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A GENERALIZED STATISTICAL CONVERGENCE IN NEUTROSOPHIC NORMED SPACE

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Abstract. In this paper, we introduce one of the generalized concepts of statistical convergence, namely ρ-statistical convergence and its boundedness in neutrosophic normed space (NNS). We investigate some fundamental properties of the newly introduced notion. Lastly, we introduce ρ-statistical convergence of order α in neutrosophic normed space and establish the relationship of the above convergence methods with some already known convergence methods in NNS.

1. Introduction

Statistical convergence was introduced by Fast [**13**] and Steinhaus [**33**] independently in the same year 1951. Though the notion was first handled as a summability method by Schoenberg [**32**] in 1959. Furthermore, in 1980, Šalát [**30**] researched some topological properties of statistical convergence for sequences of real numbers. Further, it was studied by Fridy [**14**] in 1985. In 1988, Connor [**8**] proved that a strongly p-Cesàro summable sequence for $0 < p < \infty$ is statistical convergence and the converse holds for bounded sequences. Later on, several generalizations and applications of this concept have been investigated by various authors. For more details, one may refer to [**2, 3, 7, 16, 17, 27, 31**].

On the other hand, the concept of fuzzy sets was first introduced by Zadeh [**34**] in 1965, which was an extension of the classical set-theoretical concept. Nowadays it has wide applications in different branches of science and engineering. The theory of fuzzy sets cannot always cope with the lack of knowledge of membership degrees. To overcome the drawbacks, in 1986, Atanassov [**1**] introduced intuitionistic fuzzy sets as an extension of fuzzy sets. Intuitionistic fuzzy sets have been widely used to solve various decision-making problems. Many times, decision-makers face some hesitations besides going to direct approaches (i.e., yes or no) in decision making. In addition, we can obtain a tricomponent outcome in some real events like sports, the procedure for voting, etc. For more details, one may refer to [**9, 19, 25, 26**]. Considering all in 1998, Smarandache [**29**] introduced

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the notion of neutrosophic set as a generalization of both fuzzy set and intuitionistic fuzzy set. An element belonging to a neutrosophic set consists of a triplet, namely truth-membership function (T), indeterminacy-membership function (F), and falsity-membership function (I). A neutrosophic set is determined as a set where every component of the universe has a degree of T, F and I. The notion of fuzzy normed space was introduced by Felbin [**12**] in 1992. Later on, in 2006, the concept of intuitionistic fuzzy normed spaces was introduced by Saadati and Park [**28**]. Furthermore, in 2020, Kirisçi and Simsek [**23**] introduced the notion of statistical convergence in neutrosophic normed linear spaces and investigated some of its properties. For more details, one may refer to [**20, 21, 22**].

The opinion of statistical convergence depends on the density of subsets of the set $\mathbb{N}$ of natural numbers. We say that $\delta(E)$ is the density of a subset E of $\mathbb{N}$ if the following limit exists

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k),$$

where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(\mathbb{N} \setminus E) = 1 - \delta(E)$. A sequence $x = (x_k)$ is said to be statistical convergence [**14**] to ℓ if for every $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\}) = 0.$$

ρ-density [**7**] of a set $E \subset \mathbb{N}$ is defined by

$$\delta_\rho(E) = \lim_{n\to\infty} \frac{1}{\rho_n} |\{k \leq n : k \in E\}|,$$

provided this limit exists, where afterward $\rho = (\rho_n)$ is a non-decreasing sequence of positive real numbers tending to ∞ such that

$$\limsup_n \frac{\rho_n}{n} < \infty, \quad \Delta\rho_n = O(1), \quad \text{and} \quad \Delta\rho_n = \rho_{n+1} - \rho_n, \tag{1}$$

for each positive integer n. It is clear that for $\rho_n = n$, the above definition turns to the definition of natural density. If $x = (x_k)$ is a sequence such that x_k holds property $P(k)$ for all k except a set of ρ-density zero, then we say that x_k holds $P(k)$ for "almost all k according to ρ" and we denote this by "$a.a.k$ (ρ)". A sequence $x = (x_k)$ is said to be ρ-statistical convergence [**7**] to ℓ if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{k \leq n : |x_k - \ell| \geq \varepsilon\}| = 0.$$

If $\rho_n = n$ for all $n \in \mathbb{N}$, then ρ-statistical convergence coincides with statistical convergence. The set of all ρ-statistical convergence sequences is denoted by S_ρ. Cakalli et al. [**6**] introduced the concept of ρ-statistical convergence of order β defined as

$$\lim_{n\to\infty} \frac{1}{\rho_n^\beta} |\{k \leq n : |x_k - \ell| \geq \varepsilon\}| = 0$$

for each $\varepsilon > 0$ and $0 < \beta \leq 1$. The concept of statistical boundedness was given by Fridy et al. [**15**] as follows: a real number sequence $x = (x_k)$ is statistically bounded if there exists a number $M \geq 0$ such that

$$\delta\left(\{k : |x_k| > M\}\right) = 0.$$

The set of all statistically bounded sequences is denoted by $S\,(b)$. It can be shown that every bounded sequence is statistically bounded, but the converse is not true. For this, consider a sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} k & \text{if } k \text{ is a square,} \\ 1 & \text{if } k \text{ is not a square.} \end{cases}$$

Clearly, $x = (x_k)$ is not a bounded sequence, but it is statistically bounded. For further generalization on the concept of statistical boundedness, one may refer to [**4, 5, 10, 11**].

In this paper, we introduce ρ-statistical convergence and ρ-statistical boundedness on neutrosophic normed spaces and investigate some of their properties. We also introduce ρ-statistical convergence of order α as an extension of ρ-statistical convergence.

2. Definitions and preliminaries

Definition 2.1 ([**24**])**.** A binary operation $\circ\colon [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if the following conditions are satisfied:

1. $\circ$ is associative and commutative,
2. $\circ$ is continuous,
3. $s \circ 1 = s$ for all $s \in [0,1]$,
4. $s \circ t \leq u \circ v$ whenever $s \leq u$ and $t \leq v$ for all $s,t,u,v \in [0,1]$.

Definition 2.2 ([**24**])**.** A binary operation $\bullet\colon [0,1] \times [0,1] \to [0,1]$, is said to be a continuous t-conorm if the following conditions are satisfied:

1. $\bullet$ is associative and commutative,
2. $\bullet$ is continuous,
3. $s \bullet 0 = s$ for all $s \in [0,1]$,
4. $s \bullet t \leq u \bullet v$ whenever $s \leq u$ and $t \leq v$ for all $s,t,u,v \in [0,1]$.

Definition 2.3 ([**23**])**.** Let V be a vector space and

$$\mathcal{N} = \{\langle u, \mathcal{R}(u), \mathcal{T}(u), \mathcal{W}(u)\rangle : u \in V\}$$

be a normed space (NS) such that $\mathcal{R}, \mathcal{T}, \mathcal{W}\colon V \times \mathbb{R}^+ \to [0,1]$. Let $\circ$ and $\bullet$ be the continuous t-norm and continuous t-conorm, respectively. Then the four-tuple $(V, \mathcal{N}, \circ, \bullet)$ is called neutrosophic normed space (NNS) if the following conditions hold, for all $u, v \in V$ and $\lambda, \mu > 0$ and for each $\sigma \neq 0$:

1. $0 \leq \mathcal{R}(u,\lambda) \leq 1$, $0 \leq \mathcal{T}(u,\lambda) \leq 1$, $0 \leq \mathcal{W}(u,\lambda) \leq 1$,
2. $0 \leq \mathcal{R}(u,\lambda) + \mathcal{T}(u,\lambda) + \mathcal{W}(u,\lambda) \leq 3$,
3. $\mathcal{R}(u,\lambda) = 1$ (for $\lambda > 0$) if and only if $u = 0$,
4. $\mathcal{R}(\sigma u,\lambda) = \mathcal{R}(u, \frac{\lambda}{|\sigma|})$,

5. $\mathcal{R}(u,\lambda)\circ\mathcal{R}(v,\mu)\le\mathcal{R}(u+v,\lambda+\mu)$,
6. $\mathcal{R}(u,.)$ is a continuous and non-decreasing function,
7. $\lim_{\lambda\to\infty}\mathcal{R}(u,\lambda)=1$,
8. $\mathcal{T}(u,\lambda)=0$ (for $\lambda>0$) if and only if $u=0$,
9. $\mathcal{T}(\sigma u,\lambda)=\mathcal{T}(u,\frac{\lambda}{|\sigma|})$,
10. $\mathcal{T}(u,\lambda)\bullet\mathcal{T}(v,\mu)\ge\mathcal{T}(u+v,\lambda+\mu)$,
11. $\mathcal{T}(u,.)$ is a continuous and non-increasing function,
12. $\lim_{\lambda\to\infty}\mathcal{T}(u,\lambda)=0$,
13. $\mathcal{W}(u,\lambda)=0$ (for $\lambda>0$) if and only if $u=0$,
14. $\mathcal{W}(\sigma u,\lambda)=\mathcal{W}(u,\frac{\lambda}{|\sigma|})$,
15. $\mathcal{W}(u,\lambda)\bullet\mathcal{W}(v,\mu)\ge\mathcal{W}(u+v,\lambda+\mu)$,
16. $\mathcal{W}(u,.)$ is a continuous and non-increasing function,
17. $\lim_{\lambda\to\infty}\mathcal{W}(u,\lambda)=0$,
18. if $\lambda\le 0$, then $\mathcal{R}(u,\lambda)=0,\mathcal{T}(u,\lambda)=1$ and $\mathcal{W}(u,\lambda)=1$.

Then, $\mathcal{N}=(\mathcal{R},\mathcal{T},\mathcal{W})$ is a neutrosophic norm.

Example 2.4 ([**18**])**.** Suppose $(V,\|\cdot\|)$ to be a normed space. For $s,t\in[0,1]$, define the t-norm $\circ$ and the t-conorm $\bullet$ as $s\circ t=st$ and $s\bullet t=s+t-st$, respectively. For $\lambda>\|u\|$, let

$$\mathcal{R}(u,\lambda)=\frac{\lambda-\|u\|}{\lambda+\|u\|},\quad \mathcal{T}(u,\lambda)=\frac{\|u\|}{\lambda+\|u\|},\quad \mathcal{W}(u,\lambda)=\frac{\|u\|}{\lambda}\quad \text{for all } u(\neq 0)\in V.$$

Then, $(V,\mathcal{N},\circ,\bullet)$ is a neutrosophic normed space (NNS).

Definition 2.5 ([**18**])**.** Let V be an NNS. A sequence (x_k) of V is said to be statistical convergence to ℓ with respect to the $\mathcal{N}$, if for every $0<\varepsilon<1$ and $\lambda>0$,

$$\delta(\{k\in\mathbb{N}:\mathcal{R}(x_k-\ell,\lambda)\le 1-\varepsilon\ \text{ or }\ \mathcal{T}(x_k-\ell,\lambda)\ge\varepsilon,\mathcal{W}(x_k-\ell,\lambda)\ge\varepsilon\})=0,$$

or equivalently,

$$\lim_{n\to\infty}\frac{1}{n}|\{k\le n:\mathcal{R}(x_k-\ell,\lambda)\le 1-\varepsilon\ \text{ or }\ \mathcal{T}(x_k-\ell,\lambda)\ge\varepsilon,\mathcal{W}(x_k-\ell,\lambda)\ge\varepsilon\}|=0.$$

Symbolically, it is denoted as st-$\mathcal{N}$-$\lim x_k=\ell$ or $x_k\to\ell(st\text{-}\mathcal{N})$. We denote the set of all statistical convergence sequences in NNS V by $st(\mathcal{N})$.

Definition 2.6. Let V be a NNS. A sequence (x_k) of V is said to be statistically bounded if there exists some $t_0>0$ and $b\in(0,1)$ such that

$$\delta(\{k\in\mathbb{N}:\mathcal{R}(x_k,t_0)>1-b\ \text{ or }\ \mathcal{T}(x_k,t_0)<b,\mathcal{W}(x_k,t_0)<b\})=0,$$

or equivalently,

$$\lim_{n\to\infty}\frac{1}{n}|\{k\in\mathbb{N}:\mathcal{R}(x_k,t_0)>1-b\ \text{ or }\ \mathcal{T}(x_k,t_0)<b,\mathcal{W}(x_k,t_0)<b\}|=0.$$

We denote the set of all statistically bounded sequences in a NNS V by $st^{\mathcal{N}}(b)$.

3. Main results

The main results is divided into three subsections. Firstly, we introduce ρ-statistical convergence, secondly, ρ-statistical boundedness on NNS, exploring several of their properties. Lastly, we introduce ρ-statistical convergence of order α in NNS.

3.1. ρ-statistical convergence on NNS

Definition 3.1. Let V be a NNS. A sequence (x_k) of V is said to be ρ-statistical convergence to ℓ with respect to the $\mathcal{N}$, if for every $0 < \varepsilon < 1$ and $\lambda > 0$,

$$\delta_\rho(\{k \in \mathbb{N} : \mathcal{R}(x_k - \ell, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k - \ell, \lambda) \geq \varepsilon, \mathcal{W}(x_k - \ell, \lambda) \geq \varepsilon\}) = 0,$$

or equivalently,

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{k \leq n : \mathcal{R}(x_k - \ell, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k - \ell, \lambda) \geq \varepsilon, \mathcal{W}(x_k - \ell, \lambda) \geq \varepsilon\}| = 0.$$

Symbolically, it is denoted as $st_\rho\text{-}\mathcal{N}\text{-}\lim x_k = \ell$ or $x_k \to \ell(st_\rho\text{-}\mathcal{N})$. We denote the set of all ρ-statistical convergence sequence in a NNS V by $st_\rho(\mathcal{N})$.

Definition 3.2. Let V be a NNS and (x_k) be a sequence of V. Then, (x_k) is said to be ρ-statistical Cauchy if for any $0 < \varepsilon < 1$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\delta_\rho(\{k \in \mathbb{N} : \mathcal{R}(x_k - x_N, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k - x_N, \lambda) \geq \varepsilon,\ \mathcal{W}(x_k - x_N, \lambda) \geq \varepsilon\}) = 0,$$

or equivalently,

$$\lim_{n\to\infty} \frac{1}{\rho_n}\Big|\{k \leq n : \mathcal{R}(x_k - x_N, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k - x_N, \lambda) \geq \varepsilon, \mathcal{W}(x_k - x_N, \lambda) \geq \varepsilon\}\Big| = 0.$$

Example 3.3. Let $V = \mathbb{R}$. For all $s, t \in [0, 1]$, define the continuous t-norm $s \circ t = st$ and the continuous t-conorm $s \bullet t = \min\{s + t, 1\}$. We take $\mathcal{R}, \mathcal{T}, \mathcal{W}$ in Example 2.4 for all $\lambda > 0$. Then, we define the sequence (x_k) as

$$x_k = \begin{cases} k, & [\rho_n] - 1 < k \leq [\rho_n],\ n = 1, 2, 3, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $x_k \to 0(st_\rho\text{-}\mathcal{N})$.

Justification: For every $0 < \varepsilon < 1$, we have

$$K_\varepsilon = \{k \in \mathbb{N} : \mathcal{R}(x_k - 0, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k - 0, \lambda) \geq \varepsilon, \mathcal{W}(x_k - 0, \lambda) \geq \varepsilon\}.$$

This implies

$$\begin{aligned} K_\varepsilon &= \left\{k \in \mathbb{N} : \frac{\lambda - |x_k|}{\lambda + |x_k|} \leq 1 - \varepsilon \text{ or } \frac{|x_k|}{\lambda + |x_k|} \geq \varepsilon,\ \frac{|x_k|}{\lambda} \geq \varepsilon \right\} \\ &= \left\{k \in \mathbb{N} : |x_k| \geq \frac{\lambda\varepsilon}{2 - \varepsilon} \text{ or } |x_k| \geq \frac{\lambda\varepsilon}{1 - \varepsilon},\ |x_k| \geq \lambda\varepsilon \right\} \\ &\subseteq \left\{k \in \mathbb{N} : |x_k| \geq \frac{\lambda\varepsilon}{1 - \varepsilon} \text{ or } |x_k| \geq \lambda\varepsilon \right\} \end{aligned}$$

$$\begin{aligned}&\subseteq \left(\left\{k : |x_k| > \frac{\lambda\varepsilon}{1-\varepsilon}\right\}\right)\\&= \{k \in \mathbb{N} : x_k = k\}\\&= \{k \in \mathbb{N} : [\rho_n] - 1 < k \leq [\rho_n]\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&|K_\varepsilon| \leq |\{k \in \mathbb{N} : [\rho_n] - 1 < k \leq [\rho_n]\}|\\\implies &\frac{1}{\rho_n}|K_\varepsilon| \leq \frac{1}{\rho_n}|\{k \in \mathbb{N} : [\rho_n] - 1 < k \leq [\rho_n]\}|\\\implies &\lim_{n\to\infty}\frac{1}{\rho_n}|K_\varepsilon| \leq \lim_{n\to\infty}\frac{1}{\rho_n}|\{k \in \mathbb{N} : [\rho_n] - 1 < k \leq [\rho_n]\}| = 0.\end{aligned}$$

Hence, $x_k \to 0(st_\rho\text{-}\mathcal{N})$.

Theorem 3.4. *Let $\rho = (\rho_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ and satisfying condition (1). Let $x = (x_n)$ and $y = (y_n)$ be the sequences of a NNS V. Then*

(i) $x_k \to \ell(st_\rho\text{-}\mathcal{N})$ *implies* $cx_k \to c\,\ell(st_\rho\text{-}\mathcal{N})$,

(ii) $x_k \to \ell_1(st_\rho\text{-}\mathcal{N})$ *and* $y_k \to \ell_2(st_\rho\text{-}\mathcal{N})$ *imply* $(x_k + y_k) \to (\ell_1 + \ell_2)(st_\rho\text{-}\mathcal{N})$.

Proof. (i) Proof is clear for $c = 0$. Let $c \neq 0$. We can write

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{\rho_n}|\{k \leq n : \mathcal{R}(cx_k - c\,\ell, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(cx_k - c\,\ell, \lambda) \geq \varepsilon,\\&\qquad \mathcal{W}(cx_k - c\,\ell, \lambda) \geq \varepsilon\}|\\&\leq \lim_{n\to\infty}\frac{1}{\rho_n}\left|\left\{k \leq n : \mathcal{R}(x_k - \ell, \lambda) \leq \frac{1-\varepsilon}{|c|} \text{ or } \mathcal{T}(x_k - \ell, \lambda) \geq \frac{\varepsilon}{|c|},\right.\right.\\&\qquad \left.\left.\mathcal{W}(x_k - \ell, \lambda) \geq \frac{\varepsilon}{|c|}\right\}\right|,\end{aligned}$$

so $x_k \to \ell(st_\rho - \mathcal{N})$ implies $cx_k \to c\,\ell(st_\rho - \mathcal{N})$.

(ii)

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{\rho_n}|\{k \leq n : \mathcal{R}((x_k + y_k) - (\ell_1 + \ell_2), \lambda) \leq 1 - \varepsilon \text{ or}\\&\qquad \mathcal{T}((x_k + y_k) - (\ell_1 + \ell_2), \lambda) \geq \varepsilon,\\&\qquad \mathcal{W}((x_k + y_k) - (\ell_1 + \ell_2), \lambda) \geq \varepsilon\}|\\&\leq \lim_{n\to\infty}\frac{1}{\rho_n}\left|\left\{k \leq n : \mathcal{R}(x_k - \ell_1, \lambda) \leq 1 - \frac{\varepsilon}{2} \text{ or } \mathcal{T}(x_k - \ell_2, \lambda) \geq \frac{\varepsilon}{2},\right.\right.\\&\qquad \left.\left.\mathcal{W}(x_k - \ell_2, \lambda) \geq \frac{\varepsilon}{2}\right\}\right|\\&\quad + \lim_{n\to\infty}\frac{1}{\rho_n}\left|\left\{k \leq n : \mathcal{R}(y_k - \ell_1, \lambda) \leq 1 - \frac{\varepsilon}{2} \text{ or } \mathcal{T}(y_k - \ell_2, \lambda) \geq \frac{\varepsilon}{2},\right.\right.\\&\qquad \left.\left.\mathcal{W}(y_k - \ell_2, \lambda) \geq \frac{\varepsilon}{2}\right\}\right|.\end{aligned}$$

So, we have $x_k \to \ell_1(st_\rho\text{-}\mathcal{N})$ and $y_k \to \ell_2(st_\rho\text{-}\mathcal{N})$, that implies $(x_k + y_k) \to (\ell_1 + \ell_2)(st_\rho\text{-}\mathcal{N})$. □

Lemma 3.5. *Let V be a NNS. Then, for any $0 < \varepsilon < 1$, the following statements are equivalent:*

1. $x_k \to \ell(st_\rho\text{-}\mathcal{N})$;
2. $$\begin{aligned}\delta_\rho(\{k \in \mathbb{N} : \mathcal{R}(x_k - \ell, \lambda) \leq 1 - \varepsilon\}) &= \delta_\rho(\{k \in \mathbb{N} : \mathcal{T}(x_k - \ell, \lambda) \geq \varepsilon\}) \\ &= \delta_\rho(\{k \in \mathbb{N} : \mathcal{W}(x_k - \ell, \lambda) \geq \varepsilon\}) = 0;\end{aligned}$$
3. $$\begin{aligned}\delta_\rho(\{k \in \mathbb{N} : \mathcal{R}(x_k - \ell, \lambda) > 1 - \varepsilon\}) &= \delta_\rho(\{k \in \mathbb{N} : \mathcal{T}(x_k - \ell, \lambda) < \varepsilon\}) \\ &= \delta_\rho(\{k \in \mathbb{N} : \mathcal{W}(x_k - \ell, \lambda) < \varepsilon\}) = 1;\end{aligned}$$
4. $\mathcal{R}(x_k - \ell, \lambda) \to 1(st_\rho\text{-}\mathcal{N})$, $\mathcal{T}(x_k - \ell, \lambda) \to 0(st_\rho\text{-}\mathcal{N})$, $\mathcal{W}(x_k - \ell, \lambda) \to 0(st_\rho\text{-}\mathcal{N})$.

Theorem 3.6. *Let $\rho = (\rho_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ and satisfying condition (1). If $\lim_n \inf(\frac{\rho_n}{n}) \geq 1$, then $st(\mathcal{N}) \subseteq st_\rho(\mathcal{N})$ with respect to neutrosophic norm $\mathcal{N}$.*

Proof. If $x_k \to \ell(st\text{-}\mathcal{N})$, then for every $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : \mathcal{R}(x_k - \ell, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k - \ell, \lambda) \geq \varepsilon,\ \mathcal{W}(x_k - \ell, \lambda) \geq \varepsilon\}| = 0.$$

Now,

$$\begin{aligned}&\frac{1}{n} |\{k \leq n : \mathcal{R}(x_k - \ell, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k - \ell, \lambda) \geq \varepsilon,\ \mathcal{W}(x_k - \ell, \lambda) \geq \varepsilon\}| \\ &= \frac{\rho_n}{n} \frac{1}{\rho_n} |\{k \leq n : \mathcal{R}(x_k - \ell, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k - \ell, \lambda) \geq \varepsilon,\ \mathcal{W}(x_k - \ell, \lambda) \geq \varepsilon\}| \\ &\geq \inf(\frac{\rho_n}{n}) \frac{1}{\rho_n} |\{k \leq n : \mathcal{R}(x_k - \ell, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k - \ell, \lambda) \geq \varepsilon,\ \mathcal{W}(x_k - \ell, \lambda) \geq \varepsilon\}| \\ &\geq \frac{1}{\rho_n} |\{k \leq n : \mathcal{R}(x_k - \ell, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k - \ell, \lambda) \geq \varepsilon,\ \mathcal{W}(x_k - \ell, \lambda) \geq \varepsilon\}| .\end{aligned}$$

Therefore,

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : \mathcal{R}(x_k - \ell, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k - \ell, \lambda) \geq \varepsilon,\ \mathcal{W}(x_k - \ell, \lambda) \geq \varepsilon\}| \\ &\geq \lim_{n\to\infty} \frac{1}{\rho_n} |\{k \leq n : \mathcal{R}(x_k - \ell, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k - \ell, \lambda) \geq \varepsilon,\ \mathcal{W}(x_k - \ell, \lambda) \geq \varepsilon\}|.\end{aligned}$$

Hence, $st(\mathcal{N}) \subseteq st_\rho(\mathcal{N})$. □

Theorem 3.7. *Let $\rho = (\rho_n)$ and $\tau = (\tau_n)$ be two sequences, both satisfying condition (1), and $\rho_n \leq \tau_n$ for all $n \in \mathbb{N}$. If (x_k) is a sequence in a NNS V such that $x_k \to \ell(st_\rho\text{-}\mathcal{N})$, then $x_k \to \ell(st_\tau\text{-}\mathcal{N})$.*

Proof. By our assumption, for any $0 < \varepsilon < 1$,

$$\delta_\rho(\{k \in \mathbb{N} : \mathcal{R}(x_k - \ell, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k - \ell, \lambda) \geq \varepsilon,\ \mathcal{W}(x_k - \ell, \lambda) \geq \varepsilon\}) = 0, \text{ i.e.,}$$

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{k \leq n : \mathcal{R}(x_k - \ell, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k - \ell, \lambda) \geq \varepsilon,\ \mathcal{W}(x_k - \ell, \lambda) \geq \varepsilon\}| = 0.$$

Now, since $\rho_n \leq \tau_n$ holds for all $n \in \mathbb{N}$, so we must have

$$\frac{1}{\tau_n}\left|\{k \leq n : \mathcal{R}(x_k-\ell,\lambda) \leq 1-\varepsilon \text{ or } \mathcal{T}(x_k-\ell,\lambda) \geq \varepsilon,\ \mathcal{W}(x_k-\ell,\lambda) \geq \varepsilon\}\right|$$
$$\leq \frac{1}{\rho_n}\left|\{k \leq n : \mathcal{R}(x_k-\ell,\lambda) \leq 1-\varepsilon \text{ or } \mathcal{T}(x_k-\ell,\lambda) \geq \varepsilon,\ \mathcal{W}(x_k-\ell,\lambda) \geq \varepsilon\}\right|.$$

Hence, $x_k \to \ell(st_\tau\text{-}\mathcal{N})$. □

Theorem 3.8. *In a NNS V, limit of ρ-statistical convergence sequence is unique.*

Proof. If possible, suppose $x_k \to \ell_1(st_\rho\text{-}\mathcal{N})$ and $x_k \to \ell_2(st_\rho\text{-}\mathcal{N})$ for $l_1 \neq l_2$. Then, for a given $0 < \varepsilon < 1$, we can choose $\mu > 0$ such that $(1-\varepsilon)\circ(1-\varepsilon) > 1-\mu$ and $\varepsilon \bullet \varepsilon < \mu$. Now, for any $\lambda > 0$, we define the following sets:

$$K_{\mathcal{R}_1}(\varepsilon,\lambda) = \left\{k \in \mathbb{N} : \mathcal{R}\Big(x_k-\ell_1,\frac{\lambda}{2}\Big) \leq 1-\varepsilon\right\},$$
$$K_{\mathcal{R}_2}(\varepsilon,\lambda) = \left\{k \in \mathbb{N} : \mathcal{R}\Big(x_k-\ell_2,\frac{\lambda}{2}\Big) \leq 1-\varepsilon\right\},$$
$$K_{\mathcal{T}_1}(\varepsilon,\lambda) = \left\{k \in \mathbb{N} : \mathcal{T}\Big(x_k-\ell_1,\frac{\lambda}{2}\Big) \geq \varepsilon\right\},$$
$$K_{\mathcal{T}_2}(\varepsilon,\lambda) = \left\{k \in \mathbb{N} : \mathcal{T}\Big(x_k-\ell_2,\frac{\lambda}{2}\Big) \geq \varepsilon\right\},$$
$$K_{\mathcal{W}_1}(\varepsilon,\lambda) = \left\{k \in \mathbb{N} : \mathcal{W}\Big(x_k-\ell_1,\frac{\lambda}{2}\Big) \geq \varepsilon\right\},$$
$$K_{\mathcal{W}_2}(\varepsilon,\lambda) = \left\{k \in \mathbb{N} : \mathcal{W}\Big(x_k-\ell_2,\frac{\lambda}{2}\Big) \geq \varepsilon\right\}.$$

Since $x_k \to \ell_1(st_\rho\text{-}\mathcal{N})$, by Lemma 3.5, for any $\lambda > 0$, we have

$$\delta_\rho(K_{\mathcal{R}_1}(\varepsilon,\lambda)) = \delta_\rho(K_{\mathcal{T}_1}(\varepsilon,\lambda)) = \delta_\rho(K_{\mathcal{W}_1}(\varepsilon,\lambda)) = 0.$$

Again, since $x_k \to \ell_2(st_\rho\text{-}\mathcal{N})$, by Lemma 3.5, for any $\lambda > 0$, we have

$$\delta_\rho(K_{\mathcal{R}_2}(\varepsilon,\lambda)) = \delta_\rho(K_{\mathcal{T}_2}(\varepsilon,\lambda)) = \delta_\rho(K_{\mathcal{W}_2}(\varepsilon,\lambda)) = 0.$$

Now, let $K(\varepsilon,\lambda) = (K_{\mathcal{R}_1}(\varepsilon,\lambda) \cup K_{\mathcal{R}_2}(\varepsilon,\lambda)) \cap (K_{\mathcal{T}_1}(\varepsilon,\lambda) \cup K_{\mathcal{T}_2}(\varepsilon,\lambda)) \cap (K_{\mathcal{W}_1}(\varepsilon,\lambda) \cup K_{\mathcal{W}_2}(\varepsilon,\lambda))$. Then, we have $\delta_\rho(K(\varepsilon,\lambda)) = 0$ and eventually $\delta_\rho(\mathbb{N} \setminus K(\varepsilon,\lambda)) \neq 0$, and therefore, $\mathbb{N} \setminus K(\varepsilon,\lambda)$ is non-empty. Choose $p \in \mathbb{N} \setminus K(\varepsilon,\lambda)$. Then, there are three possibilities:

(i) $p \in (\mathbb{N} \setminus (K_{\mathcal{R}_1}(\varepsilon,\lambda)) \cap (\mathbb{N} \setminus (K_{\mathcal{R}_2}(\varepsilon,\lambda))$,
(ii) $p \in (\mathbb{N} \setminus (K_{\mathcal{T}_1}(\varepsilon,\lambda)) \cap (\mathbb{N} \setminus (K_{\mathcal{T}_2}(\varepsilon,\lambda))$, and
(iii) $p \in (\mathbb{N} \setminus (K_{\mathcal{W}_1}(\varepsilon,\lambda)) \cap (\mathbb{N} \setminus (K_{\mathcal{W}_2}(\varepsilon,\lambda))$.

If we consider case (i), then we have the following:

$$\mathcal{R}(\ell_1-\ell_2,\lambda) \geq \mathcal{R}\Big(x_k-\ell_1,\frac{\lambda}{2}\Big) \circ \mathcal{R}\Big(x_k-\ell_2,\frac{\lambda}{2}\Big) > (1-\varepsilon)\circ(1-\varepsilon) > 1-\mu. \tag{2}$$

Now, since μ is arbitrary, so from equation (2), for any $\lambda > 0$, we obtain $\mathcal{R}(\ell_1 - \ell_2, \lambda) = 1$, i.e., $\ell_1 = \ell_2$.

If we consider case (ii), then we have the following:

$$\mathcal{T}(\ell_1 - \ell_2, \lambda) \leq \mathcal{T}\Big(x_k - \ell_1, \frac{\lambda}{2}\Big) \bullet \mathcal{T}\Big(x_k - \ell_2, \frac{\lambda}{2}\Big) < \varepsilon \bullet \varepsilon < \mu. \tag{3}$$

Now, since μ is arbitrary, so from equation (3), for any $\lambda > 0$, we obtain $\mathcal{T}(\ell_1 - \ell_2, \lambda) = 1$, i.e., $\ell_1 = \ell_2$.

Again, if we consider case (iii), then we have the following

$$\mathcal{W}(\ell_1 - \ell_2, \lambda) \leq \mathcal{W}\Big(x_k - \ell_1, \frac{\lambda}{2}\Big) \bullet \mathcal{W}\Big(x_k - \ell_2, \frac{\lambda}{2}\Big) < \varepsilon \bullet \varepsilon < \mu. \tag{4}$$

Now, since μ is arbitrary, so from equation (4), for any $\lambda > 0$, we obtain $\mathcal{W}(\ell_1 - \ell_2, \lambda) = 0$, i.e., $\ell_1 = \ell_2$. This completes the proof. □

3.2. ρ-statistical boundedness on NNS

Definition 3.9. A sequence (x_k) in a NNS V is said to be ρ-statistically bounded if there exists some $t_0 > 0$ and $b \in (0,1)$ such that

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{k \in \mathbb{N} : \mathcal{R}(x_k, t_0) > 1 - b \text{ or } \mathcal{T}(x_k, t_0) < b,\ \mathcal{W}(x_k, t_0) < b\}| = 0.$$

We denote the set of all ρ-statistically bounded sequences in a NNS V by $st_\rho^{\mathcal{N}}(b)$.

Theorem 3.10. *Every ρ-statistical convergence sequence (x_k) in a NNS V is ρ-statistically bounded.*

Proof. Let (x_k) be ρ-statistical convergence to $x \in V$ with respect to neutrosophic norm $\mathcal{N}$. Then for every $\lambda > 0$, we have

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{k \in \mathbb{N} : \mathcal{R}(x_k - x, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k - x, \lambda) \geq \varepsilon,\ \mathcal{W}(x_k - x, \lambda) \geq \varepsilon\}| = 0.$$

Let m be any element of $\{k \in \mathbb{N} : \mathcal{R}(x_k, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k, \lambda) \geq \varepsilon, \mathcal{W}(x_k, \lambda) \geq \varepsilon\}$. Then,

$$\mathcal{R}(x_m, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_m, \lambda) \geq \varepsilon, \mathcal{W}(x_m, \lambda) \geq \varepsilon.$$

Now,

$$\mathcal{R}(x_m - x, \lambda) \leq \mathcal{R}(x_m, \lambda) \bullet \mathcal{R}(x, 0) \leq (1 - \varepsilon) \bullet 0 \leq 1 - \varepsilon, \tag{5}$$

$$\mathcal{T}(x_m - x, \lambda) \geq \mathcal{T}(x_m, \lambda) \circ \mathcal{T}(x, 0) \geq \varepsilon \circ 1 \geq \varepsilon, \text{ and} \tag{6}$$

$$\mathcal{W}(x_m - x, \lambda) \geq \mathcal{W}(x_m, \lambda) \circ \mathcal{W}(x, 0) \geq \varepsilon \circ 1 \geq \varepsilon. \tag{7}$$

From (5), (6), and (7), we have that m is an element of

$$\{k \in \mathbb{N} : \mathcal{R}(x_k - x, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k - x, \lambda) \geq \varepsilon, \mathcal{W}(x_k - x, \lambda) \geq \varepsilon\}.$$

Thus,

$$\begin{aligned}&\{k \in \mathbb{N} : \mathcal{R}(x_k, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k, \lambda) \geq \varepsilon,\ \mathcal{W}(x_k, \lambda) \geq \varepsilon\}\\ &\subseteq \{k \in \mathbb{N} : \mathcal{R}(x_k - x, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k - x, \lambda) \geq \varepsilon,\ \mathcal{W}(x_k - x, \lambda) \geq \varepsilon\},\end{aligned}$$

which implies

$$\lim_{n\to\infty}\frac{1}{\rho_n}\left|\{k\in\mathbb{N}:\mathcal{R}(x_k,\lambda)\leq 1-\varepsilon \text{ or } \mathcal{T}(x_k,\lambda)\geq\varepsilon,\ \mathcal{W}(x_k,\lambda)\geq\varepsilon\}\right|$$
$$\leq \lim_{n\to\infty}\frac{1}{\rho_n}\left|\{k\in\mathbb{N}:\mathcal{R}(x_k-x,\lambda)\leq 1-\varepsilon \text{ or } \mathcal{T}(x_k-x,\lambda)\geq\varepsilon,\ \mathcal{W}(x_k-x,\lambda)\geq\varepsilon\}\right|.$$

So that,

$$\lim_{n\to\infty}\frac{1}{\rho_n}\left|\{k\in\mathbb{N}:\mathcal{R}(x_k,\lambda)>1-\varepsilon \text{ or } \mathcal{T}(x_k,\lambda)<\varepsilon,\ \mathcal{W}(x_k,\lambda)<\varepsilon\}\right|=0,$$

for a.a.k, i.e., (x_k) ρ-statistically bounded in a NNS V. □

But, the converse of the above theorem is not true. For this, let us consider the following example.

Example 3.11. Let $V=\mathbb{R}$, and consider a real sequence (x_k) defined by

$$x_k=\begin{cases}4k & \text{if k is an odd square,}\\ 3 & \text{if k is an even square,}\\ 1 & \text{if k is an odd nonsquare,}\\ 0 & \text{if k is an even nonsquare.}\end{cases}$$

We take $\mathcal{R},\mathcal{T},\mathcal{W}$ as in Example 2.4, i.e., $\mathcal{R}(x_k,\lambda)=\frac{\lambda-|x_k|}{\lambda+|x_k|}$, $\mathcal{T}(x_k,\lambda)=\frac{|x_k|}{\lambda+|x_k|}$, $\mathcal{W}(x_k,\lambda)=\frac{|x_k|}{\lambda}$ for all $x_k(\neq 0)\in\mathbb{R}$.

The above sequence is clearly unbounded with respect to $\mathcal{N}$. On the other hand, it is ρ-statistically bounded with respect to $\mathcal{N}$. For this, we have

$$\begin{aligned}&\delta_\rho(\{k:\mathcal{R}(x_k,\lambda)>1-b \text{ or } \mathcal{T}(x_k,\lambda)<b,\mathcal{W}(x_k,\lambda)<b\})\\ &=\delta_\rho\left(\left\{\frac{\lambda-|x_k|}{\lambda+|x_k|}>1-b \text{ or } \frac{|x_k|}{\lambda+|x_k|}<b,\frac{|x_k|}{\lambda}<b\right\}\right)\\ &\leq\delta_\rho\Big(\Big\{k:|x_k|>\frac{b\lambda}{1-b}\ or\ |x_k|>b\lambda\Big\}\Big)\\ &\leq\delta_\rho\Big(\Big\{k:|x_k|>\frac{b\lambda}{1-b}\Big\}\Big).\end{aligned}$$

Choose $\lambda=\frac{2(1-b)}{b}$. Then for $\lambda>0$, we have

$$\begin{aligned}&\delta_\rho(\{k:\mathcal{R}(x_k,\lambda)<1-b \text{ or } \mathcal{T}(x_k,\lambda)>b,\ \mathcal{W}(x_k,\lambda)>b\})\\ &=\delta_\rho\Big(\Big\{k:|x_k|>\frac{b}{1-b}\times\frac{2(1-b)}{b}=2\Big\}\Big)\\ &=\delta_\rho\left(\{k:|x_k|>2\}\right)=0.\end{aligned}$$

Hence, it is ρ-statistically bounded with respect to $\mathcal{N}$. In a NN space, every statistical convergence sequence is ρ-statistical convergence. So, if we take $\rho_n=n$, then for every $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{1}{n}\left|\{k\in\mathbb{N}:\mathcal{R}(x_k-0,\lambda)\leq 1-\varepsilon \text{ or } \mathcal{T}(x_k-0,\lambda)\geq\varepsilon,\ \mathcal{W}(x_k-0,\lambda)\geq\varepsilon\}\right|\neq 0,$$

$$\lim_{n\to\infty}\frac{1}{n}|\{k\in\mathbb{N}:\mathcal{R}(x_k-1,\lambda)\le 1-\varepsilon \text{ or } \mathcal{T}(x_k-1,\lambda)\ge\varepsilon,\ \mathcal{W}(x_k-1,\lambda)\ge\varepsilon\}|\neq 0,$$

$$\text{and } \lim_{n\to\infty}\frac{1}{n}|\{k\in\mathbb{N}:\mathcal{R}(x_k-3,\lambda)\le 1-\varepsilon \text{ or } \mathcal{T}(x_k-3,\lambda)\ge\varepsilon,\ \mathcal{W}(x_k-3,\lambda)\ge\varepsilon\}|\neq 0.$$

Therefore, the given sequence is not statistically convergent with respect to $\mathcal{N}$ which implies that it is not ρ-statistical convergence with respect to $\mathcal{N}$.

Theorem 3.12. *Let $\rho=(\rho_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ and satisfying condition* (1) *such that $\limsup_n \frac{\rho_n}{n}=M<\infty$. Then $st^{\mathcal{N}}(b)\subseteq st^{\mathcal{N}}_\rho(b)$.*

Proof. Let (x_k) be a sequence in $st^{\mathcal{N}}(b)$. Then for a given $\lambda>0$, $b\in(0,1)$, we have

$$\lim_{n\to\infty}\frac{1}{n}\left|\{k\in\mathbb{N}:\mathcal{R}(x_k,t_0)>1-b \text{ or } \mathcal{T}(x_k,t_0)<b,\ \mathcal{W}(x_k,t_0)<b\}\right|=0.$$

Now,

$$\frac{1}{n}|\{k\le n:\mathcal{R}(x_k,\lambda)\le 1-\varepsilon \text{ or } \mathcal{T}(x_k,\lambda)\ge\varepsilon,\ \mathcal{W}(x_k,\lambda)\ge\varepsilon\}|$$
$$\le \limsup\left(\frac{\rho_n}{n}\right)\frac{1}{\rho_n}\left|\{k\le n:\mathcal{R}(x_k,\lambda)\le 1-\varepsilon \text{ or } \mathcal{T}(x_k,\lambda)\ge\varepsilon,\ \mathcal{W}(x_k,\lambda)\ge\varepsilon\}\right|,$$

$$\lim_{n\to\infty}\frac{1}{n}\left|\{k\le n:\mathcal{R}(x_k,\lambda)\le 1-\varepsilon \text{ or } \mathcal{T}(x_k,\lambda)\ge\varepsilon,\ \mathcal{W}(x_k,\lambda)\ge\varepsilon\}\right|$$
$$\le \lim_{n\to\infty}\limsup(\frac{\rho_n}{n})\frac{1}{\rho_n}|\{k\le n:\mathcal{R}(x_k,\lambda)\le 1-\varepsilon \text{ or } \mathcal{T}(x_k,\lambda)\ge\varepsilon,\ \mathcal{W}(x_k,\lambda)\ge\varepsilon\}|$$
$$\le \lim_{n\to\infty}\frac{M}{\rho_n}\left|\{k\le n:\mathcal{R}(x_k,\lambda)\le 1-\varepsilon \text{ or } \mathcal{T}(x_k,\lambda)\ge\varepsilon,\ \mathcal{W}(x_k,\lambda)\ge\varepsilon\}\right|.$$

As $\limsup_n \frac{\rho_n}{n}=M<\infty$, we get $st^{\mathcal{N}}(b)\subseteq st^{\mathcal{N}}_\rho(b)$. □

3.3. ρ-statistical convergence of order α in NNS

Definition 3.13. Let V be a NNS. A sequence (x_k) of V is said to be ρ-statistical convergence of order α with $0<\alpha\le 1$, if for each $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{1}{\rho_n^\alpha}\left|\{k\le n:\mathcal{R}(x_k-\ell,\lambda)\le 1-\varepsilon \text{ or } \mathcal{T}(x_k-\ell,\lambda)\ge\varepsilon,\ \mathcal{W}(x_k-\ell,\lambda)\ge\varepsilon\}\right|=0,$$

where $\rho=(\rho_n)$ is a non-decreasing sequence of positive real numbers tending to ∞ and satisfying the condition (1). In this case, we write st^α_ρ-$\mathcal{N}$-$\lim x_k=\ell$ or $x_k\to\ell(st^\alpha_\rho$-$\mathcal{N})$. We denote the set of all ρ-statistical convergence of order α sequence by $st^\alpha_\rho(\mathcal{N})$. If $\rho_n=n$, then ρ-statistical convergence of order α in a NNS V coincides with the statistical convergence of order α in a NNS V, in addition, if $\alpha=1$, it coincides with statistical convergence in a NNS V.

Example 3.14. Let $V=\mathbb{R}$. For all $s,t\in[0,1]$, define the continuous t-norm $s\circ t=st$ and the continuous t-conorm $s\bullet t=\min\{s+t,1\}$. We take $\mathcal{R},\mathcal{T},\mathcal{W}$ as in Example 2.4, for all $\lambda > 0$. Define the sequence (x_k) as

$$x_k = \begin{cases} 1, & k = p^3 \ (p \in \mathbb{N}), \\ 0, & \text{otherwise}. \end{cases}$$

Then, $x_k \to 0(st_\rho^\alpha\text{-}\mathcal{N})$, taking $\rho_n^\alpha = n^\alpha$ for $\alpha \in (\frac{1}{3}, 1]$.
Justification: For every $0 < \varepsilon < 1$, we have

$$K_\varepsilon = \{k \leq n : \mathcal{R}(x_k - 0, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k - 0, \lambda) \geq \varepsilon,\ \mathcal{W}(x_k - 0, \lambda) \geq \varepsilon\}.$$

This implies

$$\begin{aligned} K_\varepsilon &= \left\{k \in \mathbb{N} : \frac{\lambda - |x_k|}{\lambda + |x_k|} \leq 1 - \varepsilon \text{ or } \frac{|x_k|}{\lambda + |x_k|} \geq \varepsilon, \frac{|x_k|}{\lambda} \geq \varepsilon \right\} \\ &= \left\{k \in \mathbb{N} : |x_k| \geq \frac{\lambda\varepsilon}{2 - \varepsilon} \text{ or } |x_k| \geq \frac{\lambda\varepsilon}{1 - \varepsilon}, |x_k| \geq \lambda\varepsilon \right\} \\ &\subseteq \left\{k \in \mathbb{N} : |x_k| \geq \frac{\lambda\varepsilon}{1 - \varepsilon} \text{ or } |x_k| \geq \lambda\varepsilon \right\} \\ &= \{k \leq n : x_k = 1\}. \end{aligned}$$

Then,

$$\delta_\rho^\alpha(K_\varepsilon) = \lim_{n\to\infty} \frac{|K_\varepsilon|}{n^\alpha} \leq \lim_{n\to\infty} \frac{\sqrt[3]{n}}{n^\alpha} = 0 \quad \text{for } \alpha \in (\frac{1}{3}, 1].$$

Hence, $x_k \to 0(st_\rho^\alpha\text{-}\mathcal{N})$.

Theorem 3.15. *In an NNS, the limit of ρ-statistical convergence sequence of order α is unique.*

Proof. The proof is similar to Theorem 3.8, so omitted. □

Remark. The ρ-statistical convergence of order α is well defined for $0 < \alpha \leq 1$, but it is not well defined for $\alpha > 1$, in general. For this let us consider $V = \mathbb{R}$. We take $\mathcal{R}, \mathcal{T}, \mathcal{W}$ as in Example 2.4, then

$$x_k = \begin{cases} 1 & \text{if} \quad k = 2n, \\ 0 & \text{if} \quad k \neq 2n. \end{cases}$$

For $\alpha > 1$, where $\rho_n = n$ and $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} |\{k \leq n : \mathcal{R}(x_k - 1, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k - 1, \lambda) \geq \varepsilon, \mathcal{W}(x_k - 1, \lambda) \geq \varepsilon\}|$$
$$\leq \lim_{n\to\infty} \frac{n}{2\rho_n^\alpha} = 0,$$

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} |\{k \leq n : \mathcal{R}(x_k - 0, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k - 0, \lambda) \geq \varepsilon, \mathcal{W}(x_k - 0, \lambda) \geq \varepsilon\}|$$
$$\leq \lim_{n\to\infty} \frac{n}{2\rho_n^\alpha} = 0.$$

Thus, $x_k \to 1(st_\rho\text{-}\mathcal{N})$ and $x_k \to 0(st_\rho\text{-}\mathcal{N})$, which is impossible. So, $0 < \alpha \leq 1$.

Theorem 3.16. *Let $0 < \alpha \leq \beta \leq 1$. Then $st_\rho^\alpha(\mathcal{N}) \subseteq st_\rho^\beta(\mathcal{N})$ and the inclusion is strict.*

Proof. If $0 < \alpha \leq \beta \leq 1$, then

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} |\{k \in \mathbb{N} : \mathcal{R}(x_k - \ell, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k - \ell, \lambda) \geq \varepsilon,\ \mathcal{W}(x_k - \ell, \lambda) \geq \varepsilon\}|$$

$$\leq \lim_{n\to\infty} \frac{1}{\rho_n^\beta} |\{k \in \mathbb{N} : \mathcal{R}(x_k - \ell, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k - \ell, \lambda) \geq \varepsilon,\ \mathcal{W}(x_k - \ell, \lambda) \geq \varepsilon\}|$$

which implies $st_\rho^\alpha(\mathcal{N}) \subseteq st_\rho^\beta(\mathcal{N})$. □

Corollary 3.17. *If a sequence is ρ-statistical convergence of order α in NNS V, then it is ρ-statistical convergence in NNS V, that is, $st_\rho^\alpha(\mathcal{N}) \subseteq st_\rho(\mathcal{N})$ for each $\alpha \in (0,1]$, and the inclusion is strict.*

Theorem 3.18. *Let $\rho = (\rho_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ and satisfying condition (1). Let $x = (x_n)$ and $y = (y_n)$ be the sequences of NNS V. Then*

(i) *$x_k \to \ell(st_\rho^\alpha\text{-}\mathcal{N})$ and $c \in \mathbb{R}$ imply $cx_k \to c\,\ell(st_\rho^\alpha\text{-}\mathcal{N})$,*

(ii) *$x_k \to \ell_1(st_\rho^\alpha\text{-}\mathcal{N})$, $y_k \to \ell_2(st_\rho^\alpha\text{-}\mathcal{N})$ and $c \in \mathbb{R}$ imply $(x_k + y_k) \to (\ell_1 + \ell_2)(st_\rho^\alpha\text{-}\mathcal{N})$.*

Proof. The proof is similar to the proof of Theorem 3.4, so omitted. □

Theorem 3.19. *Let $\rho = (\rho_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ and satisfying condition (1). If $\liminf_n (\frac{\rho_n}{n})^\alpha > 1$, then $st^\alpha(\mathcal{N}) \subseteq st_\rho^\alpha(\mathcal{N})$ with respect to neutrosophic norm $\mathcal{N}$.*

Proof. The proof is similar to Theorem 3.6, so omitted. □

Lemma 3.20. *Let V be an NNS. Then, for any $0 < \varepsilon < 1$, the following statements are equivalent:*

1. *$x_k \to \ell(st_\rho^\alpha\text{-}\mathcal{N})$;*
2. *$\delta_\rho^\alpha(\{k \in \mathbb{N} : \mathcal{R}(x_k - \ell, \lambda) \leq 1 - \varepsilon\}) = \delta_\rho^\alpha(\{k \in \mathbb{N} : \mathcal{T}(x_k - \ell, \lambda) \geq \varepsilon\})$*
   $= \delta_\rho^\alpha(\{k \in \mathbb{N} : \mathcal{W}(x_k - \ell, \lambda) \geq \varepsilon\}) = 0$;
3. *$\delta_\rho^\alpha(\{k \in \mathbb{N} : \mathcal{R}(x_k - \ell, \lambda) > 1 - \varepsilon \text{ or } \mathcal{T}(x_k - \ell, \lambda) < \varepsilon, \mathcal{W}(x_k - \ell, \lambda) < \varepsilon\}) = 1$;*
4. *$\mathcal{R}(x_k - \ell, \lambda) \to 1(st_\rho^\alpha\text{-}\mathcal{N})$, $\mathcal{T}(x_k - \ell, \lambda) \to 0(st_\rho^\alpha\text{-}\mathcal{N})$,*
   $\mathcal{W}(x_k - \ell, \lambda) \to 0(st_\rho^\alpha\text{-}\mathcal{N})$.

Theorem 3.21. *Let (x_k) and (y_k) be two sequences in a NNS V such that $y_k \to \ell(\mathcal{N})$ and $\delta_\rho^\alpha(\{k \in \mathbb{N} : x_k \neq y_k\}) = 0$. Then, $x_k \to \ell(st_\rho^\alpha\text{-}\mathcal{N})$.*

Proof. Suppose $\delta_\rho^\alpha(\{k \in \mathbb{N} : x_k \neq y_k\}) = 0$ holds and $y_k \to \ell(\mathcal{N})$. Then, by definition, for every $0 < \varepsilon < 1$, the set $K_\varepsilon = \{k \in \mathbb{N} : \mathcal{R}(y_k - \ell, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(y_k - \ell, \lambda) \geq \varepsilon, \mathcal{W}(y_k - \ell, \lambda) \geq \varepsilon\}$ contains almost finite number of elements, and consequently, $\delta_\rho^\alpha(K_\varepsilon) = 0$. Now, since the inclusion

$$K'_\varepsilon = \{k \in \mathbb{N} : \mathcal{R}(x_k - \ell, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{T}(x_k - \ell, \lambda) \geq \varepsilon,\ \mathcal{W}(x_k - \ell, \lambda) \geq \varepsilon\}$$
$$\subseteq K_\varepsilon \cap \{k \in \mathbb{N} : x_k \neq y_k\}$$

holds, so we must have $\delta_\rho^\alpha(K'_\varepsilon) = 0$. Hence, $x_k \to \ell(st_\rho^\alpha\text{-}\mathcal{N})$. □

Theorem 3.22. *Let $\rho = (\rho_n)$ and $\tau = (\tau_n)$ be two sequences, both satisfying condition* (1) *and $\rho_n \leq \tau_n$ for all $n \in \mathbb{N}$. If (x_k) be a sequence in a NNS V such that $x_k \to \ell(st^{\alpha}_{\rho}\text{-}\mathcal{N})$, then $x_k \to \ell(st^{\alpha}_{\tau}\text{-}\mathcal{N})$.*

Proof. The proof is similar to Theorem 3.7, so omitted. □

4. Conclusion

In this paper, we have presented and explored the concept of ρ-statistical convergence in neutrosophic normed spaces (NNS). We have investigated its fundamental properties and established its boundedness within the NNS. Additionally, we have extended our analysis to include ρ-statistical convergence of order α in NNS, further enriches our understanding of convergence in this context. Furthermore, we have explored the relation between ρ-statistical convergence and existing convergence methods in NNS, thereby contributing to a more inclusive understanding of convergence theory within neutrosophic normed spaces. Our findings not only deepen the theoretical basis of statistical convergence in NNS. Moving forward, these observations can serve as a foundation for further research and exploration in the domain of convergence theory and its applications in neutrosophic normed spaces.

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