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PROPERTIES OF THE CONE OF NON-NEGATIVE POLYNOMIALS AND DUALITY

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Abstract. Polynomial optimization problems are problems of optimizing a multivariate polynomial over the feasible set defined by a finite number of polynomial inequalities. It encompasses many problems within various fields of mathematics, e.g., binary optimization, mixed-integer linear programming, global optimization and partial differential inequalities. Problems of polynomial optimization can be equivalently reformulated as problems over the convex cone of non-negative polynomials. In this paper, the geometric and topological properties of a cone of polynomials non-negative on a given set and the respective dual cone are studied.

1. Introduction

Consider a polynomial optimization problem in the following form

$$\begin{aligned} &\min p(x) \\ &\quad x \in K, \end{aligned} \tag{1}$$

where p is a multivariate polynomial and $K \subseteq \mathbb{R}^n$ is a non-empty set. By defining a new variable γ, which will serve as the lower bound of p on K, the problem (1) can be equivalently formulated as

$$\begin{aligned} &\max \gamma \\ &\quad p(x) - \gamma \geq 0 \quad \text{for all } x \in K. \end{aligned} \tag{2}$$

The constraint in problem (2) requires a polynomial to be non-negative on a given set K.

Formulation (2) gives rise to a few questions such as: whether one can optimize over the set of polynomials non-negative on K, what the structure of that set is, whether one can test if a polynomial is non-negative on K and whether such testing can be done efficiently.

In [**10**], it was shown that testing whether a polynomial of degree at least 4 is non-negative on a basic semialgebraic set K is NP-hard, even if $K = \mathbb{R}^n$. Moreover, it was shown that unconstrained optimization of a quartic polynomial,

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optimization of a cubic polynomial over the sphere and optimization of a quadratic polynomial over the simplex are all NP-hard problems (see [**5**, Section 5], [**3**, Section 2]). As a consequence, formulation (2) provides motivation for examining the structure of a set of non-negative multivariate polynomials on a non-empty set K.

Non-negativity of polynomials on $K = \mathbb{R}^n$ has been widely examined for more than 100 years. It is obvious that if a polynomial can be represented as a finite sum of squares of other polynomials with lower degree, it is non-negative on $\mathbb{R}^n$. However, it is not obvious whether the converse holds. In fact, David Hilbert in [**4**, p. 344] proved that there are only three cases when the converse holds: for all odd-degree univariate polynomials, for all quadratic polynomials, and for all two-variable quartic polynomials. Thus, in general, the set of SOS (sum-of-squares) polynomials is a proper subset of the set of polynomials non-negative on $\mathbb{R}^n$.

In fact, both these sets are proper cones (see [**1**, Section 1.1], [**9**, Theorem 1]). The most important difference between these two cones is their structure. The cone of SOS polynomials is closely linked with semidefinite programming. More precisely, testing whether a given polynomial is SOS can be transformed into solving a feasibility problem of semidefinite programming (see [**7**, Section 2.1]). On the other hand, no simple and tractable characterization of the cone of non-negative polynomials is known (see [**6**, Section 1.1]). Therefore, the SOS cone serves as a computational substitute for the cone of non-negative polynomials. More details on the SOS cone, its geometry and applications can be found in [**11**] and [**12**].

Non-negativity of polynomials and polynomial optimization have been studied within the context of convex and conic optimization and real algebra. In [**18**], it was proposed that a convex optimization technique be used to minimize an unconstrained multivariate polynomial. In [**9**], the author discussed the duality of cones of non-negative polynomials and moment cones. More specifically, a moment cone was shown to be characterized by semidefinite constraints or, in other words, by linear matrix inequalities on condition that the corresponding dual cone, the cone of non-negative polynomials, was SOS-representable. Non-negativity of polynomials with use of real algebraic results was discussed in [**13**]. Finally, with the real algebra result of Putinar (see [**16**, Section 3]), Lasserre [**8**] constructed a sequence of semidefinite program relaxations with optima converging to the optimum of a polynomial optimization problem, known as SOS or Lasserre hierarchy.

In this paper, we concentrate on analyzing the properties of the set of polynomials non-negative on a given non-empty set $K \subseteq \mathbb{R}^n$ by means of convex analysis and linear algebra results, slightly extending the results in [**1**]: whereas in [**1**] the authors formulated the results for $K = \mathbb{R}^n$, the results presented in this paper are formulated for a general non-empty set K. Moreover, we formulate the so-called dual cone theorem and demonstrate its use when searching for the explicit characterizations of the set of polynomials non-negative on $K = [-1, 1]$. Our characterizations are in concordance with the characterizations of Fekete, Lukács and

Markov (see [**6**, Theorem 2.4], [**15**, Problem 46, p. 78]), however, applying a different method. While Fekete, Lukács and Markov used a strictly algebraic method to directly derive the characterization of the set of polynomials non-negative on $[-1, 1]$, our approach relies on convex analysis and linear algebra results, providing an interesting geometrical insight into the problem.

The paper is organized as follows: Basic definitions, notations, basic results and auxiliary propositions are contained in Section 2. Section 3 concernes with the properties of the cone of polynomials non-negative on the set K. The respective dual cone and the dual cone theorem are presented in Section 4. Section 5 provides explicit characterizations of the cone of polynomials of degree at most 2 non-negative on $K = [-1, 1]$ and its dual cone. Proofs of a few auxiliary propositions from Section 2 can be found in Appendix.

2. Preliminaries

In this section, we include standard definitions, notations and basic results concerning cones and multivariate polynomials.

We denote $\mathbb{R}[x]_d$ the real vector space of n-variate polynomials ($x \in \mathbb{R}^n$) with degree at most d and $\mathbb{R}[x]$ the real vector space of n-variate polynomials. The standard (or canonical) basis of $\mathbb{R}[x]_d$ consists of all monomials of degree at most d, namely,

$$1, x_1, \dots, x_n, x_1^2, x_1x_2, \dots, x_{n-1}x_n, \dots, x_n^2, \dots, x_1^d, \dots, x_n^d.$$

For instance, for $n = 2$ and $d = 3$, the canonical basis consists of monomials

$$1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3.$$

In fact, there are $s(d) := \sum_{i=0}^{d} \binom{n+i-1}{i} = \binom{n+d}{d}$ monomials of degree at most d ($d \in \mathbb{N}_0$). This implies that $\dim(\mathbb{R}[x]_d) = s(d)$, and thus, vector spaces $\mathbb{R}[x]_d$ and $\mathbb{R}^{s(d)}$ are isomorphic (donated by $\simeq$), i.e., $\mathbb{R}[x]_d \simeq \mathbb{R}^{s(d)}$. Apparently, any polynomial $p \in \mathbb{R}[x]_d$ can be represented as a linear combination of canonical basis vectors.

Introducing the standard multi-index notation, for

$$\mathbb{N}^n \ni \alpha := \{(\alpha_1, \alpha_2, \dots, \alpha_n) \mid \alpha_i \in \mathbb{N}_0,\ i = 1, 2, \dots, n\},$$

we set $|\alpha| = \sum_{i=1}^{n} \alpha_i$ and $\mathbb{N}_d^n := \{\alpha \in \mathbb{N}^n \mid |\alpha| \le d\}$. We set $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$. Now, every polynomial $p \in \mathbb{R}[x]_d$ can be expressed in the form

$$p(x) = \sum_{\alpha \in \mathbb{N}_d^n} p_\alpha x^\alpha, \quad x \in \mathbb{R}^n,$$

where $p_\alpha \in \mathbb{R}$ are coefficients.

The inner product $\langle \cdot, \cdot \rangle \colon \mathbb{R}[x]_d \times \mathbb{R}[x]_d \to \mathbb{R}$ is defined as follows:

$$\langle p, q \rangle = \sum_{\alpha \in \mathbb{N}_d^n} p_\alpha q_\alpha, \quad p, q \in \mathbb{R}[x]_d. \tag{3}$$

The norm induced by inner product (3) takes the following form

$$\|p\| = \Big(\sum_{\alpha\in\mathbb{N}_d^n} p_\alpha^2\Big)^{\frac{1}{2}}. \tag{4}$$

The norm (4) induces a topology on $\mathbb{R}[x]_d$. A set $\mathcal{O} \subseteq \mathbb{R}[x]_d$ is open if

$$\forall p \in \mathcal{O}\ \exists r > 0 : \mathcal{B}(p,r) := \{q \in \mathbb{R}[x]_d \mid \|p-q\| < r\} \subset \mathcal{O}.$$

Note that all norms on $\mathbb{R}[x]_d$ are equivalent, and therefore, they define the same open sets of $\mathbb{R}[x]_d$. A set $\mathcal{C} \subseteq \mathbb{R}[x]_d$ is closed if the set $\mathbb{R}[x]_d \setminus \mathcal{C}$ is open. The closure of the set $S \subseteq \mathbb{R}[x]_d$, denoted $\mathrm{cl}(S)$, is the smallest closed set containing S. The interior of S, denoted $\mathrm{int}(S)$, is the largest open set contained in S.

Note that a sequence $\{p_j\}_{j=1}^{\infty} \subseteq \mathbb{R}[x]_d$ converges to $p \in \mathbb{R}[x]_d$, denoted

$$\lim_{j\to\infty} p_j = p,$$

if

$$\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall j > n_0 : \|p_j - p\| < \varepsilon.$$

Vector space $\mathbb{R}[x]_d$ equipped with the norm $\|\cdot\|$ is a normed space, and therefore, it is first-countable. It means that for any subset $S \subseteq \mathbb{R}[x]_d$, it holds that $x \in \mathrm{cl}(S)$ if and only if there exists a sequence $\{x_j\}_{j=1}^{\infty} \subseteq S$ such that $\lim_{j\to\infty} x_j = x$.

We denote

$$m_d(x) = \big(1, x_1, \dots, x_n, x_1^2, x_1x_2, \dots, x_{n-1}x_n, \dots, x_n^2, \dots, x_1^d, \dots, x_n^d\big)^T$$

for every $x \in \mathbb{R}^n$. Note that $m_d\colon \mathbb{R}^n \to \mathbb{R}^{s(d)}$. We formulate two auxiliary propositions – Proposition 1 and Proposition 2 – which will be useful in the succeeding section. Their proofs can be found in Appendix.

Proposition 1. *Let $p \in \mathbb{R}[x]_d$. Then*

$$\forall x \in \mathbb{R}^n : |p(x)| \le \|p\| \|m_d(x)\|_2,$$

where $\|\cdot\|_2$ denotes the Euclidean norm.

Proposition 2. *For every $x \in \mathbb{R}^n$ and $d \in \mathbb{N}$, the following inequality holds*

$$\|m_d(x)\|_2^2 \ge \|m_{2d}(x)\|_2.$$

Finally, a set $C \subseteq \mathbb{R}[x]_d$ is called a cone in $\mathbb{R}[x]_d$ if it is closed under non-negative scalar multiplication, i.e., $cp \in C$ for all $c \ge 0$ and $p \in C$.
A cone C is called:

- a convex cone if for all $c_1, c_2 \ge 0$ and $p, q \in C$, it holds that $c_1p + c_2q \in C$;
- a closed cone if C is a closed set;
- a solid cone if $\mathrm{int}(C) \neq \emptyset$;
- a pointed cone if $p \in C$ and $(-p) \in C$ implies $p \equiv 0$.

A closed, convex, solid and pointed cone is called a proper cone. The conic hull of a set $S \subseteq \mathbb{R}[x]_d$, denoted cone($S$), is the set of all finite conic combinations of elements included in S, i.e.,

$$\operatorname{cone}(S) := \left\{ \sum_{i=1}^{k} c_i p_i \mid k \in \mathbb{N},\ p_i \in S,\ c_i \geq 0,\ i = 1, 2, \ldots, k \right\},$$

or, equivalently, as the smallest convex cone containing S.

3. Cone of polynomials non-negative on a non-empty set

In this section, we concentrate on the properties of the set of polynomials non-negative on a non-empty set $K \subseteq \mathbb{R}^n$. We show that this set is a convex, closed and solid cone. Moreover, if $\operatorname{int}(K) \neq \emptyset$, it is a pointed cone, and hence a proper cone. More details on (convex) cones can be found in [**2**, Section 2.4].

We denote $C_{n,d}(K)$ the set of all n-variate polynomials with degree at most d which are non-negative on the set K,

$$C_{n,d}(K) := \{p \in \mathbb{R}[x]_d \mid p(x) \geq 0 \ \text{ for all } x \in K\}.$$

The following proposition states that the set $C_{n,d}(K)$ has a conic structure.

Proposition 3. *Let $K \subseteq \mathbb{R}^n$ be a non-empty set. Then $C_{n,d}(K)$ is a convex cone.*

Proof. For arbitrary $p \in C_{n,d}(K)$ and $c \geq 0$, it holds that $cp(x) \geq 0$ for all $x \in K$, and therefore, $cp \in C_{n,d}(K)$. Moreover, for arbitrary $c_1, c_2 \geq 0$ and $p, q \in C_{n,d}(K)$, it holds that $c_1 p(x) + c_2 q(x) \geq 0$ for all $x \in K$, and therefore, $c_1 p + c_2 q \in C_{n,d}(K)$. We have shown that $C_{n,d}(K)$ is a convex cone. □

Another interesting property of $C_{n,d}(K)$ is the nesting property: one can observe that polynomials with lower degree than d which are non-negative on K are also included in $C_{n,d}(K)$.

Proposition 4. *Let $K \subseteq \mathbb{R}^n$ be a non-empty set. Then*

$$C_{n,d}(K) \supseteq C_{n,d-1}(K) \supseteq C_{n,d-2}(K) \supseteq \cdots \supseteq C_{n,0}(K).$$

Remark 1. In some cases it may happen that $C_{n,d}(K) = C_{n,d-1}(K)$, for example, $C_{1,3}(\mathbb{R}) = C_{1,2}(\mathbb{R})$.

In the following proposition, we show that $C_{n,d}(K)$ is a closed solid cone. Moreover, under the additional condition placed on the set K, it is also a pointed cone. Part a) and c) in Proposition 5 can be found in [**12**, Section 4.2] and [**1**, Section 1.1] for $K = \mathbb{R}^n$.

Proposition 5. *Let $d \in \mathbb{N}$. The convex cone $C_{n,d}(K)$ is*

a) *closed in* $\mathbb{R}[x]_d$,
b) *pointed if* $\operatorname{int}(K) \neq \emptyset$,
c) *solid.*

Proof. a) Consider an arbitrary sequence $\{p_j\}_{j=1}^{\infty} \subseteq C_{n,d}(K)$ such that $p_j \to p$ for $j \to \infty$. We have $p_j(x) \geq 0$ for all $x \in K$ and all $j \in \mathbb{N}$, and thus, $\lim_{j\to\infty} p_j(x) = p(x) \geq 0$ for all $x \in K$, which implies that $p \in C_{n,d}(K)$.

b) Assume by contradiction that $\mathrm{int}(K) \neq \emptyset$, but there exists a non-zero polynomial $p \in C_{n,d}(K)$ such that $-p \in C_{n,d}(K)$. It immediately follows that $p(x) = 0$ for all $x \in K$. Choose a point $\bar{x} \in \mathrm{int}(K)$. Then there exists $r > 0$ such that $\mathcal{B}(\bar{x}, r) \subset K$, which means that $p(x) = 0$ for all $x \in \mathcal{B}(\bar{x}, r)$. Set

$$g(x) := p(\bar{x} - x) \quad \text{for all } x \in \mathbb{R}^n.$$

It is obvious that $g(x)$ is a multivariate polynomial with degree at most d, and thus, it can be expressed in the form

$$g(x) = \sum_{\alpha \in \mathbb{N}_d^n} g_\alpha x^\alpha.$$

Moreover, g is infinitely many times differentiable on $\mathbb{R}^n$ and $g(x) = 0$ for all $x \in \mathcal{B}(0, r)$. Note that

$$\frac{\partial^{|\alpha|} g}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}(0, 0, \dots, 0) = \alpha_1! \cdot \alpha_2! \cdot \dots \cdot \alpha_n! \cdot g_{(\alpha_1, \alpha_2, \dots, \alpha_n)}, \quad \alpha \in \mathbb{N}_d^n.$$

Since $\frac{\partial^{|\alpha|} g}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}(0, 0, \dots, 0) = 0$, we obtain $g_\alpha = 0$ for all $\alpha \in \mathbb{N}_d^n$, which implies that $g \equiv 0$. Note that $p(x) = g(\bar{x} - x)$, and thus, $p \equiv 0$, which is a contradiction.

c) We show that $C_{n,2k}(K)$ is a solid cone for any $k \in \mathbb{N}$. Note that if d is divisible by 2, then set $d = 2k$ to show that $C_{n,d}(K)$ is a solid cone. If d is not divisible by 2, recall that by Proposition 4, we have $C_{n,d-1}(K) \subseteq C_{n,d}(K)$ with $d-1$ being divisible by 2. Since $\emptyset \neq \mathrm{int}(C_{n,d-1}(K)) \subseteq \mathrm{int}(C_{n,d}(K))$, we eventually have $\mathrm{int}(C_{n,d}(K)) \neq \emptyset$.

We show that a polynomial $q(x) = m_k(x)^T m_k(x)$ is an interior point of $C_{n,2k}(K)$ for any non-empty set $K \subseteq \mathbb{R}^n$. Obviously, for any polynomial $p \in \mathbb{R}[x]_{2k}$, we have

$$p(x) = q(x) + p(x) - q(x) \quad \text{for all } x \in \mathbb{R}^n.$$

By Proposition 1, we have $|p(x) - q(x)| = |(p - q)(x)| \leq \|p - q\| \|m_{2k}(x)\|_2$ for all $x \in \mathbb{R}^n$. We have

$$\forall x \in \mathbb{R}^n : p(x) \geq q(x) - \|p - q\| \|m_{2k}(x)\|_2.$$

We choose r such that for all $x \in K$, it holds that $q(x) - r\|m_{2k}(x)\|_2 \geq 0$ by setting

$$r = \frac{1}{2} \inf_{x \in K} \left\{ \frac{m_k(x)^T m_k(x)}{\sqrt{m_{2k}(x)^T m_{2k}(x)}} \right\}.$$

Since $\frac{m_k(x)^T m_k(x)}{\sqrt{m_{2k}(x)^T m_{2k}(x)}} \geq 1$ for all $x \in \mathbb{R}^n$ (see Proposition 2), $r > 0$ is indeed well defined.

Note that for any $p \in \mathbb{R}[x]_{2k}$ such that $r > \|p - q\|$, it holds that

$$0 \leq q(x) - r\|m_{2k}(x)\|_2 < q(x) - \|p - q\| \|m_{2k}(x)\|_2 \leq p(x) \quad \text{for all } x \in K,$$

implying that if $p \in \mathcal{B}(q,r)$, then $p \in C_{n,2k}(K)$.

By this construction of $r > 0$, we have shown that $\mathcal{B}(q,r) = \{p \in \mathbb{R}[x]_{2k} \mid \|p - q\| < r\} \subset C_{n,2k}(K)$. □

Example 1. The assumption of non-empty interior of K in Proposition 5 b) cannot be disposed of. Consider $K = \{(x_1,x_2) \in \mathbb{R}^2 \mid (x_2-1)^2 \leq 0\} = \{(x_1,1) \mid x_1 \in \mathbb{R}\}$ and $C_{2,2}(K) = \{p \in \mathbb{R}[x_1,x_2]_2 \mid p(x_1,1) \geq 0 \text{ for all } x_1 \in \mathbb{R}\}$. Clearly, $\operatorname{int}(K) = \emptyset$ the polynomial p, defined as $p(x_1,x_2) = (x_2-1)x_1$, is included in $C_{2,2}(K)$ but also $-p \in C_{2,2}(K)$. It shows that $C_{2,2}(K)$ is not pointed.

4. Dual cone and the dual cone theorem

In this section, we introduce the dual cone of $C_{n,d}(K)$ and the dual cone theorem. Note that the (algebraic) dual cone of $C_{n,d}(K)$ by definition consists of linear functionals $\ell\colon \mathbb{R}[x]_d \to \mathbb{R}$ such that $\ell(p) \geq 0$ for all $p \in C_{n,d}(K)$ (see, e.g., [**1**] or [**6**]), and thus, $\ell \in \mathbb{R}[x]_d^*$, which is the dual vector space of $\mathbb{R}[x]_d$. However, since $\mathbb{R}[x]_d$ is finite dimensional, it holds $\mathbb{R}[x]_d \simeq \mathbb{R}[x]_d^*$. Therefore, the dual cone of $C_{n,d}(K)$ can be represented as follows:

$$C_{n,d}(K)^* = \{q \in \mathbb{R}[x]_d \mid \langle p,q\rangle \geq 0, \quad p \in C_{n,d}(K)\}. \tag{5}$$

Note that the representation of $C_{n,d}(K)^*$ in [**6**] and [**1**] differs from the representation that we have introduced.

The properties of $C_{n,d}(K)^*$ follow directly from the general theory of dual cones (see [**2**, Section 2.6.1] and the bipolar theorem (Theorem 14.1) in [**17**]) and Proposition 5. They are included in the following proposition. Note that a similar statement to part c) is mentioned in [**6**, Lemma 4.6].

Proposition 6. *For the dual cone $C_{n,d}(K)^*$, the following statements hold:*

a) *$C_{n,d}(K)^*$ is a closed convex cone in $\mathbb{R}[x]_d$,*
b) *$C_{n,d}(K)^*$ is pointed,*
c) *if $\operatorname{int}(K) \neq \emptyset$, then $C_{n,d}(K)^*$ is solid,*
d) *$C_{n,d}(K)^{**} = C_{n,d}(K)$.*

The dual cone $C_{n,d}(K)^*$ represented by (5), can be characterized as a closure of a conic hull of polynomials of the form $\sum_{\alpha \in \mathbb{N}_d^n} t^\alpha x^\alpha$, where $t \in K$. We state and prove this characterization of $C_{n,d}(K)^*$ in the following theorem. A similar characterization theorem of $C_{n,d}(\mathbb{R}^n)^*$ in terms of linear functionals was proved in e.g., [**12**, Lemma 4.11] or [**1**, Lemma 2.1]. Another characterization $C_{n,d}(K)^*$ in terms of vectors of $\mathbb{R}^{s(d)}$ having a finite representing measure with support contained in K was proved in [**6**, Lemma 4.7] with additional assumption on K being compact.

Theorem 1. *Let $C_{n,d}(K)$ be a cone of non-negative polynomials on K of degree at most d ($d \in \mathbb{N}$). Then*

$$C_{n,d}(K)^* = \mathrm{cl}\left(\mathrm{cone}\left[\sum_{\alpha\in\mathbb{N}_d^n} t^\alpha x^\alpha \mid t \in K\right]\right).$$

Proof. $\supseteq$: Take $q \in \mathrm{cone}\left[\sum_{\alpha\in\mathbb{N}_d^n} t^\alpha x^\alpha \mid t \in K\right]$. It means that there exist a number $m \in \mathbb{N}$, coefficients $c_1, c_2, \dots, c_m \geq 0$ and vectors $t_1, t_2, \dots, t_m \in K$ such that

$$q(x) = \sum_{i=1}^m c_i \sum_{\alpha\in\mathbb{N}_d^n} t_i^\alpha x^\alpha = \sum_{\alpha\in\mathbb{N}_d^n} \sum_{i=1}^m c_i t_i^\alpha x^\alpha.$$

Now take an arbitrary $p \in C_{n,d}(K)$ to show that

$$\begin{aligned}\langle p, q\rangle &= \sum_{\alpha\in\mathbb{N}_d^n} p_\alpha q_\alpha = \sum_{\alpha\in\mathbb{N}_d^n} p_\alpha \sum_{i=1}^m c_i t_i^\alpha \\ &= \sum_{i=1}^m c_i \sum_{\alpha\in\mathbb{N}_d^n} p_\alpha t_i^\alpha = \sum_{i=1}^m c_i p(t_i) \geq 0.\end{aligned}$$

Note that $p(t_i) \geq 0$ for all $i = 1, 2, \dots, m$, because $t_i \in K$ and p is non-negative on K. Since p is chosen arbitrarily, we obtain that $q \in C_{n,d}(K)^*$.

Now suppose that $q \notin \mathrm{cone}\left[\sum_{\alpha\in\mathbb{N}_d^n} t^\alpha x^\alpha \mid t \in K\right]$, but

$$q \in \mathrm{cl}\left(\mathrm{cone}\left[\sum_{\alpha\in\mathbb{N}_d^n} t^\alpha x^\alpha \mid t \in K\right]\right).$$

There exists a sequence of polynomials $\{q_j\}_{j=1}^\infty \subseteq \mathrm{cone}\left[\sum_{\alpha\in\mathbb{N}_d^n} t^\alpha x^\alpha \mid t \in K\right]$ such that $\lim_{j\to\infty} q_j = q$. Using the argument above, it is obvious that for any $p \in C_{n,d}(K)$, it holds that $\langle p, q_j\rangle \geq 0$ for all $j \in \mathbb{N}$. With inner product being continuous, by limit transition, we have $\langle p, q\rangle \geq 0$, and hence, $q \in C_{n,d}(K)$.

$\subseteq$: We use the separating hyperplane theorem (see, e.g., [**2**, Chapter 2.5]). Suppose that $q \in C_{n,d}(K)^*$, but

$$q \notin \mathrm{cl}\left(\mathrm{cone}\left[\sum_{\alpha\in\mathbb{N}_d^n} t^\alpha x^\alpha \mid t \in K\right]\right).$$

Since $\mathrm{cl}\left(\mathrm{cone}\left[\sum_{\alpha\in\mathbb{N}_d^n} t^\alpha x^\alpha \mid t \in K\right]\right)$ is a closed convex cone, there exists a separating polynomial $v \in \mathbb{R}[x]_d$ such that

$$\begin{aligned}&\langle v, q\rangle < 0 \quad \text{and} \\ &\langle v, r\rangle \geq 0 \quad \text{for all } r \in \mathrm{cl}\left(\mathrm{cone}\left[\sum_{\alpha\in\mathbb{N}_d^n} t^\alpha x^\alpha \mid t \in K\right]\right).\end{aligned}$$

For arbitrary $\bar{t} \in K$, set $r(x) = \sum_{\alpha\in\mathbb{N}_d^n} \bar{t}^\alpha x^\alpha$. Note that $\langle v, r\rangle \geq 0$, and thus, we have

$$\langle v, r\rangle = \sum_{\alpha\in\mathbb{N}_d^\alpha} v_\alpha \bar{t}^\alpha = v(\bar{t}) \geq 0.$$

Since $\bar{t}$ was chosen arbitrarily, we have $v(t) \geq 0$ for all $t \in K$, and thus, $v \in C_{n,d}(K)$. But this is in contradiction with $\langle v, q\rangle < 0$. □

5. Characterization of $C_{1,2}([-1,1])^*$ and $C_{1,2}([-1,1])$

In this section, we demonstrate the use of the dual cone theorem in finding explicit characterizations of the cones $C_{1,2}([-1,1])^*$ and $C_{1,2}([-1,1])$. Note that the general characterization of $C_{n,d}(K)$ is not known.

According to Theorem 1, we have

$$C_{1,2}([-1,1])^* = \mathrm{cl}\left(\mathrm{cone}\left[1 + tx + t^2x^2 \mid t \in [-1,1]\right]\right).$$

It means that for every polynomial in cone $\left[1 + tx + t^2x^2 \mid t \in [-1,1]\right]$, there exist a number $k \in \mathbb{N}$, $t_1, t_2, \ldots, t_k \in [-1,1]$ and $c_1, c_2, \ldots, c_k \geq 0$ such that

$$q(x) = \underbrace{\left(\sum_{i=1}^k c_i\right)}_{=:q_0} + \underbrace{\left(\sum_{i=1}^k c_i t_i\right)}_{=:q_1} x + \underbrace{\left(\sum_{i=1}^k c_i t_i^2\right)}_{=:q_2} x^2.$$

It can be easily verified that

$$\mathrm{cone}\left[1 + tx + t^2x^2 \mid t \in [-1,1]\right] \subseteq \left\{q \in \mathbb{R}[x]_2 \mid q_2 \geq 0,\ q_0 \geq q_2,\ q_0q_2 \geq q_1^2\right\}.$$

Now, if $q \in C_{1,2}([-1,1])^*$, but $q \notin \mathrm{cone}\left[1 + tx + t^2x^2 \mid t \in [-1,1]\right]$, there exists a sequence of polynomials $\{q^{(j)}\}_{j=1}^\infty$ such that $\lim_{j\to\infty} q^{(j)} = q$. Note that for all $j \in \mathbb{N}$, it holds that $q_2^{(j)} \geq 0$, $q_0^{(j)} \geq q_2^{(j)}$ and $q_0^{(j)}q_2^{(j)} \geq (q_1^{(j)})^2$. Calculating the limits, we obtain that $q \in \{q \in \mathbb{R}[x]_2 \mid q_2 \geq 0,\ q_0 \geq q_2,\ q_0q_2 \geq q_1^2\}$, which shows that $C_{1,2}([-1,1])^* \subseteq \{q \in \mathbb{R}[x]_2 \mid q_2 \geq 0,\ q_0 \geq q_2,\ q_0q_2 \geq q_1^2\}$.

To show the converse inclusion, consider an arbitrary polynomial $q(x) = q_0 + q_1x + q_2x^2 \in \{q \in \mathbb{R}[x]_2 \mid q_2 \geq 0,\ q_0 \geq q_2,\ q_0q_2 \geq q_1^2\}$. We need to show that $\langle p, q\rangle = p_0q_0 + p_1q_1 + p_2q_2 \geq 0$ for all polynomials $p(x) = p_0 + p_1x + p_2x^2 \in C_{1,2}([-1,1])$.

Note that we may assume that $q_0 \neq 0$. If $q_0 = 0$, then we also have $q_2 = 0$ and $q_1 = 0$, and thus, $q \equiv 0$ and $\langle p, q\rangle = 0$ for all $p \in C_{1,2}([-1,1])$.

Also note that since $q_2 \geq 0$, we have $q_0 \geq 0$ and $q_0^2 \geq q_1^2$, which implies that $q_0 \geq |q_1|$. Hence, if $q_0 > 0$, we have $-1 \leq \frac{q_1}{q_0} \leq 1$.

Now, take an arbitrary polynomial $p \in C_{1,2}([-1,1])$. There are three cases to consider.

1. $p_2 \geq 0$.

In this case, we have

$$\begin{aligned}\langle p, q\rangle &= q_0\left(p_0 + p_1\frac{q_1}{q_0} + p_2\frac{q_2}{q_0}\right)\\ &\geq q_0\left(p_0 + p_1\frac{q_1}{q_0} + p_2\frac{q_1^2}{q_0^2}\right)\\ &= q_0 p\left(\frac{q_1}{q_0}\right) \geq 0.\end{aligned}$$

2. $p_2 < 0$ and $p_1 < 0$.

In this case, we define $P(x) = (-p_1 - p_2) + p_1 x + p_2 x^2$. It holds that $p(x) - P(x) = p_0 + p_1 + p_2 = p(1) \geq 0$ for all $x \in \mathbb{R}$, and thus, $\langle p - P, q\rangle = q_0 p(1) \geq 0$, from which we have $\langle p, q\rangle = \langle P, q\rangle + q_0 p(1)$. Now,

$$\begin{aligned}\langle P, q\rangle &= q_0(-p_1 - p_2) + q_1 p_1 + q_2 p_2\\ &= p_1(q_1 - q_0) + p_2(q_2 - q_0) \geq 0\end{aligned}$$

since $p_1 < 0$, $q_1 - q_0 \leq 0$ and $p_2 < 0$, $q_2 - q_0 \leq 0$. Hence,

$$\langle p, q\rangle = \langle P, q\rangle + q_0 p(1) \geq 0.$$

3. $p_2 < 0$ and $p_1 \geq 0$.

In this case, we define $P(x) = (p_1 - p_2) + p_1 x + p_2 x^2$. Again, $p(x) - P(x) = p_0 - p_1 + p_2 = p(-1) \geq 0$ for all $x \in \mathbb{R}$, and again $\langle p - P, q\rangle = q_0 p(-1) \geq 0$, from which we have $\langle p, q\rangle = \langle P, q\rangle + q_0 p(-1)$. Now,

$$\begin{aligned}\langle P, q\rangle &= q_0(p_1 - p_2) + q_1 p_1 + q_2 p_2\\ &= p_1(q_0 + q_1) + p_2(q_2 - q_0) \geq 0\end{aligned}$$

since $p_1 \geq 0$, $q_0 + q_1 \geq 0$ and $p_2 < 0$, $q_2 - q_0 \leq 0$. Hence,

$$\langle p, q\rangle = \langle P, q\rangle + q_0 p(-1) \geq 0.$$

Since p was chosen arbitrarily, we have shown that $\langle p, q\rangle \geq 0$ for all $p \in C_{1,2}([-1,1])$, and thus, $q \in C_{1,2}([-1,1])^*$.

We have found the explicit characterization of $C_{1,2}([-1,1])^*$. In fact,

$$C_{1,2}([-1,1])^* = \left\{q \in \mathbb{R}[x]_2 \mid q_2 \geq 0,\ q_0 \geq q_2,\ q_0 q_2 \geq q_1^2\right\}. \tag{6}$$

From the geometrical point of view, it can be said that $C_{1,2}([-1,1])^*$ is the intersection of a cone isomorphic to the cone of 2×2 symmetric positive semidefinite matrices and a polyhedral cone. More specifically,

$$\begin{aligned}C_{1,2}([-1,1])^* = {}&\{q_0 + q_1 x + q_2 x^2 \in \mathbb{R}[x]_2 \mid q_0, q_2 \geq 0,\ q_0 q_2 \geq q_1^2\}\\ &\cap \{q_0 + q_1 x + q_2 x^2 \in \mathbb{R}[x]_2 \mid q_2 \geq 0,\ q_0 \geq q_2,\ q_1 \in \mathbb{R}\}.\end{aligned} \tag{7}$$

Since $C_{1,2}([-1,1])$ is a closed convex cone (see Propositions 3 and 5), it holds $C_{1,2}([-1,1]) = C_{1,2}([-1,1])^{**}$. Thus the explicit characterization of $C_{1,2}([-1,1])$ can be found by taking the dual of both sides in (7). More specifically,

$$C_{1,2}([-1,1]) = \mathrm{cl}\Big(\big\{q \in \mathbb{R}[x]_2 \mid q_0, q_2 \geq 0,\ q_0q_2 \geq q_1^2\big\}^* + \big\{q \in \mathbb{R}[x]_2 \mid q_2 \geq 0,\ q_0 \geq q_2,\ q_1 \in \mathbb{R}\big\}^*\Big), \tag{8}$$

or after calculating the dual cones,

$$C_{1,2}([-1,1]) = \mathrm{cl}\Big(\big\{p \in \mathbb{R}[x]_2 \mid p_0, p_2 \geq 0,\ p_0p_2 \geq p_1^2/4\big\} + \{p \in \mathbb{R}[x]_2 \mid p_0 \geq 0,\ p_0 + p_2 \geq 0,\ p_1 = 0\}\Big). \tag{9}$$

In fact, the closure operator in (8) and (9) is not needed since the sum of these two cones is closed. Note that the sum of two closed convex cones is closed if the intersection of their relative interiors is non-empty (for more details, see, e.g., [**14**, Section 2.7]). It can be easily verified that $1+x^2$ belongs to the (relative) interiors of both cones.

We finally obtain the characterization of $C_{1,2}([-1,1])$ in the following form:

$$C_{1,2}([-1,1]) = \big\{(p_0 + r_0) + p_1x + (p_2 + r_2)x^2 \in \mathbb{R}[x]_2 \mid p_0, p_2, r_0 \geq 0,\ p_0p_2 \geq p_1^2/4,\ r_0 + r_2 \geq 0\big\}. \tag{10}$$

Note that in (10), one can write $r_0 + r_2x^2 = r_0(1 - x^2) + (r_0 + r_2)x^2$ with $r_0 \geq 0$ and $r_0 + r_2 \geq 0$. Since the set $\{p \in \mathbb{R}[x]_2 \mid p_0, p_2 \geq 0,\ p_0p_2 \geq p_1^2/4\}$ is a convex cone, it holds that if $p_0, p_2 \geq 0$, $p_0p_2 \geq p_1^2/4$, $r_0 \geq 0$, $r_0 + r_2 \geq 0$, then

$$(p_0 + p_1x + p_2x^2) + (0 + 0x + (r_0 + r_2)x^2) \in \{p \in \mathbb{R}[x]_2 \mid p_0, p_2 \geq 0,\ p_0p_2 \geq p_1^2/4\}.$$

Hence, one can rewrite (10) as follows:

$$C_{1,2}([-1,1]) = \big\{p_0 + p_1x + p_2x^2 + r(1 - x^2) \in \mathbb{R}[x]_2 \mid p_0, p_2 \geq 0,\ p_0p_2 \geq p_1^2/4,\ r \geq 0\big\}. \tag{11}$$

Note that from the characterization (11), it is possible to find the characterization of $C_{1,2}([a,b])$, where $a < b$ ($a, b \in \mathbb{R}$), by using an affine change of variables

$$x \mapsto \frac{2}{b-a}x - \frac{a+b}{b-a}.$$

It can be easily verified that if $p \in C_{1,2}([-1,1])$, then

$$q(x) := p\left(\frac{2}{b-a}x - \frac{a+b}{b-a}\right) \geq 0 \quad \text{for all } x \in [a,b], \tag{12}$$

and thus, $q \in C_{1,2}([a,b])$. On the other hand, for every $q \in C_{1,2}([a,b])$, we may observe that

$$p(x) := q\left(\frac{b-a}{2}x + \frac{b+a}{2}\right) \geq 0 \quad \text{for all } x \in [-1,1],$$

and thus, $p \in C_{1,2}([-1,1])$. Thus $q \in C_{1,2}([a,b])$ if and only if q can be written in the form (12) for some $p \in C_{1,2}([-1,1])$.

Now from (11), it follows that $p \in C_{1,2}([-1,1])$ if and only if p can be written as

$$p(x) = s(x) + r(1 - x^2) \quad \text{for all } x \in \mathbb{R},$$

where $s(x) := p_0 + p_1 x + p_2 x^2$ and $p_0, p_2 \geq 0,\ p_0 p_2 \geq p_1^2/4,\ r \geq 0$. Note that $s \in C_{1,2}(\mathbb{R})$, and thus,

$$s\left(\frac{2}{b-a}x - \frac{a+b}{b-a}\right) = s_0 + s_1 x + s_2 x^2 \geq 0 \quad \text{for all } x \in \mathbb{R},$$

so it also holds that $s_0, s_2 \geq 0$ and $s_0 s_2 \geq s_1^2/4$. We obtain that $q \in C_{1,2}([a,b])$ if and only if q can be written in the form

$$\begin{aligned} q(x) &= p\left(\frac{2}{b-a}x - \frac{a+b}{b-a}\right) \\ &= s\left(\frac{2}{b-a}x - \frac{a+b}{b-a}\right) + \frac{4r}{(b-a)^2}(b-x)(x-a) \\ &= s_0 + s_1 x + s_2 x^2 + R(b-x)(a-x) \quad \text{for all } x \in \mathbb{R}, \end{aligned}$$

where $s_0, s_2 \geq 0,\ s_0 s_2 \geq s_1^2/4,\ R \geq 0$. We conclude that

$$\begin{aligned} C_{1,2}([a,b]) = \big\{ & s_0 + s_1 x + s_2 x^2 + R(b-x)(x-a) \in \mathbb{R}[x]_2 \mid \\ & s_0, s_2 \geq 0,\ s_0 s_2 \geq s_1^2/4,\ R \geq 0 \big\}. \end{aligned} \tag{13}$$

Note that we have derived the characterizations (11) using the dual cone theorem and the basic convex analysis and linear algebra results. First, we derived the characterization of $C_{1,2}([-1,1])^*$, and subsequently, we derived the characterization of $C_{1,2}([-1,1])$ by dualizing the characterization of $C_{1,2}([-1,1])^*$. However, usually different methods, requiring powerful results on trigonometric polynomials and complex analysis or cumbersome algebraic manipulations with polynomials (see, e.g., [**15**, p. 259]), are applied to derive such characterizations. The characterizations (11) and (13) are in concordance with the characterizations of Fekete (see, e.g., [**6**, Theorem 2.4] or [**15**, Problem 46, p. 78]).

6. Conclusion

In this paper, we have analyzed the properties of the set of polynomials non-negative on a given non-empty set $K \subseteq \mathbb{R}^n$, which can be encountered e.g. in polynomial optimization problems. We have shown that this set is in fact a convex, closed and solid cone, which shows a link between polynomial and conic optimization. Moreover, if the interior of the set K is non-empty, it is a pointed cone and hence a proper cone.

We have introduced a representation of the respective dual cone and proved the dual cone theorem, owing to which we have managed to find the characterization of the dual cone of the cone of polynomials non-negative on K. Furthermore, we have found the characterizations of $C_{1,2}([-1,1])^*$, using the dual cone theorem; and $C_{1,2}([-1,1])$. In fact, we have shown that $C_{1,2}([-1,1])^*$ is isomorphic to the intersection of two cones: the cone of 2×2 symmetric positive semidefinite matrices and a polyhedral cone. In addition, by a change of variables, we have also found the characterization of $C_{1,2}([a,b])$, where $a < b$ and $a, b \in \mathbb{R}$. These characterizations correspond to the results of Fekete (see [**6**], [**15**]).

Even though no tractable characterization of the cone $C_{n,d}(K)$ is known, a similar approach to the one included in Section 5 may be applied to finding the explicit characterization of at least some of these cones, for instance, $C_{n,2}([-1,1]^n)$.

Appendix

Proof of Proposition 1.
The claim follows from Cauchy-Schwarz inequality applied to a vector of coefficients $(p_\alpha)_{\alpha \in \mathbb{N}_d^n}$ and the vector of monomial basis $m_d(x)$ since

$$p(x) = m_d(x)^T (p_\alpha)_{\alpha \in \mathbb{N}_d^n} \quad \text{for all } x \in \mathbb{R}^n. \qquad \square$$

Proof of Proposition 2.
Since both right-hand side and left-hand side of the inequality are non-negative numbers, we can equivalently prove $(m_d(x)^T m_d(x))^2 \geq m_{2d}(x)^T m_{2d}(x)$ for all $x \in \mathbb{R}^n$ and $d \in \mathbb{N}$. Note that

$$m_{2d}(x)^T m_{2d}(x) = \sum_{\gamma \in \mathbb{N}_{2d}^n} x_1^{2\gamma_1} x_2^{2\gamma_2} \cdots x_n^{2\gamma_n} \quad \text{for all } x \in \mathbb{R}^n,$$

and

$$(m_d(x)^T m_d(x))^2 = \sum_{\alpha, \beta \in \mathbb{N}_d^n} x_1^{2\alpha_1 + 2\beta_1} x_2^{2\alpha_2 + 2\beta_2} \cdots x_n^{2\alpha_n + 2\beta_n} \quad \text{for all } x \in \mathbb{R}^n.$$

Fix an arbitrary $x \in \mathbb{R}^n$ and $d \in \mathbb{N}$. We show that every term included in $m_{2d}(x)^T m_{2d}(x)$ is also included in $(m_d(x)^T m_d(x))^2$. Since both $m_{2d}(x)^T m_{2d}(x)$ and $(m_d(x)^T m_d(x))^2$ are sums of non-negative numbers for any given $x \in \mathbb{R}^n$, we prove that $(m_d(x)^T m_d(x))^2 \geq m_{2d}(x)^T m_{2d}(x)$ for all $x \in \mathbb{R}^n$ and $d \in \mathbb{N}$.

More specifically, we want to show that for any $\gamma \in \mathbb{N}_{2d}^n$, there exist $\alpha, \beta \in \mathbb{N}_d^n$ such that $\alpha + \beta = \gamma$. For an arbitrary but fixed $\gamma \in \mathbb{N}_{2d}^n$, we construct $\alpha \in \mathbb{N}_d^n$ by setting

$$\alpha_i = \begin{cases} \frac{\gamma_i}{2}, & \gamma_i \equiv 0 \pmod 2, \\ \frac{\gamma_i - 1}{2}, & \gamma_i \equiv 1 \pmod 2 \ \wedge \ \sum_{j=1}^{i-1} \alpha_j > \sum_{j=1}^{i-1} (\gamma_j - \alpha_j), \\ \frac{\gamma_i + 1}{2}, & \gamma_i \equiv 1 \pmod 2 \ \wedge \ \sum_{j=1}^{i-1} \alpha_j \leq \sum_{j=1}^{i-1} (\gamma_j - \alpha_j), \end{cases}$$

$i = 1, 2, \ldots, n$. Then $\beta_i = \gamma_i - \alpha_i$, $i = 1, 2, \ldots, n$. We need to show that $\alpha, \beta \in \mathbb{N}_d^n$. It is obvious that $\alpha + \beta = \gamma$ and that $\alpha_i, \beta_i \in \mathbb{N}_0$, $i = 1, 2, \ldots, n$, and therefore, it suffices to show that $\sum_{i=1}^n \alpha_i \leq d$ and $\sum_{i=1}^n \beta_i \leq d$.

Denote o_1 the number of cases when $\gamma_i \equiv 1 \pmod 2 \wedge \sum_{j=1}^{i-1} \alpha_j \leq \sum_{j=1}^{i-1}(\gamma_j - \alpha_j)$, $i = 1, 2, \ldots, n$ and o_2 the number of cases when $\gamma_i \equiv 1 \pmod 2 \wedge \sum_{j=1}^{i-1} \alpha_j > \sum_{j=1}^{i-1}(\gamma_j - \alpha_j)$, $i = 1, 2, \ldots, n$. Then it follows that

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \frac{\gamma_i}{2} + \frac{1}{2}(o_1 - o_2),$$
$$\sum_{i=1}^{n} \beta_i = \sum_{i=1}^{n} \frac{\gamma_i}{2} + \frac{1}{2}(o_2 - o_1).$$

Firstly, we show that $o_1 - o_2 \in \{0, 1\}$. Suppose that there are k odd numbers among $\gamma_1, \gamma_2, \ldots, \gamma_n$, where $k \in \{0, 1, 2, \ldots, n\}$. If $k = 0$, then obviously $o_1 = o_2 = 0$, and thus, $o_1 - o_2 \in \{0, 1\}$. If $k \neq 0$, denote these odd numbers $\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_k}$. If $k \equiv 1 \pmod 2$, then $k = 2l - 1$ for some $l \in \mathbb{N}$ and by construction of α we have $\alpha_{i_1} = \frac{\gamma_{i_1}+1}{2}$, $\alpha_{i_2} = \frac{\gamma_{i_2}-1}{2}$, $\ldots$, $\alpha_{i_{2l-2}} = \frac{\gamma_{i_{2l-2}}-1}{2}$, $\alpha_{i_{2l-1}} = \frac{\gamma_{i_{2l-1}}+1}{2}$. Therefore, $o_1 = l$ and $o_2 = l - 1$, and therefore, $o_1 - o_2 = 1 \in \{0, 1\}$. If $k \equiv 0 \pmod 2$, then $k = 2l$ for some $l \in \mathbb{N}$. Again, by construction of α, we have $\alpha_{i_1} = \frac{\gamma_{i_1}+1}{2}$, $\alpha_{i_2} = \frac{\gamma_{i_2}-1}{2}$, $\ldots$, $\alpha_{i_{2l-1}} = \frac{\gamma_{i_{2l-1}}+1}{2}$, $\alpha_{i_{2l}} = \frac{\gamma_{i_{2l}}-1}{2}$. Therefore, $o_1 = o_2 = l$, and therefore, $o_1 - o_2 = 0 \in \{0, 1\}$.

Since $o_1 - o_2 \in \{0, 1\}$, we have shown that $\sum_{i=1}^{n} \alpha_i \geq \sum_{i=1}^{n} \beta_i$. Now, we show that $\sum_{i=1}^{n} \alpha_i \leq d$. It is evident that $\sum_{i=1}^{n} \alpha_i \leq \sum_{i=1}^{n} \frac{\gamma_i}{2} + \frac{1}{2}$. Moreover, since $\gamma \in \mathbb{N}_{2d}^{n}$, we have $\sum_{i=1}^{n} \gamma_i \leq 2d$. There are two cases to consider:

1. $\sum_{i=1}^{n} \gamma_i \leq 2d - 1 < 2d$.
   It follows automatically that $\sum_{i=1}^{n} \frac{\gamma_i}{2} + \frac{1}{2} \leq d$, and therefore, $\sum_{i=1}^{n} \alpha_i \leq d$.
2. $\sum_{i=1}^{n} \gamma_i = 2d$.
   However, that is possible if and only if $k \equiv 0 \pmod 2$, which means that $o_1 - o_2 = 0$, and therefore, $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \frac{\gamma_i}{2} = d$.

We have finally shown that $d \geq \sum_{i=1}^{n} \alpha_i \geq \sum_{i=1}^{n} \beta_i$, which completes the proof. □

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