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TWO-WEIGHTED NORM INEQUALITIES OF SINGULAR INTEGRAL OPERATORS ON LOCAL MORREY SPACES

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Abstract. In this paper, we extend the two-weighted norm inequalities of singular integral operators to local Morrey spaces. This gives the mapping properties of singular integral operators from weighted local Morrey spaces to other local Morrey spaces with different weight. As application of this result, we extend the Córdoba-Fefferman inequalities to weighted local Morrey spaces.

1. Introduction

This paper aims to extend the two-weighted norm inequalities for singular integral operators to local Morrey spaces. That is, we obtain the boundedness of singular integral operators from a weighted local Morrey space to another weighted local Morrey space with different weight.

The two-weighted norm inequalities on Lebesgue spaces [**4, 8, 10, 28, 34, 35, 39**] are extensions of the weighted norm inequalities [**11**]. The two-weighted norm inequalities offer applications on some results related with the study of partial differential equations such as the eigenvalues estimation [**24**], the regularity of degenerate second order elliptic differential operator [**7, 9**] and the unique continuation for the differential inequality [**6**].

Recently, the two-weighted norm inequalities for operators were generalized to some extensions of Lebesgue spaces such as the Morrey spaces [**15, 20, 22, 43**] and the Herz spaces [**46**]. The Morrey spaces were introduced by Morrey in [**27**], to study the solutions of some quasi-elliptic partial differential equations and the Herz spaces were introduced by Herz to study the Fourier series. For the studies and the recent developments of Herz spaces and Morrey spaces, the reader is referred to [**26, 40**], respectively.

The studies of Morrey spaces has been further extended to the local Morrey spaces recently. The mapping properties of the Hardy-Littlewood maximal operator, the singular integral operators, the fractional integral operators and the spherical maximal operators were extended to the local Morrey spaces in [**2, 3, 44, 45**].

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As motivated by the above results on the two-weighted norm inequalities on Lebesgue spaces and the studies of local Morrey space, we investigate the extensions of the two-weighted norm inequality to local Morrey spaces.

In this paper, we find that the conditions imposed on the weights used in the two-weighted norm inequalities on Lebesgue spaces can also be used in the two-weighted norm inequalities on local Morrey spaces. Moreover, our main result also extends the Córdoba-Fefferman inequalities [**4**] to local Morrey spaces.

This paper is organized as follows. The definition of the weighted local Morrey spaces and some preliminary results on the two-weighted norm inequalities are given in Section 2. The main result of this paper, the two-weighted norm inequalities of singular integral operators on local Morrey spaces is given in Section 3. The Córdoba-Fefferman inequalities on local Morrey spaces are also presented in this section.

2. Preliminaries and definitions

Let $B(z,r)=\{x\in\mathbb{R}^n:|x-z|<r\}$ denote the open ball with center $z\in\mathbb{R}^n$ and radius $r>0$. Let $\mathbb{B}=\{B(z,r):z\in\mathbb{R}^n,\,r>0\}$. For any Lebesgue measurable set E, let $|E|$ and χ_E be the Lebesgue measure and the characteristic function of E, respectively.

For any non-negative locally integrable function u, we call u a weight. The weighted Lebesgue spaces consist of those Lebesgue measurable functions f satisfying

$$\|f\|_{L_p(u)}=\Big(\int_{\mathbb{R}^n}|f(x)|^p u(x)\mathrm{d}x\Big)^{1/p}<\infty.$$

Notice that $L_p(u)$ is not necessarily a Banach function space [**1**, Chapter 1, Definition 1]. More specifically, for any unbounded Lebesgue measurable set E with finite measure $|E|<\infty$, $\|\chi_E\|_{L_p(u)}$ is not necessarily to be finite. On the other hand, it is a ball Banach function space introduced and studied in [**41**].

Recall that the Hardy-Littlewood maximal operator M is defined as

$$(\mathrm{M}\,f)(x)=\sup_{B\ni x}\frac{1}{|B|}\int_B|f(y)|\mathrm{d}y,\quad f\in L^1_{\mathrm{loc}},$$

where the supremum is taken over all balls $B\in\mathbb{B}$ containing x. Now we define the class of pairs of weights (v,u) such that M is bounded from $L_p(v)$ to $L_p(u)$.

Definition 2.1. Let $1<p<\infty$. For any non-negative locally integrable functions u,v, we write $(v,u)\in\mathbb{M}_p$ if the Hardy-Littlewood maximal operator M is bounded from $L_p(v)$ to $L_p(u)$.

For instance, for any $1<p<\infty$ and any weight u, in view of [**11**, Chapter II, Theorem 2.12], we have $(\mathrm{M}\,u,u)\in\mathbb{M}_p$.

For any $1<p<\infty$, $(u,u)\in\mathbb{M}_p$ is equivalent with $u\in A_p$, where A_p is the Muckenhoupt class of weight functions [**42**, Chapter V, Section 1]. The reader is referred to [**23**] for the proof of this result.

We have a precise characterization of the condition $(v,u)\in\mathbb{M}_p$ by Sawyer [**38**].

Theorem 2.1. *Let $1<p<\infty$. For any non-negative locally integrable functions u,v, $(v,u)\in\mathbb{M}_p$ if and only if there exists a constant $C>0$ such that for any cube Q,*

$$\int_Q (\mathrm{M}(v^{1-p'}\chi_Q)(x))^p u(x)\mathrm{d}x \le C\int_Q v(x)^{1-p'}\mathrm{d}x<\infty. \tag{1}$$

Notice that whenever (v,u) satisfies (1), $v^{1-p'}=v^{-p'/p}$ is locally integrable.

We recall an estimate from [**15**, Lemma 2.4].

Lemma 2.2. *Let $1<p<\infty$ and v,u be non-negative locally integrable functions. If $(v,u)\in\mathbb{M}_p$, then there exists a constant $C>0$ such that for any $B\in\mathbb{B}$, we have*

$$\|\chi_B\|_{L_{p'}(v^{-p'/p})}\|\chi_B\|_{L_p(u)}\le C|B|. \tag{2}$$

Now, we give the definition of the weighted local Morrey spaces.

Definition 2.2. Let $1<p<\infty$, $\omega\colon(0,\infty)\to(0,\infty)$ and u be a non-negative locally integrable function. The weighted local Morrey space $LM^p_\omega(u)$ consists of those Lebesgue measurable functions f satisfying

$$\|f\|_{LM^p_\omega(u)}=\sup_{r>0}\frac{1}{\omega(r)}\|f\chi_{B(0,r)}\|_{L_p(u)}<\infty.$$

The above definition for the weighted local Morrey space is similar to the definition in [**32**]. The definition given in [**32**], is set in terms of the cubes and the above definition uses the balls. When $\omega(r)=r^{\frac{n}{p}-\frac{n}{q}}$ with $1\le q\le p<\infty$, the weighted local Morrey spaces become the ones defined in [**36, 37**]. When $\omega(r)=u(B(0,r))^{\frac{1}{p}-\frac{1}{q}}$, it reduces to the ones studied in [**25**].

The weighted local Morrey spaces are extensions of weighted Morrey spaces. The reader is referred to [**12, 13, 19, 25, 30, 31**] for the studies of weighted Morrey spaces.

3. Main result

We obtain the two-weight norm inequalities for singular integral operators on Morrey spaces in this section. We first recall the conditions imposed on the weight ω from [**15**].

Definition 3.1. Let $1<p<\infty$ and u be a non-negative locally integrable function. For any $\omega\colon(0,\infty)\to(0,\infty)$, we write $\omega\in\mathcal{L}W_{p,u}$ if there exists a constant $C>0$ such that for any $r>0$,

$$\omega(2r)\le C\omega(B(r)), \tag{3}$$

$$\sum_{j=0}^{\infty}\frac{\|\chi_{B(0,r)}\|_{L_p(u)}}{\|\chi_{B(0,2^{j+1})}\|_{L_p(u)}}\omega(2^{j+1}r)\le C\omega(r). \tag{4}$$

For any $a \in \mathbb{R}$, whenever $u(x) = |x|^a$, write $\mathcal{L}W_{p,u}$ as $\mathcal{L}W_{p,a}$.

Let $\lambda \in (0, 1/p)$. When $u \equiv 1$ and $\omega(r)$ satisfies the condition

$$\omega(2r) \le 2^{\lambda}\omega(r)$$

for all $r > 0$, then $\omega \in \mathcal{L}W_{p,u}$.

Now, we obtain some estimates used to show that the singular integral operators are well defined on the weighted local Morrey spaces. We use the idea from [**17**].

Proposition 3.1. *Let $1 < p < \infty$, u, v be non-negative locally integrable functions. Let K be a Lebesgue measurable function that satisfies*

$$|K(x,y)| \le C_0 \frac{1}{|x-y|^n}, \quad x \ne y, \tag{5}$$

for some $C_0 > 0$. If $\omega \in \mathcal{L}W_{p,u}$, $(v,u) \in \mathbb{M}_p$, then for any $f \in LM^p_\omega(v)$, $r > 0$ and $x \in B(0,r)$,

$$\int_{\mathbb{R}^n \setminus B(0,2r)} |K(x,y)f(y)|\mathrm{d}y \le C\|f\|_{LM^p_\omega(v)} \frac{\omega(r)}{\|\chi_{B(0,r)}\|_{L_p(u)}} \tag{6}$$

for some $C > 0$.

Proof. Let $f \in LM^p_\omega(v)$ and $r > 0$. Write $f(x) = f_0(x) + \sum_{j=1}^{\infty} f_j(x)$, where $f_0 = \chi_{B(0,2r)}f$ and $f_j = \chi_{B(0,2^{j+1}r)\setminus B(0,2^j r)}f$, $j \in \mathbb{N} \setminus \{0\}$.

According to (5), there exists a constant $C > 0$ such that for any $R > 0$, $z \in \mathbb{R}^n$ and Lebesgue measurable function f with $\mathrm{supp} f \subseteq \mathbb{R}^n \setminus B(z,R)$,

$$\int_{\mathbb{R}^n \setminus B(0,2r)} |K(z,y)f(y)|\mathrm{d}y \le C\frac{1}{R^n}\int_{\mathbb{R}^n} |f(y)|\mathrm{d}y. \tag{7}$$

Then, (7) guarantees that there is a constant $C > 0$ such that for any $j \ge 1$ and $x \in B(0,r)$,

$$\int_{\mathbb{R}^n \setminus B(0,2r)} |K(z,y)f_j(y)|\mathrm{d}y \le C2^{-jn}r^{-n}\int_{B(0,2^{j+1}r)} |f(y)|\mathrm{d}y. \tag{8}$$

The Hölder inequality ensures that

$$\begin{aligned}
&\int_{B(0,2^{j+1}r)} |f(y)|\mathrm{d}y \\
&\le \Big(\int_{\mathbb{R}^n} |f(x)|^p v(x)\mathrm{d}x\Big)^{1/p}\Big(\int_{\mathbb{R}^n} |\chi_{B(0,2^{j+1}r)}(x)|^{p'} v(x)^{-p'/p}\mathrm{d}x\Big)^{1/p'} \\
&\le C\big\|\chi_{B(0,2^{j+1}r)}f\big\|_{L_p(v)}\|\chi_{B(0,2^{j+1}r)}\|_{L_{p'}(v^{-p'/p})}
\end{aligned}$$

for some $C > 0$, where p' is conjugate of p.

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^n \setminus B(0,2r)} |K(z,y)f_j(y)|\mathrm{d}y \\
&\le C2^{-jn}r^{-n}\big\|\chi_{B(0,2^{j+1}r)}f\big\|_{L_p(v)}\|\chi_{B(0,2^{j+1}r)}\|_{L_{p'}(v^{-p'/p})}.
\end{aligned} \tag{9}$$

Applying (2) with $B = B(0, 2^{j+1}r)$, we have

$$\|\chi_{B(0,2^{j+1}r)}\|_{L_{p'}(v^{-p'/p})} \le C\frac{2^{(j+1)n}r^n}{\|\chi_{B(0,2^{j+1}r)}\|_{L_p(u)}}.$$

Using the above inequality on (9), we obtain

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B(0,2r)} |K(z,y)f_j(y)|\mathrm{d}y &\le C2^{-jn}r^{-n}\frac{\left\|\chi_{B(0,2^{j+1}r)}f\right\|_{L_p(v)}2^{(j+1)n}r^n}{\|\chi_{B(0,2^{j+1}r)}\|_{L_p(u)}}\\ &\le C\frac{1}{\|\chi_{B(0,2^{j+1}r)}\|_{L_p(u)}}\left\|\chi_{B(0,2^{j+1}r)}f\right\|_{L_p(v)}\end{aligned}$$

for some $C > 0$. Thus,

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus B(0,2r)} |K(z,y)f_j(y)|\mathrm{d}y\\ &\le C\frac{1}{\|\chi_{B(0,2^{j+1}r)}\|_{L_p(u)}}\frac{\omega(2^{j+1}r)}{\omega(2^{j+1}r)}\|\chi_{B(0,2^{j+1}r)}f\|_{L_p(v)}\\ &\le C\frac{1}{\|\chi_{B(0,2^{j+1}r)}\|_{L_p(u)}}\omega(2^{j+1}r)\sup_{R>0}\frac{1}{\omega(R)}\|\chi_{B(0,R)}f\|_{L_p(v)}\\ &= C\frac{1}{\|\chi_{B(0,2^{j+1}r)}\|_{L_p(u)}}\omega(2^{j+1}r)\|f\|_{LM^p_\omega(v)}.\end{aligned} \tag{10}$$

As $\omega \in \mathcal{L}W_{p,u}$, (4) and (10) yield

$$\sum_{j=1}^{\infty}\int_{\mathbb{R}^n\setminus B(0,2r)} |K(z,y)f_j(y)|\mathrm{d}y \le C\|f\|_{LM^p_\omega(v)}\frac{\omega(r)}{\|\chi_{B(0,r)}\|_{L_p(u)}}. \qquad \square$$

Now, we are ready to show that the singular integral operator is well defined on the weighted local Morrey spaces and obtain the two-weighted norm inequalities for the singular integral operators on local Morrey spaces.

Theorem 3.2. *Let $1 < p < \infty$ and u, v be non-negative locally integrable functions. Let K be a Lebesgue measurable function that satisfies*

$$|K(x,y)| \le C\frac{1}{|x-y|^n}, \quad x \ne y, \tag{11}$$

for some $C > 0$. Let

$$Tf(x) = \lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} K(x,y)f(y)\mathrm{d}y.$$

If $\omega \in \mathcal{L}W_{p,u}$, $(v,u) \in \mathbb{M}_p$ and $T\colon L_p(v) \to L_p(u)$ is bounded, then T can be extended to be a bounded operator from $LM^p_\omega(v)$ to $LM^p_\omega(u)$.

Proof. Let $f \in LM^p_\omega(v)$. For any $r > 0$ and $|x| < r$, define

$$Tf(x) = (T\chi_{B(0,2r)}f)(x) + \int_{\mathbb{R}^n\setminus B(0,2r)} K(x,y)f(y)\mathrm{d}y.$$

First, we show that T is well defined. As $\chi_{B(0,2r)}f \in L_p(v)$ and $T\colon L_p(v) \to L_p(u)$ is bounded, each term in the above sum is well defined. Proposition 3.1 assures that the second term is a well defined Lebesgue measurable function.

It remains to show that the definition is independent of r. Let $R > r > 0$ and $x \in B(0,r)$. In view of Proposition 3.1, we have

$$(T\chi_{B(0,2R)\setminus B(0,2r))}f)(x) = \int_{B(0,2R)\setminus B(0,2r)} K(x,y)f(y)\mathrm{d}y.$$

Consequently,

$$\begin{aligned}
&(T\chi_{B(0,2R)}f)(x) + \int_{\mathbb{R}^n\setminus B(0,2R)} K(x,y)f(y)\mathrm{d}y \\
&= (T\chi_{B(0,2r)}f)(x) + (T\chi_{B(0,2R)\setminus B(0,2r))}f)(x) + \int_{\mathbb{R}^n\setminus B(0,2R)} K(x,y)f(y)\mathrm{d}y \\
&= (T\chi_{B(0,2r)}f)(x) + \int_{B(0,2R)\setminus B(0,2r)} K(x,y)f(y)\mathrm{d}y + \int_{\mathbb{R}^n\setminus B(0,2R)} K(x,y)f(y)\mathrm{d}y \\
&= (T\chi_{B(0,2r)}f)(x) + \int_{\mathbb{R}^n\setminus B(0,2r)} K(x,y)f(y)\mathrm{d}y.
\end{aligned}$$

Therefore, the definition of $Tf(x)$, $x \in B(0,r)$ is independent of $r > 0$. Hence, T is well defined on $LM^p_\omega(v)$.

Since $T\colon L_p(v) \to L_p(u)$ is a bounded linear operator, we find that $\|Tf_0\|_{L_p(u)} \le C\|f_0\|_{L_p(v)}$. Consequently, (3) and Proposition 3.1 guarantee that

$$\begin{aligned}
&\frac{1}{\omega(r)}\|\chi_{B(0,r)}(Tf)\|_{L_p(u)} \\
&\le \frac{1}{\omega(r)}\|\chi_{B(0,r)}(Tf\chi_{B(0,2r)})\|_{L_p(u)} + \left\|\chi_{B(0,r)}\int_{\mathbb{R}^n\setminus B(0,2r)} K(x,y)f(y)\mathrm{d}y\right\|_{L_p(u)} \\
&\le C\frac{1}{\omega(2r)}\|\chi_{B(z,2r)}f\|_{L_p(v)} + C\frac{\|\chi_{B(0,r)}\|_{L_p(u)}}{\omega(r)}\|f\|_{LM^p_\omega(v)}\frac{\omega(r)}{\|\chi_{B(0,r)}\|_{L_p(u)}} \\
&\le C\|f\|_{LM^p_\omega(v)}.
\end{aligned}$$

The boundedness of T follows by taking the supremum over $r > 0$ on both sides of the above inequality. □

Proposition 3.1 and Theorem 3.2 are extensions of [**17**, Theorems 3.1 and 3.2] to the local Morrey spaces.

Now, we apply Theorem 3.2 to Calderón-Zygmund operators. We say that T is a Calderón-Zygmund operator if T is a bounded linear operator on L_2 and there exists $\delta > 0$ such that for any $x, y \in \mathbb{R}^n$ with $x \ne y$ and any z with $|x-z| \le \frac{1}{2}|x-y|$, the Schwartz kernel of T, $K(x,y)$, satisfies

$$\begin{aligned}
|K(x,y)| &\le C|x-y|^{-n}, \\
|K(x,y) - K(z,y)| &\le C|x-z|^{\delta}|x-y|^{-n-\delta}, \\
|K(x,z) - K(x,y)| &\le C|y-z|^{\delta}|x-y|^{-n-\delta}.
\end{aligned}$$

Let $M^{[p]+1}$ be the $([p]+1)$-iterate of the Hardy-Littlewood maximal operator M and $[p]$ be the integer part of p.

Let u be a non-negative integrable function and $\omega \in \mathcal{W}_{p,M^{[p]+1}u}$. According to [**34**, Theorem 3.1], $T\colon L^p(M^{[p]+1}u) \to L_p(u)$ is bounded. In addition, [**11**, Chapter II, Theorem 2.12] yields

$$\begin{aligned}\int_{\mathbb{R}^n} (Mf(x))^p u(x)\mathrm{d}x &\le C\int_{\mathbb{R}^n} |f(x)|^p Mu(x)\mathrm{d}x \\ &\le C\int_{\mathbb{R}^n} |f(x)|^p M^{[p]+1}u(x)\mathrm{d}x\end{aligned}$$

for some $C > 0$. Therefore, $(M^{[p]+1}u, u) \in \mathbb{M}_p$. Theorem 3.2 assures the boundedness of T from $LM^p_\omega(\mathrm{M}^{[p]+1}\,u)$ to $LM^p_\omega(u)$.

Notice that for any non-negative locally integrable function u and any $1 < r < \infty$, we have $u \le (M(u^r)^{1/r})$. In view of [**42**, Chapter V, Proposition 8], we find that $(M(u^r))^{1/r} \in A_1$, where A_1 is the class of Muckenhoupt weight function. Consequently, [**42**, Chapter V, Section 5.2] asserts that

$$M^{[p]+1}u \le M^{[p]+1}(M(u^r)^{1/r}) \le CM(u^r)^{1/r}.$$

Thus, we obtain the Córdoba-Fefferman inequalities [**4**] on the local Morrey spaces.

Corollary 3.3. *Let $1 < r < \infty$, $1 < p < \infty$, u be a non-negative integrable function, and $\omega \in \mathcal{W}_{p,M^{[p]+1}u}$. If T is a Calderón-Zygmund operator, then for any non-negative locally integrable function u, we have*

$$\|Tf\|_{LM^p_\omega(u)} \le C\|f\|_{LM^p_\omega((\mathrm{M}(u^r))^{1/r})}$$

for some $C > 0$.

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