0} \Phi(\varpi, t) = 0$,
(h) $\Psi(\varpi, t) > 0$,
(ı) $\Psi(\varpi, t) = 0$ iff $\varpi = 0$,
(i) $\Psi(c\varpi, t) = \Psi\left(\varpi, \frac{t}{|c|}\right)$ for all $t \in \mathbb{R}^+$ and $c \neq 0$,
(j) $\Psi(\varpi, t) \diamondsuit \Psi(q, s) \geq \Psi(\varpi + q, t + s)$,
(k) $\Psi(\varpi, t) : (0, \infty) \to [0, 1]$ is continuous in t,
(l) $\lim_{t\to\infty} \Psi(\varpi, t) = 0$ and $\lim_{t\to 0} \Psi(\varpi, t) = 1$.

In this case, we call (Φ, Ψ) an IF-norm on X.

Utilizing the notion of ideals, Kostyrko et al. [**16**] determined the notion of $\mathcal{I}$ and $\mathcal{I}^*$-convergence.

Let $Y \neq \emptyset$. $\mathcal{I} \subset 2^Y$ is called an ideal on Y provided that (a) for each $U, V \in \mathcal{I}$ implies $U \cup V \in \mathcal{I}$; (b) for each $U \in \mathcal{I}$ and $V \subset P$ implies $V \in \mathcal{I}$.

Let $Y \neq \emptyset$. $\mathcal{F} \subset 2^Y$ is named a filter on Y provided that (a) for all $U, V \in \mathcal{F}$ implies $U \cap V \in \mathcal{F}$; (b) for all $U \in \mathcal{F}$ and $V \supset P$ implies $V \in \mathcal{F}$.

An ideal $\mathcal{I}$ is known as non-trivial provided that $Y \notin \mathcal{I}$ and $\mathcal{I} \neq \emptyset$. A non-trivial ideal $\mathcal{I} \subset P(Y)$ is known as an admissible ideal in Y iff $\mathcal{I} \supset \{\{w\} : w \in Y\}$. Afterwards, the filter $F = F(\mathcal{I}) = \{Y - S : S \in \mathcal{I}\}$ is named the filter connected with the ideal.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is named strongly admissible when $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$.

Throughout the work, we consider $\mathcal{I}_2$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Definition 1.2. A double sequence $\theta_2 = \theta_{ps} = \{(k_p, l_s)\}$ is named double lacunary sequence when there are two increasing sequences of integers (k_p) and (l_s)

such that

$$k_0 = 0,\ h_p = k_p - k_{p-1} \to \infty \ \text{ and } \ l_0 = 0,\ h_s = l_s - l_{s-1} \to \infty, \quad p, s \to \infty.$$

We will utilize the subsequent notation $k_{ps} := k_p l_s$, $h_{ps} := h_p h_s$, and θ_{ps} is identified by

$$I_{ps} := \{ (k,l) : k_{p-1} < k \leq k_p \text{ and } l_{s-1} < l \leq l_s \},$$
$$q_p := \frac{k_p}{k_{p-1}}, \quad q_s := \frac{l_s}{l_{s-1}}, \quad \text{and} \quad q_{ps} := q_p q_s.$$

Throughout the paper, by $\theta_2 = \theta_{ps} = \{(k_p, l_s)\}$ we will indicate a double lacunary sequence.

Definition 1.3. Presume (Y, ρ) is a metric space. A double sequence $\varpi = (\varpi_{uv})$ is named to be $\mathcal{I}_2$-convergent to α, provided that for any $\gamma > 0$, we acquire

$$A(\gamma) = \{(u,v) \in \mathbb{N} \times \mathbb{N} : \rho(w_{uv}, \alpha) \geq \gamma\} \in \mathcal{I}_2.$$

We write $\mathcal{I}_2\text{-}\lim_{u,v\to\infty} \varpi_{uv} = \alpha$.

Definition 1.4. A double sequence $\varpi = (\varpi_{uv})$ is named to be $\mathcal{I}_2$-lacunary statistical convergent or $S_{\theta_2}(\mathcal{I}_2)$-convergent to α, provided that for all μ, $\gamma > 0$,

$$\left\{ (p,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{ps}} \left| \{(u,v) \in I_{ps} : |\varpi_{uv} - \alpha| \geq \mu\} \right| \geq \gamma \right\} \in \mathcal{I}_2.$$

In that case, we denote $\varpi_{uv} \to \alpha \, (S_{\theta_2}(\mathcal{I}_2))$ or $S_{\theta_2}(\mathcal{I}_2)\text{-}\lim\limits_{u,v\to\infty} \varpi_{uv} = \alpha$.

Moore [**22**] established the double series certain theorems concerning relationships between Nörlund means for single series given by Nörlund [**29**].

Let $f = (f_{uv})_{u,v\in\mathbb{N}}$ be a double sequence of nonnegative numbers with $f_{00} > 0$. Its partial sum is determined as

$$T_{ps} = \sum_{u,v=0}^{p,s} f_{uv} \text{ for each } p, s \in \mathbb{N}.$$

Take $\varpi = (\varpi_{uv}) \in \mathcal{M}_u$ as a double sequence of complex numbers. The Nörlund means $\mathcal{N}_{ps}^f(\varpi)$ are identified by

$$\mathcal{N}_{ps}^f(\varpi) := \frac{1}{T_{ps}} \sum_{u,v=0}^{p,s} f_{p-u,s-v} \varpi_{uv}. \tag{1}$$

A sequence $\varpi = (\varpi_{uv}) \in \mathcal{M}_u$ is said to be intuitionistic Nörlund $\mathcal{I}_2$-statistically convergent to $\alpha \in \mathbb{R}$ w.r.t. IFN (Φ, Ψ), provided that for all $\sigma, \gamma > 0$ and $\mu \in (0,1)$,

$$K_1 := \Big\{ (p,s) \in \mathbb{N}^2 : \frac{1}{ps} \big| \{ u \leq p, v \leq s : \Phi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \alpha, \sigma\right) \leq 1 - \mu \text{ or } \Psi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \alpha, \sigma\right) \geq \mu \} \big| \geq \gamma \Big\} \in \mathcal{I}_2.$$

Symbolically, we denote $\mathcal{N}_{ps}^f\text{-}\mathcal{I}_2\text{-}st\lim \varpi_{uv} = \alpha$ or $\varpi_{uv} \to \alpha(S[\mathcal{I}_2^{(\Phi,\Psi)}(\mathcal{N}_{ps}^f)])$.

Throughout the article, we assume that the sequences $\varpi = (\varpi_{uv}) \in \mathcal{M}_u$ and $\mathcal{N}^f_{ps}(\varpi_{uv})$ are connected as shown in (1), and $\mathcal{I}_2$ is an admissible ideal of a subset of $\mathbb{N}^2$.

2. Nörlund $\mathcal{I}_2$-lacunary statistical convergence

In this section, we present our findings. We begin with the following definitions which play a crucial role throughout the paper.

Definition 2.1. A sequence $\varpi = (\varpi_{uv}) \in \mathcal{M}_u$ is named to be Nörlund $\mathcal{I}_2$-lacunary statistical convergent to $\alpha \in \mathbb{R}$, provided that for all μ, $\gamma > 0$,

$$T_1 := \Big\{ (p,s) \in \mathbb{N}^2 : \frac{1}{h_{ps}} \big|\big\{(u,v) \in I_{ps} : \big|\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha\big| \geq \mu\big\}\big| \geq \gamma \Big\} \in \mathcal{I}_2.$$

Definition 2.2. A sequence $\varpi = (\varpi_{uv}) \in \mathcal{M}_u$ is named to be intuitionistic Nörlund $\mathcal{I}_2$-statistically convergent to $\alpha \in \mathbb{R}$ w.r.t. IFN (Φ, Ψ), provided that for all σ, $\gamma > 0$ and $\mu \in (0,1)$,

$$\begin{aligned} K_1 := \Big\{ (p,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{ps} \big|\big\{ u \leq p, v \leq s : \Phi\big(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\big) \leq 1 - \mu \\ \text{or } \Psi\big(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\big) \geq \mu \big\}\big| \geq \gamma \Big\} \in \mathcal{I}_2. \end{aligned}$$

Symbolically, we indicate $\mathcal{N}^f_{ps}$-$\mathcal{I}_2$-$st\lim \varpi_{uv} = \alpha$.

Definition 2.3. A sequence $\varpi = (\varpi_{uv}) \in \mathcal{M}_u$ is named to be intuitionistic Nörlund $\mathcal{I}_2$-lacunary statistical convergent to $\alpha \in \mathbb{R}$ w.r.t. IFN (Φ, Ψ), provided that for all σ, $\gamma > 0$ and $\mu \in (0,1)$,

$$\begin{aligned} K_1 := \Big\{ (p,s) \in \mathbb{N}^2 : \frac{1}{h_{ps}} \big|\big\{ (u,v) \in I_{ps} : \Phi\big(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\big) \leq 1 - \mu \\ \text{or } \Psi\big(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\big) \geq \mu \big\}\big| \geq \gamma \Big\} \in \mathcal{I}_2. \end{aligned}$$

In this case, we denote $\mathcal{N}^f_{ps}$-$\mathcal{I}_{\theta_2}$-$st\lim \varpi_{uv} = \alpha$ or $\varpi_{uv} \to \alpha(S_{\theta_2}[\mathcal{I}_2^{(\Phi,\Psi)}(\mathcal{N}^f_{ps})])$. The set of all Nörlund $\mathcal{I}_2$-lacunary statistical convergent sequences in IFNS is denoted by $S_{\theta_2}[\mathcal{I}_2^{(\Phi,\Psi)}(\mathcal{N}^f_{ps})]$.

Definition 2.4. A sequence $\varpi = (\varpi_{uv}) \in \mathcal{M}_u$ is named to be intuitionistic Nörlund strongly $\mathcal{I}_2$-lacunary summable to $\alpha \in \mathbb{R}$ or $N_{\theta_2}[\mathcal{I}_2^{(\Phi,\Psi)}(\mathcal{N}^f_{ps})]$-summable to $\alpha \in \mathbb{R}$ w.r.t. IFN (Φ, Ψ), provided that for all $\sigma > 0$ and $\mu \in (0,1)$,

$$\begin{aligned} K_1 := \Big\{ (p,s) \in \mathbb{N}^2 : \frac{1}{h_{ps}} \sum_{(u,v) \in I_{ps}} \Phi\big(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\big) \leq 1 - \mu \\ \text{or } \frac{1}{h_{ps}} \sum_{(u,v) \in I_{ps}} \Psi\big(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\big) \geq \mu \Big\} \in \mathcal{I}_2. \end{aligned}$$

In this case, we denote $\varpi_{uv} \overset{N_{\theta_2}[\mathcal{I}_2^{(\Phi,\Psi)}(\mathcal{N}^f_{ps})]}{\to} \alpha$ or $\varpi_{uv} \to \alpha(N_{\theta_2}[\mathcal{I}_2^{(\Phi,\Psi)}(\mathcal{N}^f_{ps})])$.

Definition 2.5. A sequence $\varpi = (\varpi_{uv}) \in \mathcal{M}_u$ is named to be intuitionistic Nörlund strongly $\mathcal{I}_2$-lacunary Cauchy w.r.t. IFN (Φ, Ψ), provided that for all $\sigma > 0$ and $\eta \in (0,1)$, there are $r, t \in \mathbb{N}$ such that the set K_2 belongs to $\mathcal{I}_2$, where

$$K_2 := \Big\{ (p,s) \in \mathbb{N}^2 : \frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv}) - \mathcal{N}_{ps}^{f}(\varpi_{rt}), \sigma\right) \leq 1-\mu$$
$$\text{or } \frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Psi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv}) - \mathcal{N}_{ps}^{f}(\varpi_{rt}), \sigma\right) \geq \mu \Big\} \in \mathcal{I}_2.$$

Theorem 2.6. *Take θ_2 as a double lacunary sequence. At that case,*

$$\varpi_{uv} \to \alpha\left(N_{\theta_2}\left[\mathcal{I}_2^{(\Phi,\Psi)}\left(\mathcal{N}_{ps}^{f}\right)\right]\right) \Longrightarrow \varpi_{uv} \to \alpha\left(S_{\theta_2}\left[\mathcal{I}_2^{(\Phi,\Psi)}\left(\mathcal{N}_{ps}^{f}\right)\right]\right).$$

Proof. Assume $\varpi_{uv} \to \alpha(N_{\theta_2}[\mathcal{I}_2^{(\Phi,\Psi)}(\mathcal{N}_{ps}^{f})])$. Afterwards, for all $\sigma > 0$ and $\mu \in (0,1)$, we get

$$\sum_{(u,v)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv}) - \alpha, \sigma\right) \text{ or } \Psi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv}) - \alpha, \sigma\right)$$
$$\geq \sum_{\substack{(u,v)\in I_{ps} \\ \Phi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv})-\alpha,\sigma\right)\leq 1-\mu \text{ or} \\ \Psi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv})-\alpha,\sigma\right)\geq \mu}} \Phi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv}) - \alpha, \sigma\right) \text{ or } \Psi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv}) - \alpha, \sigma\right)$$
$$\geq \mu \cdot \left|\Big\{(u,v) \in I_{ps} : \Phi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv}) - \alpha, \sigma\right) \leq 1-\mu \text{ or } \Psi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv}) - \alpha, \sigma\right) \geq \mu\Big\}\right|,$$

and so,

$$\frac{1}{\mu h_{ps}} \sum_{(u,v)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv}) - \alpha, \sigma\right) \text{ or } \Psi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv}) - \alpha, \sigma\right)$$
$$\geq \frac{1}{h_{ps}} \left|\Big\{(u,v) \in I_{ps} : \Phi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv}) - \alpha, \sigma\right) \leq 1-\mu \text{ or } \Psi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv}) - \alpha, \sigma\right) \geq \mu\Big\}\right|.$$

Then, for any σ, $\gamma > 0$ and $\mu \in (0,1)$,

$$\Big\{ (p,s) \in \mathbb{N}^2 : \frac{1}{h_{ps}} \big|\{ (u,v) \in I_{ps} : \Phi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv}) - \alpha, \sigma\right) \leq 1-\mu$$
$$\text{or } \Psi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv}) - \alpha, \sigma\right) \geq \mu\}\big| \geq \gamma \Big\}$$
$$\subseteq \Big\{ (p,s) \in \mathbb{N}^2 : \frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv}) - \alpha, \sigma\right) \leq (1-\mu)\cdot\gamma$$
$$\text{or } \frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Psi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv}) - \alpha, \sigma\right) \geq \mu.\gamma \Big\} \in \mathcal{I}_2.$$

As a result, we acquire $\varpi_{uv} \to \alpha(S_{\theta_2}[\mathcal{I}_2^{(\Phi,\Psi)}(\mathcal{N}_{ps}^{f})])$. □

Theorem 2.7. *Presume θ_2 is a lacunary sequence. Then (ϖ_{uv}) is bounded $((\varpi_{uv}) \in \mathcal{M}_u)$ and*

$$\varpi_{uv} \to \alpha \left(S_{\theta_2}\left[\mathcal{I}_2^{(\Phi,\Psi)}\left(\mathcal{N}_{ps}^f\right)\right]\right) \Longrightarrow \varpi_{uv} \to \alpha\left(N_{\theta_2}\left[\mathcal{I}_2^{(\Phi,\Psi)}\left(\mathcal{N}_{ps}^f\right)\right]\right).$$

Proof. Suppose that $\varpi_{uv} \to \alpha(S_{\theta_2}[\mathcal{I}_2^{(\Phi,\Psi)}(\mathcal{N}_{ps}^f)])$ and $(\varpi_{uv}) \in \mathcal{M}_u$. So, there is an $U > 0$ such that

$$\Phi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \alpha, \sigma\right) \geq 1 - U \quad \text{or} \quad \Psi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \alpha, \sigma\right) \leq U$$

for all $\sigma > 0$. Given $\mu \in (0,1)$, we have

$$\begin{aligned}
&\frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \alpha, \sigma\right) \text{ or } \Psi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \alpha, \sigma\right) \\
&= \frac{1}{h_{ps}} \sum_{\substack{(u,v)\in I_{ps} \\ \Phi\left(\mathcal{N}_{ps}^f(\varpi_{uv})-\alpha,\sigma\right)\leq 1-\frac{\mu}{2} \text{ or} \\ \Psi\left(\mathcal{N}_{ps}^f(\varpi_{uv})-\alpha,\sigma\right)\geq \frac{\mu}{2}}} \Phi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \alpha, \sigma\right) \text{ or } \Psi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \alpha, \sigma\right) \\
&\quad + \frac{1}{h_{ps}} \sum_{\substack{(u,v)\in I_{ps} \\ \Phi\left(\mathcal{N}_{ps}^f(\varpi_{uv})-\alpha,\sigma\right)> 1-\frac{\mu}{2} \text{ or} \\ \Psi\left(\mathcal{N}_{ps}^f(\varpi_{uv})-\alpha,\sigma\right)< \frac{\mu}{2}}} \Phi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \alpha, \sigma\right) \text{ or } \Psi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \alpha, \sigma\right) \\
&\leq \frac{U}{h_{ps}}\Big|\Big\{(u,v) \in I_{ps} : \Phi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \alpha, \sigma\right) \leq 1 - \frac{\mu}{2} \\
&\qquad \text{or } \Psi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \alpha, \sigma\right) \geq \frac{\mu}{2}\Big\}\Big| + \frac{\mu}{2}.
\end{aligned}$$

As a result, for all $\sigma > 0$ and $\mu \in (0,1)$, we get

$$\begin{aligned}
&\Big\{(p,s) \in \mathbb{N}^2 : \frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \alpha, \sigma\right) \leq 1 - \mu \\
&\qquad \text{or } \frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Psi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \alpha, \sigma\right) \geq \mu\Big\} \\
&\subseteq \Big\{(p,s) \in \mathbb{N}^2 : \frac{1}{h_{ps}}\big|\big\{(u,v) \in I_{ps} : \Phi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \alpha, \sigma\right) \leq 1 - \frac{\mu}{2} \\
&\qquad \text{or } \Psi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \alpha, \sigma\right) \geq \frac{\mu}{2}\big\}\big| \geq \frac{\mu}{2U}\Big\} \in \mathcal{I}_2.
\end{aligned}$$

We obtain $\varpi_{uv} \to \alpha(N_{\theta_2}[\mathcal{I}_2^{(\Phi,\Psi)}(\mathcal{N}_{ps}^f)])$. □

Theorem 2.8. *If $\liminf_p q_p > 1$ and $\liminf_s q_s > 1$, then $\mathcal{N}_{ps}^f$-$\mathcal{I}_2$-$st \lim \varpi_{uv} = \alpha$ gives $\mathcal{N}_{ps}^f$-$\mathcal{I}_{\theta_2}$-$st \lim \varpi_{uv} = \alpha$.*

Proof. Presume that $\liminf_p q_p > 1$ and $\liminf_s q_s > 1$. Then, there are $\rho > 0$, $\tau > 0$ such that

$$q_p \geq 1 + \rho \quad \text{and} \quad q_s \geq 1 + \tau$$

for sufficiently large p, s, which yields

$$\frac{h_{ps}}{j_p k_s} \geq \frac{\rho\tau}{(1+\rho)(1+\tau)}.$$

Suppose that $\mathcal{N}^f_{ps}$-$\mathcal{I}_2$-$st\lim \varpi_{uv} = \alpha$. For all $\mu \in (0,1)$ and all $\sigma > 0$, we get

$$\begin{aligned}
&\frac{1}{j_p k_s}\Big|\Big\{(u,v) : u \leq j_p, v \leq k_s,\ \Phi\left(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\right) \leq 1-\mu \\
&\qquad\qquad \text{or } \Psi\left(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\right) \geq \mu\Big\}\Big| \\
&\geq \frac{1}{j_p k_s}\Big|\Big\{(u,v) \in I_{ps} : \Phi\left(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\right) \leq 1-\mu \\
&\qquad\qquad \text{or } \Psi\left(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\right) \geq \mu\Big\}\Big| \\
&= \frac{h_{ps}}{j_p k_s}\frac{1}{h_{ps}}\Big|\Big\{(u,v) \in I_{ps} : \Phi\left(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\right) \leq 1-\mu \\
&\qquad\qquad \text{or } \Psi\left(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\right) \geq \mu\Big\}\Big| \\
&\geq \frac{\rho\tau}{(1+\rho)(1+\tau)}\frac{1}{h_{ps}}\Big|\Big\{(u,v) \in I_{ps} : \Phi\left(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\right) \leq 1-\mu \\
&\qquad\qquad \text{or } \Psi\left(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\right) \geq \mu\Big\}\Big|.
\end{aligned}$$

So, for any $\gamma > 0$,

$$\begin{aligned}
&\Big\{(p,s) \in \mathbb{N}^2 : \frac{1}{h_{ps}}\Big|\Big\{(u,v) \in I_{ps} : \Phi\left(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\right) \leq 1-\mu \\
&\qquad\qquad \text{or } \Psi\left(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\right) \geq \mu\Big\}\Big| \geq \gamma\Big\} \\
&\subseteq \Big\{(p,s) \in \mathbb{N}^2 : \frac{1}{j_p k_s}\Big|\Big\{(u,v) : u \leq j_p, v \leq k_s, \Phi\left(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\right) \leq 1-\mu \\
&\qquad\qquad \text{or } \Psi\left(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\right) \geq \mu\Big\}\Big| \geq \frac{\rho\tau\gamma}{(1+\rho)(1+\tau)}\Big\}.
\end{aligned}$$

So, according to our assumption, the set on the right side belongs to $\mathcal{I}_2$, clearly the set on the left side belongs to $\mathcal{I}_2$. As a result, we get $\mathcal{N}^f_{ps}$-$\mathcal{I}_{\theta_2}$-$st\lim \varpi_{uv} = \alpha$. □

Theorem 2.9. *If* $\limsup_p q_p < \infty$ *and* $\limsup_s q_s < \infty$, *then* $\mathcal{N}^f_{ps}$-$\mathcal{I}_{\theta_2}$-$st\lim \varpi_{uv} = \alpha$ *gives* $\mathcal{N}^f_{ps}$-$\mathcal{I}_2$-$st\lim \varpi_{uv} = \alpha$.

Proof. Presume that $\limsup_p q_p < \infty$ and $\limsup_s q_s < \infty$. At that case, there are U, $V > 0$ such that $q_p < U$ and $q_s < V$ for all p, s. Let $\mathcal{N}^f_{ps}$-$\mathcal{I}_{\theta_2}$-$st\lim \varpi_{uv} = \alpha$ and assume

$$\begin{aligned}
H_{ps} := &\Big|\Big\{(u,v) \in I_{ps} : \Phi\left(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\right) \leq 1-\mu \\
&\qquad\qquad \text{or } \Psi\left(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\right) \geq \mu\Big\}\Big|.
\end{aligned}$$

Since $\mathcal{N}^f_{ps}$-$\mathcal{I}_{\theta_2}$-$st\lim \varpi_{uv} = \alpha$, it supplies for all $\mu \in (0,1)$ and all $\sigma, \gamma > 0$,

$$\Big\{ (p,s) \in \mathbb{N}^2 : \frac{1}{h_{ps}} \Big| \Big\{ (u,v) \in I_{ps} : \Phi\left(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\right) \leq 1-\mu \text{ or } \Psi\left(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\right) \geq \mu \Big\} \Big| \geq \gamma \Big\}$$
$$= \Big\{ (p,s) \in \mathbb{N}^2 : \frac{H_{ps}}{h_{ps}} \geq \gamma \Big\} \in \mathcal{I}_2.$$

Hence, we can take positive integers p_0, $s_0 \in \mathbb{N}$ such that $\frac{H_{ps}}{h_{ps}} < \gamma$ for all $p \geq p_0$, $s \geq s_0$. Now, assume

$$W := \max\{H_{ps} : 1 \leq p \leq p_0,\ 1 \leq s \leq s_0\}$$

and k and l are integers satisfying $j_{p-1} < k \leq j_p$ and $k_{s-1} < l \leq k_s$. Then, for all $\mu \in (0,1)$ and all $\sigma > 0$, we get

$$\begin{aligned}
&\frac{1}{kl}\Big| \Big\{ (u,v) : u \leq k, v \leq l,\ \Phi\left(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\right) \leq 1-\mu \text{ or } \Psi\left(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\right) \geq \mu \Big\} \Big| \\
&\leq \frac{1}{j_{p-1}k_{s-1}}\Big| \Big\{ (u,v) : u \leq j_p, v \leq k_s,\ \Phi\left(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\right) \leq 1-\mu \text{ or } \Psi\left(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\right) \geq \mu \Big\} \Big| \\
&= \frac{1}{j_{p-1}k_{s-1}} \{H_{11} + H_{12} + H_{21} + H_{22} + \cdots + H_{p_0 s_0} + \cdots + H_{ps}\} \\
&\leq \frac{p_0 s_0}{j_{p-1}k_{s-1}} \left(\max_{\substack{1 \leq u \leq p_0 \\ 1 \leq v \leq s_0}} \{H_{uv}\} \right) + \frac{1}{j_{p-1}k_{s-1}} \Big\{ h_{p_0(s_0+1)} \frac{H_{p_0(s_0+1)}}{h_{p_0(s_0+1)}} \\
&\quad + h_{(p_0+1)s_0} \frac{H_{(p_0+1)s_0}}{h_{(p_0+1)s_0}} + h_{(p_0+1)(s_0+1)} \frac{H_{(p_0+1)(s_0+1)}}{h_{(p_0+1)(s_0+1)}} + \cdots + h_{ps} \frac{H_{ps}}{h_{ps}} \Big\} \\
&\leq \frac{p_0 s_0 W}{j_{p-1}k_{s-1}} + \frac{1}{j_{p-1}k_{s-1}} \left(\sup_{\substack{p > p_0 \\ s > s_0}} \frac{H_{ps}}{h_{ps}} \right) \left(\sum_{\substack{u \geq p_0 \\ v \geq s_0}} h_{uv} \right) \\
&\leq \frac{p_0 s_0 W}{j_{p-1}k_{s-1}} + \gamma \frac{(j_p - j_{p_0})(k_s - k_{s_0})}{j_{p-1}k_{s-1}} \leq \frac{p_0 s_0 W}{j_{p-1}k_{s-1}} + \gamma q_p q_s \\
&\leq \frac{p_0 s_0 W}{j_{p-1}k_{s-1}} + \gamma U V.
\end{aligned}$$

Since $j_{p-1}k_{s-1} \to \infty$ as $k, l \to \infty$, it gives that for all $\sigma > 0$ and $\eta \in (0,1)$,

$$\frac{1}{kl}\Big| \Big\{ (u,v) : u \leq k, v \leq l,\ \Phi\left(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\right) \leq 1-\mu \text{ or } \Psi\left(\mathcal{N}^f_{ps}(\varpi_{uv}) - \alpha, \sigma\right) \geq \mu \Big\} \Big| \to 0,$$

and so for any $\gamma_1 > 0$, the set

$$\Big\{ (k,l) \in \mathbb{N}\times\mathbb{N} : \frac{1}{kl}\Big|\Big\{ (u,v) : u \leq k, v \leq l,\ \Phi\left(\mathcal{N}^f_{ps}\left(\varpi_{uv}\right) - \alpha, \sigma\right) \leq 1-\mu \\ \text{or } \Psi\left(\mathcal{N}^f_{ps}\left(\varpi_{uv}\right) - \alpha, \sigma\right) \geq \mu\Big\}\Big| \geq \gamma_1 \Big\} \in \mathcal{I}_2.$$

It yields that $\mathcal{N}^f_{ps}$-$\mathcal{I}_2$-$st\lim \varpi_{uv} = \alpha$. □

Theorem 2.10. *Take θ_2 as a lacunary sequence. If*

$$1 < \liminf_p q_p < \limsup_p q_p < \infty \quad \text{and} \quad 1 < \liminf_s q_s < \limsup_s q_s < \infty,$$

then $\mathcal{N}^f_{ps}$-$\mathcal{I}_{\theta_2}$-$st\lim \varpi_{uv} = \alpha$ iff $\mathcal{N}^f_{ps}$-$\mathcal{I}_2$-$st\lim \varpi_{uv} = \alpha$.

Proof. It is clear from Theorem 2.8 and Theorem 2.9. □

Theorem 2.11. *Assume $\mathcal{I}_2$ is a strongly admissible ideal satisfying property (AP_2), $\theta_2 \in \mathcal{F}(\mathcal{I}_2)$. If*

$$(\varpi_{uv}) \in S\left[\mathcal{I}_2^{(\Phi,\Psi)}\left(\mathcal{N}^f_{ps}\right)\right] \cap S_{\theta_2}\left[\mathcal{I}_2^{(\Phi,\Psi)}\left(\mathcal{N}^f_{ps}\right)\right],$$

then

$$S\left[\mathcal{I}_2^{(\Phi,\Psi)}\left(\mathcal{N}^f_{ps}\right)\right] - \lim_{u,v\to\infty} \varpi_{uv} = S_{\theta_2}\left[\mathcal{I}_2^{(\Phi,\Psi)}\left(\mathcal{N}^f_{ps}\right)\right] - \lim_{u,v\to\infty} \varpi_{uv}.$$

Proof. Assume that

$$S\left[\mathcal{I}_2^{(\Phi,\Psi)}\left(\mathcal{N}^f_{ps}\right)\right] - \lim_{u,v\to\infty} \varpi_{uv} = \alpha_1$$

and

$$S_{\theta}\left[\mathcal{I}_2^{(\Phi,\Psi)}\left(\mathcal{N}^f_{ps}\right)\right] - \lim_{u,v\to\infty} \varpi_{uv} = \alpha_2,$$

and $\alpha_1 \neq \alpha_2$. Take

$$0 < \mu < \frac{1}{2}\Phi\left(\mathcal{N}^f_{ps}\left(\varpi_{uv}\right) - \alpha, \sigma\right) \\ \text{and} \quad 0 < \mu < \frac{1}{2}\Psi\left(\mathcal{N}^f_{ps}\left(\varpi_{uv}\right) - \alpha, \sigma\right)$$

for all $\sigma > 0$. As $\mathcal{I}_2$ satisfies the feature (AP_2), then there is $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $\left(\mathbb{N}^2\right) \setminus M \in \mathcal{I}_2$) such that for all $\sigma > 0$, $\mu \in (0,1)$ and $(k,l) \in M$,

$$\lim_{k,l\to\infty} \frac{1}{kl}\Big|\Big\{u \leq k, v \leq l : \Phi\left(\mathcal{N}^f_{ps}\left(\varpi_{uv}\right) - \alpha_1, \sigma\right) \leq 1-\mu \\ \text{or } \Psi\left(\mathcal{N}^f_{ps}\left(\varpi_{uv}\right) - \alpha_1, \sigma\right) \geq \mu\Big\}\Big| = 0.$$

Take

$$T = \Big\{u \leq k, v \leq l : \Phi\left(\mathcal{N}^f_{ps}\left(\varpi_{uv}\right) - \alpha_1, \sigma\right) \leq 1-\mu \\ \text{or } \Psi\left(\mathcal{N}^f_{ps}\left(\varpi_{uv}\right) - \alpha_1, \sigma\right) \geq \mu\Big\}$$

and

$$S = \Big\{u \leq k, v \leq l : \Phi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{uv}\right) - \alpha_2, \sigma\right) \leq 1 - \mu \\ \text{or } \Psi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{uv}\right) - \alpha_2, \sigma\right) \geq \mu\Big\}.$$

At that case, $kl = |T \cup S| \leq |T| + |S|$. This yields that $1 \leq \frac{|T|}{kl} + \frac{|S|}{kl}$. Since $\frac{|S|}{kl} \leq 1$ and $\lim_{k,l \to \infty} \frac{|T|}{kl} = 0$, so we have to get $\lim_{k,l \to \infty} \frac{|S|}{kl} = 1$.

Let $M^* = M \cap \theta_2 \in \mathcal{F}(\mathcal{I}_2)$. Afterwards, for all $\sigma > 0$, $\mu \in (0,1)$ and $(u_r, v_j) \in M^*$, the $u_r v_j$th term of the statistical limit expression

$$\frac{1}{kl}\left|\left\{u \leq k, v \leq l : \Phi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{uv}\right) - \alpha_2, \sigma\right) \leq 1 - \mu \text{ or } \Psi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{uv}\right) - \alpha_2, \sigma\right) \geq \mu\right\}\right|$$

is

$$\begin{aligned} & \frac{1}{u_r v_j}\Big|\Big\{ (u,v) \in \bigcup_{p,s=1,1}^{r,j} I_{ps} : \Phi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{uv}\right) - \alpha_2, \sigma\right) \leq 1 - \mu \\ & \qquad \text{or } \Psi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{uv}\right) - \alpha_2, \sigma\right) \geq \mu\Big\}\Big| \\ & = \frac{1}{\bigcup_{p,s=1,1}^{r,j} h_{ps}} \bigcup_{p,s=1,1}^{r,j} t_{ps} h_{ps}, \end{aligned} \tag{2}$$

where

$$t_{ps} = \frac{1}{h_{ps}}\Big|\Big\{ (u,v) \in I_{ps} : \Phi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{uv}\right) - \alpha_2, \sigma\right) \leq 1 - \mu \\ \text{or } \Psi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{uv}\right) - \alpha_2, \sigma\right) \geq \mu\Big\}\Big| \xrightarrow{\mathcal{I}_2} 0,$$

because $\mathcal{N}_{ps}^{f}$-$\mathcal{I}_{\theta_2}$-$st \lim \varpi_{uv} = \alpha_2$. Since θ_2 is a lacunary sequence, (2) is a regular weighted mean transform of t_{ps}'s, and as a result it is $\mathcal{I}_2$-convergent to 0 as $r, j \to \infty$, and also it has a subsequence which is convergent to 0 since $\mathcal{I}_2$ holds the feature (AP_2). However, since this is a subsequence of

$$\left\{\frac{1}{kl}\Big|\Big\{u \leq k, v \leq l : \Phi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{uv}\right) - \alpha_2, \sigma\right) \leq 1 - \mu \\ \text{or } \Psi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{uv}\right) - \alpha_2, \sigma\right) \geq \mu\Big\}\Big|\right\}_{(k,l) \in M},$$

we obtain that

$$\left\{\frac{1}{kl}\Big|\Big\{u \leq k, v \leq l : \Phi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{uv}\right) - \alpha_2, \sigma\right) \leq 1 - \mu \\ \text{or } \Psi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{uv}\right) - \alpha_2, \sigma\right) \geq \mu\Big\}\Big|\right\}_{(k,l) \in M}$$

is not convergent to 1, a contradiction. Hence, we cannot get $\alpha_1 \neq \alpha_2$. □

Theorem 2.12. *All intuitionistic Nörlund strongly $\mathcal{I}_2$-lacunary convergent sequences are intuitionistic Nörlund strongly $\mathcal{I}_2$-lacunary Cauchy w.r.t. IFN (Φ, Ψ).*

Proof. Assume $\varpi_{uv} \to \alpha(N_{\theta_2}[\mathcal{I}_2^{(\Phi,\Psi)}(\mathcal{N}_{ps}^f)])$. At this case, for all $\mu \in (0,1)$ and each $\sigma > 0$,

$$K_1 = \Bigg\{ (p,s) \in \mathbb{N}^2 : \frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \alpha, \sigma\right) \leq 1-\mu \\ \text{or } \frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Psi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \alpha, \sigma\right) \geq \mu \Bigg\} \in \mathcal{I}_2.$$

As $\mathcal{I}_2$ is a strongly admissible ideal, we get

$$K_1^c = \Bigg\{ (p,s) \in \mathbb{N}^2 : \frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \alpha, \sigma\right) > 1-\mu \\ \text{and } \frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Psi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \alpha, \sigma\right) < \mu \Bigg\} \in \mathcal{F}(\mathcal{I}_2).$$

Hence, we can choose the non-negative integers p, s such that $(p,s) \notin K_1$. Therefore, we obtain

$$\frac{1}{h_{ps}} \sum_{(u_0,v_0)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^f(\varpi_{u_0v_0}) - \alpha, \sigma\right) > 1-\mu \\ \text{and } \frac{1}{h_{ps}} \sum_{(u_0,v_0)\in I_{ps}} \Psi\left(\mathcal{N}_{ps}^f(\varpi_{u_0v_0}) - \alpha, \sigma\right) < \mu.$$

Now, let

$$K_2 = \Bigg\{ (p,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{ps}} \sum_{(u,v),(u_0,v_0)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \mathcal{N}_{ps}^f(\varpi_{u_0v_0}), \sigma\right) \leq 1-2\mu \\ \text{or } \frac{1}{h_{ps}} \sum_{(u,v),(u_0,v_0)\in I_{ps}} \Psi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \mathcal{N}_{ps}^f(\varpi_{u_0v_0}), \sigma\right) \geq 2\mu \Bigg\}.$$

Contemplate the inequality

$$\frac{1}{h_{ps}} \sum_{(u,v),(u_0,v_0)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \mathcal{N}_{ps}^f(\varpi_{u_0v_0}), \sigma\right) \\ \leq \frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \alpha, \sigma\right) + \frac{1}{h_{ps}} \sum_{(u_0,v_0)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^f(\varpi_{u_0v_0}) - \alpha, \sigma\right)$$

and

$$\frac{1}{h_{ps}} \sum_{(u,v),(u_0,v_0)\in I_{ps}} \Psi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \mathcal{N}_{ps}^f(\varpi_{u_0v_0}), \sigma\right) \\ \leq \frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Psi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \alpha, \sigma\right) + \frac{1}{h_{ps}} \sum_{(u_0,v_0)\in I_{ps}} \Psi\left(\mathcal{N}_{ps}^f(\varpi_{u_0v_0}) - \alpha, \sigma\right).$$

Observe that when $(p,s) \in K_2$, so

$$\frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{uv}\right)-\alpha,\sigma\right)+\frac{1}{h_{ps}} \sum_{(u_0,v_0)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{u_0v_0}\right)-\alpha,\sigma\right) \leq 1-2\mu,$$

and

$$\frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Psi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{uv}\right)-\alpha,\sigma\right)+\frac{1}{h_{ps}} \sum_{(u_0,v_0)\in I_{ps}} \Psi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{u_0v_0}\right)-\alpha,\sigma\right) \geq 2\mu.$$

From another standpoint, as $(p,s) \notin K_1$, we have

$$\frac{1}{h_{ps}} \sum_{(u_0,v_0)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{u_0v_0}\right)-\alpha,\sigma\right) > 1-\mu$$
$$\text{and} \quad \frac{1}{h_{ps}} \sum_{(u_0,v_0)\in I_{ps}} \Psi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{u_0v_0}\right)-\alpha,\sigma\right) < \mu.$$

We acquire

$$\frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{uv}\right)-\alpha,\sigma\right) \leq 1-\mu$$
$$\text{or} \quad \frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Psi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{uv}\right)-\alpha,\sigma\right) \geq \mu.$$

Hence, $(p,s) \in K_1$. This yields that $K_2 \subset K_1 \in \mathcal{I}_2$ for all $\mu \in (0,1)$ and each $\sigma > 0$. So $K_2 \in \mathcal{I}_2$. Hence, the sequence strongly $\mathcal{I}_2$-lacunary Cauchy with regards to IFN (Φ,Ψ). □

Definition 2.13. The sequence (ϖ_{uv}) is intuitionistic Nörlund strongly $\mathcal{I}_2^*$-lacunary convergent to α iff there is a set $M = \left\{(u,v) \in \mathbb{N}^2\right\}$ such that

$$M' = \left\{(p,s) \in \mathbb{N}^2 : (u,v) \in I_{ps}\right\} \in \mathcal{F}\left(\mathcal{I}_2\right)$$

for all $\sigma > 0$,

$$\lim_{p,s\to\infty} \frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{uv}\right)-\alpha,\sigma\right) = 1,$$
$$\lim_{p,s\to\infty} \frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Psi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{uv}\right)-\alpha,\sigma\right) = 0.$$

Symbolically, we indicate $\varpi_{uv} \overset{(\Phi,\Psi)}{\to} \alpha\left(N_{\theta_2}\left[\mathcal{I}_2^*\left(\mathcal{N}_{ps}^{f}\right)\right]\right)$.

Theorem 2.14. *If*

$$\varpi_{uv} \overset{(\Phi,\Psi)}{\to} \alpha\left(N_{\theta_2}\left[\mathcal{I}_2^*\left(\mathcal{N}_{ps}^{f}\right)\right]\right),$$

then

$$\varpi_{uv} \to \alpha\left(N_{\theta_2}\left[\mathcal{I}_2^{(\Phi,\Psi)}\left(\mathcal{N}_{ps}^{f}\right)\right]\right).$$

Proof. Assume that $\varpi_{uv} \overset{(\Phi,\Psi)}{\to} \alpha\left(N_{\theta_2}\left[\mathcal{I}_2^*\left(\mathcal{N}_{ps}^f\right)\right]\right)$. At that case, there is a set $M = \left\{(u,v) \in \mathbb{N}^2\right\}$ such that

$$M' = \left\{(p,s) \in \mathbb{N}^2 : (u,v) \in I_{ps}\right\} \in \mathcal{F}\left(\mathcal{I}_2\right)$$

for all $\sigma > 0$,

$$\lim_{p,s\to\infty} \frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^f\left(\varpi_{uv}\right) - \alpha, \sigma\right) > 1 - \mu,$$

$$\lim_{p,s\to\infty} \frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Psi\left(\mathcal{N}_{ps}^f\left(\varpi_{uv}\right) - \alpha, \sigma\right) < \mu$$

for all $\mu \in (0,1)$ and all $u, v \geq t_0$. In the light of this fact, we get

$$\begin{aligned} T = \Bigg\{ & (p,s) \in \mathbb{N}^2 : \frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^f\left(\varpi_{uv}\right) - \alpha, \sigma\right) \leq 1 - \mu \\ & \text{or } \frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Psi\left(\mathcal{N}_{ps}^f\left(\varpi_{uv}\right) - \alpha, \sigma\right) \geq \mu \Bigg\} \\ & \subset H \cup \left(M' \cap \left(\left(\{1,2,\ldots,(t_0-1)\} \times \mathbb{N}\right) \cup \left(\mathbb{N} \times \{1,2,\ldots,(t_0-1)\}\right)\right)\right) \end{aligned}$$

for $\left(\mathbb{N}^2\right) \setminus M' = H \in \mathcal{I}_2$. As $\mathcal{I}_2$ is an admissible ideal, we get

$$H \cup \left(M' \cap \left(\left(\{1,2,\ldots,(t_0-1)\} \times \mathbb{N}\right) \cup \left(\mathbb{N} \times \{1,2,\ldots,(t_0-1)\}\right)\right)\right) \in \mathcal{I}_2,$$

and so $T \in \mathcal{I}_2$. As a result, $\varpi_{uv} \to \alpha(N_{\theta_2}[\mathcal{I}_2^{(\Phi,\Psi)}(\mathcal{N}_{ps}^f)])$. □

Theorem 2.15. *Suppose $\mathcal{I}_2$ is a strongly admissible ideal including feature (AP_2). Then,*

$$\varpi_{uv} \to \alpha\left(N_{\theta_2}\left[\mathcal{I}_2^{(\Phi,\Psi)}\left(\mathcal{N}_{ps}^f\right)\right]\right)$$

gives

$$\varpi_{uv} \overset{(\Phi,\Psi)}{\to} \alpha\left(N_{\theta_2}\left[\mathcal{I}_2^*\left(\mathcal{N}_{ps}^f\right)\right]\right).$$

Definition 2.16. The sequence $\{F_{wq}\}$ is said to be intuitionistic Nörlund strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence if for all $\mu \in (0,1)$ and all $\sigma > 0$, there is a set $M = \left\{(u,v) \in \mathbb{N}^2\right\}$ such that

$$M' = \left\{(p,s) \in \mathbb{N}^2 : (u,v) \in I_{ps}\right\} \in \mathcal{F}\left(\mathcal{I}_2\right)$$

and $T \in \mathbb{N}$ so that

$$\frac{1}{h_{ps}} \sum_{(u,v),(r,t)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^f\left(\varpi_{uv}\right) - \mathcal{N}_{ps}^f\left(\varpi_{rt}\right), \sigma\right) > 1 - \mu$$

$$\text{and} \quad \frac{1}{h_{ps}} \sum_{(u,v),(r,t)\in I_{ps}} \Psi\left(\mathcal{N}_{ps}^f\left(\varpi_{uv}\right) - \mathcal{N}_{ps}^f\left(\varpi_{rt}\right), \sigma\right) < \mu$$

for all $u, v, r, t \geq T$.

Theorem 2.17. *All intuitionistic Nörlund strongly $\mathcal{I}_2^*$-lacunary Cauchy sequences are intuitionistic Nörlund strongly $\mathcal{I}_2$-lacunary Cauchy in IFNS w.r.t.* (Φ, Ψ).

Proof. When the hypothesis is satisfied, then for all $\mu \in (0,1)$ and all $\sigma > 0$, there is a set $M = \{(u,v) \in \mathbb{N}^2\}$ such that

$$M' = \{(p,s) \in \mathbb{N}^2 : (u,v) \in I_{ps}\} \in \mathcal{F}(\mathcal{I}_2)$$

and $T \in \mathbb{N}$ such that

$$\frac{1}{h_{ps}} \sum_{(u,v),(r,t) \in I_{ps}} \Phi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \mathcal{N}_{ps}^f(\varpi_{rt}), \sigma\right) > 1 - \mu$$

$$\text{and} \quad \frac{1}{h_{ps}} \sum_{(u,v),(r,t) \in I_{ps}} \Psi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \mathcal{N}_{ps}^f(\varpi_{rt}), \sigma\right) < \mu$$

for all $u, v, r, t \geq T$. Take $H = (\mathbb{N}^2) \setminus M'$. It is obvious that $H \in \mathcal{I}_2$ and

$$\begin{aligned} P = \Bigg\{ (p,s) \in \mathbb{N}^2 : & \frac{1}{h_{ps}} \sum_{(u,v),(r,t) \in I_{ps}} \left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \mathcal{N}_{ps}^f(\varpi_{rt}), \sigma\right) \leq 1 - \mu \\ & \text{or } \frac{1}{h_{ps}} \sum_{(u,v),(r,t) \in I_{ps}} \Psi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \mathcal{N}_{ps}^f(\varpi_{rt}), \sigma\right) \geq \mu \Bigg\} \\ \subset H \cup (M' \cap ((\{1,2,\ldots,(N-1)\} \times \mathbb{N}) & \cup (\mathbb{N} \times \{1,2,\ldots,(N-1)\}))). \end{aligned}$$

As $\mathcal{I}_2$ is a strongly admissible ideal, then

$$H \cup (M' \cap ((\{1,2,\ldots,(T-1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\ldots,(T-1)\}))) \in \mathcal{I}_2.$$

As a result, we get $P \in \mathcal{I}_2$, namely, (ϖ_{uv}) is intuitionistic Nörlund strongly $\mathcal{I}_2$-lacunary Cauchy w.r.t. (Φ, Ψ). □

Theorem 2.18. *Suppose $\mathcal{I}_2$ is an admissible ideal including feature (AP_2). At that case, the notion of intuitionistic Nörlund strongly $\mathcal{I}_2$-lacunary Cauchy coincides with intuitionistic Nörlund strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence.*

Proof. When a sequence is intuitionistic Nörlund strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence, then it is strongly $\mathcal{I}_2$-lacunary Cauchy sequence by Theorem 2.17, where $\mathcal{I}_2$ need not to have the property (AP_2).

So, it is enough to denote that a sequence (ϖ_{uv}) is a intuitionistic Nörlund strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence under presumption that it is a intuitionistic Nörlund strongly $\mathcal{I}_2$-lacunary Cauchy sequence. Assume (ϖ_{uv}) is a intuitionistic Nörlund strongly $\mathcal{I}_2$-lacunary Cauchy sequence. Afterwards, for all $\mu \in (0,1)$ and all $\sigma > 0$, there exist $r, t \in \mathbb{N}$ such that

$$\begin{aligned} K := \Bigg\{ (p,s) \in \mathbb{N}^2 : & \frac{1}{h_{ps}} \sum_{(u,v) \in I_{ps}} \Phi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \mathcal{N}_{ps}^f(\varpi_{rt}), \sigma\right) \leq 1 - \mu \\ & \text{or } \frac{1}{h_{ps}} \sum_{(u,v) \in I_{ps}} \Psi\left(\mathcal{N}_{ps}^f(\varpi_{uv}) - \mathcal{N}_{ps}^f(\varpi_{rt}), \sigma\right) \geq \mu \Bigg\} \in \mathcal{I}_2. \end{aligned}$$

Let

$$T_i = \Bigg\{ (p,s) \in \mathbb{N}^2 : \frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv}) - \mathcal{N}_{ps}^{f}(\varpi_{r_i t_i}), \sigma\right) > 1 - \frac{1}{i}$$
$$\text{or} \quad \frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Psi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv}) - \mathcal{N}_{ps}^{f}(\varpi_{r_i t_i}), \sigma\right) < \frac{1}{i} \Bigg\},$$

where $r(i) = r\left(\frac{1}{i}\right)$, $t(i) = t\left(\frac{1}{i}\right)$, $i = 1, 2, \ldots$. Obviously, for $i = 1, 2, \ldots$, $T_i \in \mathcal{F}(\mathcal{I}_2)$. As $\mathcal{I}_2$ has the feature (AP_2), afterwards by [**7**, Theorem 3.3], there is $T \subset \mathbb{N}^2$ such that $T \in \mathcal{F}(\mathcal{I}_2)$ and $T \setminus T_i$ is finite for all i. Now, we prove that

$$\lim_{u,v,r,t\to\infty} \frac{1}{h_{ps}} \sum_{(u,v),(r,t)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv}) - \mathcal{N}_{ps}^{f}(\varpi_{rt}), \sigma\right) = 1$$
$$\text{and} \quad \lim_{u,v,r,t\to\infty} \frac{1}{h_{ps}} \sum_{(u,v),(r,t)\in I_{ps}} \Psi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv}) - \mathcal{N}_{ps}^{f}(\varpi_{rt}), \sigma\right) = 0$$

for all $\sigma > 0$ and $(u,v), (r,t) \in I_{ps}$. To denote these, assume $\mu \in (0,1)$ and $m \in \mathbb{N}$, so that $m > \frac{2}{\mu}$. When $(u,v), (r,t) \in T$, then $T \setminus T_m$ is a finite set; so, there exists $l = l(m)$ such that

$$\frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv}) - \mathcal{N}_{ps}^{f}(\varpi_{r_m t_m}), \sigma\right) > 1 - \frac{1}{m},$$
$$\frac{1}{h_{us}} \sum_{(r,t)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^{f}(\varpi_{rt}) - \mathcal{N}_{ps}^{f}(\varpi_{r_m t_m}), \sigma\right) > 1 - \frac{1}{m},$$

and

$$\frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Psi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv}) - \mathcal{N}_{ps}^{f}(\varpi_{r_m t_m}), \sigma\right) < \frac{1}{m},$$
$$\frac{1}{h_{us}} \sum_{(r,t)\in I_{ps}} \Psi\left(\mathcal{N}_{ps}^{f}(\varpi_{rt}) - \mathcal{N}_{ps}^{f}(\varpi_{r_m t_m}), \sigma\right) < \frac{1}{m},$$

for all $u, v, r, t > l(m)$. Hence, it gives that

$$\begin{aligned}
&\frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv}) - \mathcal{N}_{ps}^{f}(\varpi_{rt}), \sigma\right) \\
&\leq \frac{1}{h_{ps}} \sum_{(u,v)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^{f}(\varpi_{uv}) - \mathcal{N}_{ps}^{f}(\varpi_{r_m t_m}), \sigma\right) \\
&\quad + \frac{1}{h_{ps}} \sum_{(r,t)\in I_{ps}} \Phi\left(\mathcal{N}_{ps}^{f}(\varpi_{rt}) - \mathcal{N}_{ps}^{f}(\varpi_{r_m t_m}), \sigma\right) \\
&> \left(1 - \frac{1}{m}\right) + \left(1 - \frac{1}{m}\right) > 1 - \mu,
\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{h_{ps}}\sum_{(u,v)\in I_{ps}}\Psi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{uv}\right)-\mathcal{N}_{ps}^{f}\left(\varpi_{rt}\right),\sigma\right)\\ &\le\frac{1}{h_{ps}}\sum_{(u,v)\in I_{ps}}\Psi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{uv}\right)-\mathcal{N}_{ps}^{f}\left(\varpi_{r_mt_m}\right),\sigma\right)\\ &\quad+\frac{1}{h_{us}}\sum_{(r,t)\in I_{us}}\Psi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{rt}\right)-\mathcal{N}_{ps}^{f}\left(\varpi_{r_mt_m}\right),\sigma\right)<\frac{1}{m}+\frac{1}{m}<\mu.\end{aligned}$$

For any $\mu\in(0,1)$, there exists $l=l\left(\mu\right)$ such that $p,s\ge l\left(\mu\right)$ and $\left(u,v\right),\ \left(r,t\right)\in T\in\mathcal{F}\left(I_2\right)$, we obtain

$$\Bigg\{(p,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{ps}}\sum_{(u,v)\in I_{ps}}\Phi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{uv}\right)-\mathcal{N}_{ps}^{f}\left(\varpi_{rt}\right),\sigma\right)\le 1-\mu$$
$$\text{or}\quad\frac{1}{h_{ps}}\sum_{(u,v)\in I_{ps}}\Psi\left(\mathcal{N}_{ps}^{f}\left(\varpi_{uv}\right)-\mathcal{N}_{ps}^{f}\left(\varpi_{rt}\right),\sigma\right)\ge\mu\Bigg\}\in\mathcal{I}_2.$$

This means that (ϖ_{uv}) is intuitionistic Nörlund strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence. □

References

1. Atanassov K. T., *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems **20**(1) (1986), 87–96.
2. Başar F., *Summability Theory and its Applications*, 2nd ed., CRC Press/Taylor & Francis Group, Boca Raton, London, New York, 2022.
3. Başar F. and Savaşcı M. Y., *Double Sequence Spaces and Four Dimensional Matrices*, 2nd ed., Monographs and Research Notes in Mathematics, CRC Press/Taylor & Francis Group, Boca Raton, London, New York, 2022.
4. Belen C. and Yıldırım M., *On generalized statistical convergence of double sequences via ideals*, Ann. Univ. Ferrara **58**(1) (2012), 11–20.
5. Çakan C., Altay B. and Çoşkun H., *Double lacunary density and lacunary statistical convergence of double sequences*, Studia Sci. Math. Hungar. **47**(1) (2010), 35–45.
6. Das P., Kostyrko P., Wilczyński W. and Malik P., *$\mathcal{I}$ and $\mathcal{I}^*$-convergence of double sequences*, Math. Slovaca **58**(5) (2008), 605–620.
7. Dündar E. and Altay B., *$\mathcal{I}_2$-convergence and $\mathcal{I}_2$-Cauchy double sequences*, Acta Math. Sci. **34**(2) (2014), 343–353.
8. Fast H., *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.
9. Fridy J. A. and Orhan C., *Lacunary statistical convergence*, Pacific J. Math. **160(1)** (1993), 43–51.
10. Hardy G. H., *Divergent Series*, AMS Chelsea Publishing, 2000.
11. Kadak U. and Mohiuddine S. A., *Generalized statistically almost convergence based on the difference operator which includes the (p,q)-Gamma function and related approximation theorems*, Results Math. **73**(9) (2018), 1–31.
12. Kaleva O. and Seikkala S., *On fuzzy metric spaces*, Fuzzy sets and Systems **12** (1984), 215–229.
13. Karakuş S., Demirci K. and Duman O., *Statistical convergence on intuitionistic fuzzy normed spaces*, Chaos Solitons Fractals **35** (2008), 763–769.
14. Khan V. A., Abdullah S. A. A. and Alshlool K. M. A. S., *A study of Nörlund ideal convergent sequence spaces*, Yugosl. J. Oper. Res. **31**(4) (2021), 483–494.
15. Khan V. A. and Khan I. A., *Spaces of intuitionistic fuzzy Nörlund $\mathcal{I}$-convergent sequences*, Afr. Mat. **33** (2022), Art. No. 18.

16. Kostyrko P., Šalát T. and Wilczyński W., *$\mathcal{I}$-convergence*, Real Anal. Exchange **26**(2) (2000), 669–686.

17. Kramosil I. and Michalek J., *Fuzzy metric and statistical metric spaces*, Kybernetika (Prague) **11** (1975), 336–344.

18. Kumar S., Kumar V. and Bhatia S. S., *On ideal version of lacunary statistical convergence of double sequences*, Gen. Math. Notes **17**(1) (2013), 32–44.

19. Lael F. and Nourouzi K., *Some results on the IF-normed spaces*, Chaos Solitons Fractals **37** (2008), 931–939.

20. Menger K., *Statistical metrics*, Proc. Natl. Acad. Sci. **28**(12) (1942), 535–537.

21. Mohiuddine S. A., Asiri A. and Hazarika B., *Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorem*, Int. J. Gen. Syst. **48**(5) (2019), 492–506.

22. Moore C. N., *On relationships between Nörlund means for double series*, Proc. Amer. Math. Soc. **5**(6) (1954), 957–963.

23. Mohiuddine S. A. and Danish Lohani Q. M., *On generalized statistical convergence in intuitionistic fuzzy normed space*, Chaos Solitons Fractals **42** (2009), 1731–1737.

24. Mursaleen M. and Edely O. H. H., *Statistical convergence of double sequences*, J. Math. Anal. Appl. **288**(1) (2003), 223–231.

25. Mursaleen M. and Mohiuddine S. A., *On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space*, J. Comput. Appl. Math. **233**(2) (2009), 142–149.

26. Mursaleen M. and Mohiuddine S. A., *Statistical convergence of double sequences in intuitionistic fuzzy normed spaces*, Chaos Solitons Fractals **41** (2009), 2414–2421.

27. Mursaleen M., Mohiuddine S. A. and Edely O. H. H., *On the ideal convergence of double sequences in intuitionistic fuzzy normed spaces*, Comput. Math. Appl. **59** (2010), 603–611.

28. Nabiev A. A., Pehlivan S. and Gürdal M., *On $\mathcal{I}$-Cauchy sequences*, Taiwanese J. Math. **11**(2) (2007), 569–566.

29. Nörlund N. E., *Sur une application des fonctions permutables*, Lund. Universities Arsskrift **16** (1919), 1–10.

30. Park J. H., *Intuitionistic fuzzy metric spaces*, Chaos Solitons Fractals **22** (2004), 1039–1046.

31. Rath D. and Tripathy B. C., *Matrix maps on sequence spaces associated with sets of integers*, Indian J. Pure Appl. Math. **27**(2) (1996), 197–206.

32. Savaş E. and Das P., *A generalized statistical convergence via ideals*, Appl. Math. Lett. **24** (2011), 826–830.

33. Savaş E. and Gürdal M., *A generalized statistical convergence in intuitionistic fuzzy normed spaces*, Sci. Asia **41** (2015), 289–294.

34. Savaş E. and Gürdal M., *Certain summability methods in intuitionistic fuzzy normed spaces*, J. Intell. Fuzzy Systems **27**(4) (2014), 1621–1629.

35. Tripathy B. C., Hazarika B. and Choudhary B., *Lacunary $I-$convergent sequences*, Kyungpook Math. J. **52**(4) (2012), 473–482.

36. Tuğ O. and Başar F., *On the spaces of Nörlund null and Nörlund convergent sequences*, TWMS J. Pure Appl. Math. **7**(1) (2016), 76–87.

37. Wang C. S., *On Nörlund sequence spaces,* Tamkang J. Math. **9**(1) (1978), 269–274.

38. Zadeh L. A., *Fuzzy Sets*, Inf. Control **8**(3) (1965), 338–353.

Ö. Kişi, Department of Mathematics, Bartın University, Bartın, Turkey,
e-mail: `okisi@bartin.edu.tr`

M. Gürdal, Department of Mathematics, Süleyman Demirel University, 32260, Isparta, Turkey,
e-mail: `gurdalmehmet@sdu.edu.tr`

C. Choudhury, Department of Mathematics, Tripura University (A Central University), Suryamaninagar, Agartala, India,
e-mail: `chiranjibchoudhury123@gmail.com`