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REPRESENTATIONS OF MENGER HYPERCOMPOSITIONAL ALGEBRAS BY SOME TYPES OF COMMUTATIVE HYPEROPERATIONS

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Abstract. We present an abstract characterization of diagonal semihypergroups derived from any Menger hypercompositional algebra. We also prove that the set of all k-commutative hyperoperations forms a Menger algebra. The necessary and sufficient conditions under which a Menger hypercompositional algebra of rank $n>1$ is embeddable into an algebra of k-commutative hyperoperations are proposed.

1. Introduction and preliminaries

The study of Menger algebras was initiated by K. Menger [**21**] in the middle of last century. It has many applications in various areas, for example, cybernetics, multi-valued computations, and modern universal algebra [**2, 5, 23**]. In fact, for a fixed positive integer n, a Menger algebra of rank n is a pair of a nonempty set G and an $(n+1)$-ary operation on G such that the superassociative law holds. Recently, such algebras were investigated in different topics, for example, partial Menger algebras of terms [**6**], Menger algebras of full terms defined by transformations which preserve a partition [**19**], ternary Menger algebras [**22**]. Algebraic properties of Menger algebras were recently examined in [**16**]. We now present two babic examples of Menger algebras. The first one is the set $\mathbb{R}^+$ of all positive real numbers with the operation $\circ\colon (\mathbb{R}^+)^{n+1} \to \mathbb{R}^+$, defined by $\circ(x_0,\dots,x_n) = x_0 \sqrt[n]{x_1 \cdots x_n}$. Another one is the set of all real numbers $\mathbb{R}$ with the following $(n+1)$-ary operation $\circ$, which is defined by $\circ(x, y_1 \dots, y_n) = x + \frac{y_1+\ldots+y_n}{n}$ for all $x, y_1 \dots, y_n \in \mathbb{R}$. In a view of extensions, a Menger algebra of rank $n = 1$ is a semigroup. This means that a Menger algebra of rank n is a generalized structure of semigroups, too.

It is well known in the theory of representations that semigroups and groups can be isomorphically represented by functions of one variable. Representations of other structures can be seen, for example, in [**1, 13, 25**]. Generally, Menger

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algebras of some types are also studied in the same direction. It turned out that some types of Menger algebras of rank n can be represented by n-ary functions. In fact, let A^n be the Cartesian product of a nonempty set A. By a *full n-ary function* on A or an *n-ary operation* on A, we mean any mapping from A^n to A in which it is defined for all elements of A^n. On the set $T(A^n, A)$ of all such mappings, one can consider the *Menger's superposition*, i.e., an $(n+1)$-ary operation $\mathcal{O}\colon T(A^n, A)^{n+1} \to T(A^n, A)$ defined by

$$\mathcal{O}(f, g_1, \ldots, g_n)(a_1, \ldots, a_n) = f(g_1(a_1, \ldots, a_n), \ldots, g_n(a_1, \ldots, a_n)),$$

where $f, g_1, \ldots, g_n \in T(A^n, A), a_1, \ldots, a_n \in A$. Therefore, $(T(A^n, A), \mathcal{O})$ is called a *Menger algebra of all full n-ary functions*, or a *Menger algebra of all n-ary operations.*

In recent years, algebras of operations allowing a certain fixed permutation of variables were studied. For example, an abstract characterization of the Menger algebra of all idempotent n-ary operations defined on a fixed set A was presented in [**11**]. A Menger algebra of associative and self-distributive n-ary operations was considered in [**8**]. Other classes of functions in this direction may be seen in [**9, 10**]. Weak near-unanimity functions generated by cyclic and weak near-unanimity terms were described in [**18**]. Algebras of partial multiplace functions for signatures that contain composition and operations given by the set-theoretic operations were described in [**14, 20**].

Recall from [**17**] that a nonempty set G equipped with one $(n+1)$-ary hyperoperation $\diamond$ on G satisfying the identity of the superassociativity is called a *Menger hypercomposition algebra* of rank n or a *Menger hyperalgebra* of rank n. Note that a Menger hyperalgebra can be reduced to a semihypergroup if we set $n = 1$. Recent progress in the theory of semihypergroups can be found, for example, in [**3, 15, 24**]. Furthermore, every Menger algebra is a Menger hypercomposition algebra. Like a representation in classical algebras, the situation for hypercomposition algebras has been studied. The symbol $P^*(A)$ stands for a power set of A without empty set. A mapping $\alpha\colon A^n \to P^*(A)$ is called a *multivalued full n-ary function* or an *n-ary hyperoperation* on A. One can apply the following $(n+1)$-ary operation $\bullet : T(A^n, P^*(A))^{n+1} \to T(A^n, P^*(A))$. Actually, it is defined by

$$\bullet(f, g_1, \ldots, g_n)(x_1, \ldots, x_n) = \bigcup_{\substack{y_i \in g_i(x_1, \ldots, x_n) \\ i \in \{1, \ldots, n\}}} f(y_1, \ldots, y_n)$$

for all $i = 1, \ldots, n$, where $f, g_1, \ldots, g_n \in T(A^n, P^*(A))$, $x_1, \ldots, x_n \in A$. Consequently, the set $T(A^n, P^*(A))$ of all n-ary hyperoperations on A together with an $(n+1)$-ary operation $\bullet$ forms a Menger algebra.

In this paper, following [**12, 17**], we continue to study k-commutative operations in a more general form, which means that we introduce a hyperoperation of k-commutative variables. The fact that the set of all k-commutative hyperoperations defined on a fixed set is a Menger hypercompositional algebra is proved. An abstract characterization such that any Menger hypercompositional algebra is embeddable into an algebra of k-commutative hyperoperations is given.

2. Results

We begin this section with giving some classes of semihypergroups derived from any Menger hypercompositional algebra.

In [**7**, Chapter 2], W. A. Dudek and V. S. Trokhimenko defined the notion of a diagonal semigroup of a Menger algebra. For this reason, it is interesting to proceed in the same direction but in another generalized structure.

As a general abbreviation used in the theory of n-ary algebra, in this paper, the symbol a_1^n stands for $a_1, \ldots, a_n$. By $\overset{n}{a}$, we mean $\underbrace{a, \ldots, a}_{n \text{ times}}$.

On a Menger hypercompositional algebra $(G, \diamond)$ of rank n, one can define the new binary hyperoperation $+$ on G by setting

$$x + y = \diamond(x, \overset{n}{y})$$

for all $x, y \in G$. Due to the satisfaction of an $(n+1)$-ary hyperoperation $\diamond$ with the superassociative law, consequently, it is clear that $(G, +)$ is a semihypergroup. We call $(G, +)$ the *diagonal semihypergroup* derived from $(G, \diamond)$.

Theorem 2.1. *Any semihypergroup $(G, \cdot)$ in which an n-ary hyperoperation f on G satisfies the conditions*

(1) $f(\overset{n}{a}) = \{a\}$ *for all* $a \in G$,

(2) $f(a_1^n) \cdot a = \bigcup\limits_{\substack{b_i \in a_i \cdot a \\ i \in \{1, \ldots, n\}}} f(b_1^n)$ *for all* $a, a_1, \ldots, a_n \in G$,

is a diagonal semihypergroup of some Menger hypercompositional algebra of rank n.

Proof. Assume that all conditions are satisfied. On a semihypergroup $(G, \cdot)$, we now define a hyperoperation of type $(n+1)$ by

$$\star(a, b_1^n) = a \cdot f(b_1^n)$$

for all $a, b_1, \ldots, b_n \in G$. To show that a hyperoperation $\star$ is superassociative, let $a, b_1, \ldots, b_n, c_1, \ldots, c_n$ be arbitrary elements in G. Then, we have

$$\begin{aligned}
\star(\star(a, b_1^n), c_1^n) &= \star(a \cdot f(b_1^n), c_1^n) \\
&= (a \cdot f(b_1^n)) \cdot f(c_1^n) = a \cdot (f(b_1^n) \cdot f(c_1^n)) \\
&= a \cdot \Big(\bigcup_{\substack{d_i \in b_i \cdot f(c_1^n) \\ i \in \{1, \ldots, n\}}} f(d_1^n) \Big) = a \cdot \Big(\bigcup_{\substack{d_i \in \star(b_i, c_1^n) \\ i \in \{1, \ldots, n\}}} f(d_1^n) \Big) \\
&= \bigcup_{\substack{d_i \in \star(b_i, c_1^n) \\ i \in \{1, \ldots, n\}}} \star(a, d_1^n) \\
&= \star(a, \star(b_1, c_1^n), \ldots, \star(b_n, c_1^n)),
\end{aligned}$$

which shows that $(G,\star)$ is a Menger hypercompositional algebra of rank n. Suppose that $(G,+)$ is a diagonal semihypergroup of a Menger hypercompsotional algebra $(G,\star)$. For every $a,b\in G$, we obtain $a+b=\star(a,\overset{n}{b})=a\cdot f(\overset{n}{b})=a\cdot\{b\}=a\cdot b$. Thus, $(G,+)$ and $(G,\cdot)$ are identical, which proves that $(G,\cdot)$ is a diagonal semihypergroup. $\square$

An element e of a diagonal semihypergroup $(G,+)$ is said to be *left diagonal-scalar identity* of a Menger hypercompositional algebra $(G,\diamond)$ if it is a left scalar identity of the diagonal semihypergroup of $(G,\diamond)$, i.e., it satisfies the identity $e+a=\diamond(e,a,\dots,a)=\{a\}$ for all $a\in G$. Analogously, a right-diagonal scalar identity is defined. If e is both a left and right diagonal scalar identity, then e is called a *diagonal scalar identity.*

Theorem 2.2. *A semihypergroup $(G,\cdot)$ with a left scalar identity is a diagonal semihypergroup of some Menger hypercompositional algebra with a left diagonal-scalar identity if and only if there exists an operation of type (n) on G which satisfies the condition* (1) *and* (2) *of* Theorem 2.1.

Proof. According to Theorem 2.1, $(G,\star)$ is a Menger hypercompositional algebra. Suppose first that $(G,\cdot)$ is a diagonal semihypergroup of a Menger hypercompositional algebra $(G,\star)$ with a left diagonal-scalar identity, say e. We define an n-ary operation f on G by $f(a_1,\dots,a_n)=\star(e,a_1^n)$. Clearly, f satisfies the conditions (1) and (2) of Theorem 2.1. In fact, for all $a,a_1\dots,a_n,b\in G$, we obtain $f(\overset{n}{a})=\star(e,\overset{n}{a})=\{a\}$ and $f(a_1^n)\cdot a=\star(e,a_1^n)\cdot a=\star(\star(e,a_1^n),\overset{n}{a})=\star(e,\star(a_1,\overset{n}{a}),\dots,\star(a_n,\overset{n}{a}))=\star(e,a_1\cdot a,\dots,a_n\cdot a)=\bigcup\limits_{\substack{b_i\in a_i\cdot a\\ i\in\{1,\dots,n\}}} f(b_1^n)$.

For the converse, assume that $(G,\cdot)$ is a semihypergroup and all of conditions are satisfied. It follows from Theorem 2.1 that $(G,\cdot)$ is a diagonal semihypergroup of a Menger hypercompositional algebra with the hyperoperation $\star(a,b_1^n)=a\cdot f(b_1^n)$. It is obvious that a left identity e of this semihypergroup is a left diagonal-scalar of the Menger hypercompositional algebra. $\square$

The second part of the main results is contributed to hypercompositional algebras of some type of hyperoperation. We follow the work of W. A. Dudek and V. S. Trokhimenko [**12**] but focus on an extended version in sense of its hyperoperation.

Let $n>1$ and $|A|>1$. For each $k\in\{1,\dots,n-1\}$, an n-ary hyperoperation f defined on a nonempty set A is said to be *k-commutative* if it satisfies the identity

$$f(\overset{i-1}{a},\overset{k}{b},\overset{n-i-k+1}{a})=f(\overset{i-1}{b},\overset{k}{a},\overset{n-i-k+1}{b})$$

for all $a,b\in A$ and $i\in\{1,\dots,n-k\}$. Let $\mathcal{M}_n^k(A,P^*(A))$ be the set of all k-commutative n-ary hyperoperations on A.

Lemma 2.3. *The set $\mathcal{M}_n^k(A,P^*(A))$ of all k-commutative n-ary hyperoperations on A forms a Menger algebra.*

Proof. Let $f, g_1, \ldots, g_n$ be hyperoperations in $\mathcal{M}_n^k(A, P^*(A))$. Then

$$\begin{aligned}
\bullet(f, g_1, \ldots, g_n)(\overset{i-1}{a}, \overset{k}{b}, \overset{n-i-k+1}{a}) &= \bigcup_{\substack{c_l \in g_l(\overset{i-1}{a}, \overset{k}{b}, \overset{n-i-k+1}{a}) \\ l \in \{1, \ldots, n\}}} f(c_1, \ldots, c_n) \\
&= \bigcup_{\substack{c_l \in g_l(\overset{i-1}{b}, \overset{k}{a}, \overset{n-i-k+1}{b}) \\ l \in \{1, \ldots, n\}}} f(c_1, \ldots, c_n) \\
&= \bullet(f, g_1, \ldots, g_n)(\overset{i-1}{b}, \overset{k}{a}, \overset{n-i-k+1}{b}),
\end{aligned}$$

which shows that the set $\mathcal{M}_n^k(A, P^*(A))$ is a Menger algebra with respect to the composition $\mathcal{O}$. □

Lemma 2.4. *The Menger algebra $(\mathcal{M}_n^k(A), \mathcal{O}$ of all k-commutative n-ary operations on A is embeddable into Menger algebra $(\mathcal{M}_n^k(A, P^*(A)), \bullet)$ of all k-commutative n-ary hyperoperations on A.*

Proof. For each $k > 1$, suppose first that f is a k-commutative n-ary operation on A. Define an n-ary hyperoperation $\overline{f}$ on A by setting $\overline{f}(a_1^n) = \{f(a_1^n)\}$ for all $a_1, \ldots, a_n \in A$. It is clear that $\overline{f}$ belongs to the set $\mathcal{M}_n^k(A, P^*(A))$ since f is a k-commutative n-ary operation on A. We have to prove that a mapping $\sigma\colon \mathcal{M}_n^k(A) \to \mathcal{M}_n^k(A, P^*(A))$ defined by $\sigma(f) = \overline{f}$ for all $f \in \mathcal{M}_n^k(A)$, σ is a monomorphism. To do this, let $f, g \in \mathcal{M}_n^k(A)$ and $a_1, \ldots, a_n \in A$. If $\overline{f} = \overline{g}$, then also $\overline{f}(a_1, \ldots, a_n) = \overline{g}(a_1, \ldots, a_n)$, subsequently, $\{f(a_1, \ldots, a_n)\} = \{g(a_1, \ldots, a_n)\}$. This implies $f = g$ and thus σ is injective. By the definition of the composition $\bullet$, it is not hard to verify that a mapping σ preserves the operations, i.e.,

$$\sigma(\mathcal{O}(f, g_1, \ldots, g_n)) = \bullet(\sigma(f), \sigma(g_1), \ldots, \sigma(g_n)).$$

In fact, if we let $a_1, \ldots, a_n \in A$, then

$$\begin{aligned}
&\overline{\mathcal{O}(f, g_1, \ldots, g_n)}(a_1, \ldots, a_n) \\
&= \{\mathcal{O}(f, g_1, \ldots, g_n)(a_1, \ldots, a_n)\} = \{f(g_1(a_1, \ldots, a_n), \ldots, g_n(a_1, \ldots, a_n))\} \\
&= \overline{f}(g_1(a_1, \ldots, a_n), \ldots, g_n(a_1, \ldots, a_n)) = \bigcup_{\substack{b_i \in \{g_i(a_1, \ldots, a_n)\} \\ i \in \{1, \ldots, n\}}} \overline{f}(b_1, \ldots, b_n) \\
&= \bigcup_{\substack{b_i \in \overline{g_i}(a_1, \ldots, a_n) \\ i \in \{1, \ldots, n\}}} \overline{f}(b_1, \ldots, b_n) = \bullet(\overline{f}, \overline{g_1}, \ldots, \overline{g_n})(a_1, \ldots, a_n),
\end{aligned}$$

which shows that σ is a homomorphism. □

Lemma 2.5. *On the Menger algebra $(\mathcal{M}_n^k(A, P^*(A)), \bullet)$, we have*

$$\bullet(f, \overset{i-1}{g}, \overset{k}{h}, \overset{n-i-k+1}{g}) = \bullet(f, \overset{i-1}{h}, \overset{k}{g}, \overset{n-i-k+1}{h})$$

for all $f, g, h \in \mathcal{M}_n^k(A, P^*(A))$ and $i \in \{1, \ldots, n-k\}$.

Proof. We show that

$$\begin{aligned}
\bullet(f, \overset{i-1}{g}, \overset{k}{h}, \overset{n-i-k+1}{g})(a_1^n) &= \bigcup_{\substack{c\in g(a_1^n)\\ d\in h(a_1^n)}} f(\underbrace{c,\dots,c}_{i-1 \text{ times}}, \underbrace{d,\dots,d}_{k \text{ times}}, \underbrace{c,\dots,c}_{n-i-k+1 \text{ times}}) \\
&= \bigcup_{\substack{c\in g(a_1^n)\\ d\in h(a_1^n)}} f(\underbrace{d,\dots,d}_{i-1 \text{ times}}, \underbrace{c,\dots,c}_{k \text{ times}}, \underbrace{d,\dots,d}_{n-i-k+1 \text{ times}}) \\
&= \bullet(f, \overset{i-1}{h}, \overset{k}{g}, \overset{n-i-k+1}{h})(a_1^n).
\end{aligned}$$

The proof is finished. □

Theorem 2.6. *A Menger hypercompositional algebra $(G, \diamond)$ of rank $n > 1$ is embeddable into the Menger algebra of all k-commutative n-ary hyperoperations defined on some set if and only if the identity*

$$\diamond(a, \overset{i-1}{b}, \overset{k}{c}, \overset{n-i-k+1}{b}) = \diamond(a, \overset{i-1}{c}, \overset{k}{b}, \overset{n-i-k+1}{c})$$

holds for all $a, b, c \in G$ and $i \in \{1, \dots, n-k\}$.

Proof. Suppose that there exists a monomorphism from a Menger hypercompositional algebra $(G, \diamond)$ of rank $n > 1$ to the Menger algebra of all k-commutative n-ary hyperoperations. It follows directly from Lemma 2.5 and the injectivity of such mapping that the necessity is obtained.

For the converse, assume that $(G, \diamond)$ is a Menger hypercompositional algebra of rank $n > 1$. Let $x, y \notin G$ and $x \neq y$. We extend a set G to $G' := G \cup \{x, y\}$. For each g in G', an n-ary hyperoperation on G' can be defined by the following

$$\lambda_g(a_1, \dots, a_n) = \begin{cases} \diamond(g, a_1, \dots, a_n) & \text{if } a_1, \dots, a_n \in G, \\ \{g\} & \text{if } a_j = x \text{ for all } j = 1, \dots, n, \\ \{y\} & \text{otherwise.} \end{cases}$$

For any nonempty set A of G', an n-ary extended function λ_A on G' is defined by

$$\lambda_A(a_1, \dots, a_n) = \begin{cases} \diamond(A, a_1, \dots, a_n) & \text{if } a_1, \dots, a_n \in G, \\ A & \text{if } a_j = x \text{ for all } j = 1, \dots, n, \\ \{y\} & \text{in all other cases.} \end{cases}$$

First, we show that for each g in G', λ_g is a k-commutative n-ary hyperoperation. For this, let $a, b \in G'$. We first consider the case when $a, b \in G$. According to the definition of λ_g, then by our assumption, we have

$$\begin{aligned}
\lambda_g(\overset{i-1}{a}, \overset{k}{b}, \overset{n-i-k+1}{a}) = \diamond(g, \overset{i-1}{a}, \overset{k}{b}, \overset{n-i-k+1}{a}) &= \diamond(g, \overset{i-1}{b}, \overset{k}{a}, \overset{n-i-k+1}{b}) \\
= \lambda_g(\overset{i-1}{b}, \overset{k}{a}, \overset{n-i-k+1}{b}).&
\end{aligned}$$

In the second case, if $a \in G$ and $b \in \{x, y\}$, or $a \in \{x, y\}$ and $b \in G$, then

$$\lambda_g(\overset{i-1}{a}, \overset{k}{b}, \overset{n-i-k+1}{a}) = \{y\} = \lambda_g(\overset{i-1}{b}, \overset{k}{a}, \overset{n-i-k+1}{b}).$$

Finally, if both a, b belong to $\{x, y\}$, we consider two situations. If $a = b = x$, then

$$\lambda_g(\overset{i-1}{a}, \overset{k}{b}, \overset{n-i-k+1}{a}) = \lambda_g(\underbrace{x, \dots, x}_{n \text{ times}}) = \{g\} = \lambda_g(\underbrace{x, \dots, x}_{n \text{ times}})$$
$$= \lambda_g(\overset{i-1}{b}, \overset{k}{a}, \overset{n-i-k+1}{b}).$$

On the other hand, if $a = b = y$, then we obtain

$$\lambda_g(\overset{i-1}{a}, \overset{k}{b}, \overset{n-i-k+1}{a}) = \lambda_g(\underbrace{y, \dots, y}_{n \text{ times}}) = \{y\} = \lambda_g(\underbrace{y, \dots, y}_{n \text{ times}})$$
$$= \lambda_g(\overset{i-1}{b}, \overset{k}{a}, \overset{n-i-k+1}{b}).$$

Consequently, in all cases, we conclude that a hyperoperation λ_g generated by each element g of G' is k-commutative.

Define a mapping φ which takes from a Menger hypercompositional algebra $(G, \diamond)$ to a Menger algebra of all k-commutative hyperoperations defined on an extended set G' by $\varphi(g) = \lambda_g$ for all $g \in G$. We now show that φ is a strong isomorphism between $(G, \diamond)$ and $(\mathcal{M}_n^k(G', P^*(G')), \bullet)$, i.e., the equation

$$\varphi(\diamond(g, a_1, \dots, a_n)) = \bullet(\varphi(g), \varphi(a_1), \dots, \varphi(a_n)),$$

which is equivalent to

$$\lambda_{\diamond(g, a_1, \dots, a_n)} = \bullet(\lambda_g, \lambda_{a_1}, \dots, \lambda_{a_n}),$$

holds for all $g, a_1, \dots, a_n \in G$. For this, we let $x_1, \dots, x_n$ be arbitrary elements in G'. If $x_1, \dots, x_n \in G$, then by the fact that $\diamond$ is superassociative, we have

$$\begin{aligned}
\lambda_{\diamond(g, a_1, \dots, a_n)}(x_1, \dots, x_n) &= \diamond(\diamond(g, a_1, \dots, a_n), x_1, \dots, x_n) \\
&= \diamond(g, \diamond(a_1, x_1, \dots, x_n), \dots, \diamond(a_n, x_1, \dots, x_n)) \\
&= \diamond(g, \lambda_{a_1}(x_1, \dots, x_n), \dots, \mu_{a_n}(x_1, \dots, x_n)) \\
&= \bigcup_{\substack{c_i \in \lambda_{a_i}(x_1, \dots, x_n) \\ i \in \{1, \dots, n\}}} \diamond(g, c_1, \dots, c_n) \\
&= \bigcup_{\substack{c_i \in \lambda_{a_i}(x_1, \dots, x_n) \\ i \in \{1, \dots, n\}}} \lambda_g(c_1, \dots, c_n) \\
&= \bullet(\lambda_g, \lambda_{a_1}, \dots, \mu_{a_n})(x_1, \dots, x_n).
\end{aligned}$$

Now let $x_1 = \cdots = x_n = x$, then according to the definition of an extension λ_A, we obtain

$$\begin{aligned}\lambda_{\diamond(g,a_1,\dots,a_n)}(x_1,\dots,x_n) &= \lambda_{\diamond(g,a_1,\dots,a_n)}(x,\dots,x) = \diamond(g,a_1,\dots,a_n)\\ &= \lambda_g(a_1,\dots,a_n) = \bigcup_{\substack{c_i\in\{a_i\}\\ i\in\{1,\dots,n\}}} \lambda_g(c_1,\dots,c_n)\\ &= \bigcup_{\substack{c_i\in\lambda_{a_i}(x,\dots,x)\\ i\in\{1,\dots,n\}}} \lambda_g(c_1,\dots,c_n)\\ &= \bullet(\lambda_g,\lambda_{a_1},\dots,\lambda_{a_n})(x_1,\dots,x_n),\end{aligned}$$

which means

$$\lambda_{\diamond(g,a_1,\dots,a_n)}(x,\dots,x) = \bullet(\lambda_g,\lambda_{a_1},\dots,\lambda_{a_n})(x,\dots,x).$$

In all other cases, we have

$$\lambda_{\diamond(g,a_1,\dots,a_n)}(x_1,\dots,x_n) = \{y\}$$

and

$$\begin{aligned}\bullet(\lambda_g,\lambda_{a_1},\dots,\lambda_{a_n})(x_1,\dots,x_n) &= \bigcup_{\substack{x_i\in\lambda_{a_i}(x_1,\dots,x_n)\\ i\in\{1,\dots,n\}}} \lambda_g(x_1,\dots,x_n)\\ &= \bigcup_{\substack{c_i\in\{y\}\\ i\in\{1,\dots,n\}}} \lambda_g(c_1,\dots,c_n) = \lambda_g(y,\dots,y) = \{y\},\end{aligned}$$

which means

$$\lambda_{\diamond(g,a_1,\dots,a_n)}(x_1,\dots,x_n) = \bullet(\lambda_g,\lambda_{a_1},\dots,\lambda_{a_n})(x_1,\dots,x_n).$$

Therefore, φ has a monomorphism property.

To prove that φ is injective, suppose $\varphi(g_1) = \varphi(g_2)$ for some $g_1, g_2 \in G$, which means $\lambda_{g_1} = \lambda_{g_2}$. Because x belongs to the domain of an n-ary hyperoperation λ_g for all $g \in G$, as a result, we have $\lambda_{g_1}(x,\dots,x) = \lambda_{g_2}(x,\dots,x)$, subsequently, $\{g_1\} = \{g_2\}$. Thus, $g_1 = g_2$. This finishes the proof of the theorem. □

In [**7**], various kinds of ideals in a Menger algebra were proposed. Note that these concepts may be seen as ideals in arbitrary semigroups if $n = 1$.

A nonempty set A of a Menger algebra (G, o) of rank n is called:

(1) an *s-ideal* if $o(a, g_1,\dots,g_n) \in A$ for all $a \in A, g_1,\dots,g_n \in G$,
(2) a *v-ideal* if $o(g, a_1,\dots,a_n) \in A$ for all $g \in G, a_1,\dots,a_n \in A$,
(3) an *ideal* if it is both an s-ideal and a v-ideal.

An element g of a Menger algebra (G, o) of rank n is called *left zero* of G if $o(g, a_1,\dots,a_n) = g$ for all $a_1,\dots,a_n \in G$.

We now focus on a subclass of k-commutative n-ary hyperoperations. For any element a of a nonempty set A, one can define an n-ary hyperoperation on A, denoted by f_a and called a *constant hyperoperation*, which is defined by

$$f_a(a_1, \ldots, a_n) = \{a\}$$

for all $a_1, \ldots, a_n \in A$. The set of all constant hyperoperation on A is denoted by C_A, i.e., $C_A = \{f_a \mid a \in A\}$. Furthermore, a constant hyperoperation on a nonempty set A is defined by $f_A(a_1, \ldots, a_n) = A$ and $f_a(A_1, \ldots, A_n) = \{a\}$.

Proposition 2.7. *The following statements hold:*

1) *The set C_A is an ideal of the Menger algebra of k-commutative n-ary hyperoperations on A.*
2) *Every constant hyperoperation in C_A is a left zero element in the algebra $(\mathcal{M}_n^k(A, P^*(A)), \bullet)$.*

Proof. Clearly, $C_A \subset \mathcal{M}_n^k(A, P^*(A))$. In order to show that C_A is a v-ideal, let σ be a hyperoperation in the algebra $\mathcal{M}_n^k(A, P^*(A))$ and $\delta_{a_1}, \ldots, \delta_{a_n}$ be constant hyperoperations in C_A. For every $b_1, \ldots, b_n \in A$, we obtain

$$\begin{aligned}
\bullet(\sigma, \delta_{a_1}, \ldots, \delta_{a_n})(b_1, \ldots, b_n) &= \sigma(\delta_{a_1}(b_1, \ldots, b_n), \ldots, \delta_{a_n}(b_1, \ldots, b_n)) \\
&= \bigcup_{\substack{c_i \in \delta_{a_i}(b_1, \ldots, b_n) \\ i \in \{1, \ldots, n\}}} \sigma(c_1, \ldots, c_n) \\
&= \bigcup_{\substack{c_i \in \{a_i\} \\ i \in \{1, \ldots, n\}}} \sigma(c_1, \ldots, c_n) = \sigma(a_1, \ldots, a_n) \\
&= f_{\sigma(a_1, \ldots, a_n)}(b_1, \ldots, b_n).
\end{aligned}$$

Thus, $\bullet(\sigma, \delta_{a_1}, \ldots, \delta_{a_n}) = f_{\sigma(a_1, \ldots, a_n)} \in C_A$. Assume now that f_a is a constant hyperoperation on A and $\delta_1, \ldots, \delta_n$ are k-commutative n-ary hyperoperations on A. Then, by the definition of $\bullet$, we have

$$\bullet(f_a, \delta_1, \ldots, \delta_n)(a_1, \ldots, a_n) = \bigcup_{\substack{b_i \in \delta_i(a_1, \ldots, a_n) \\ i \in \{1, \ldots, n\}}} f_a(b_1, \ldots, b_n) = \{a\} = f_a(a_1, \ldots, a_n)$$

for all $a_1, \ldots, a_n \in A$, which implies that C_A is an s-ideal of $(\mathcal{M}_n^k(A, P^*(A)), \mathcal{O})$ and each constact hyperoperation in C_A is left zero with respect to $\bullet$. The proof is completed. □

Proposition 2.8. *Let H be a nonempty subset of a Menger hypercompositional algebra (G, o) of rank $n > 1$ such that $o(h_0, h_1, \ldots, h_n) = \{h_0\}$ for all $h_0, h_1, \ldots, h_n \in H$. Then (H, o) is a Menger hypercompositional subalgebra of (G, o) which is isomorphic to a Menger subalgebra $(C_H, \bullet)$ of $(C^n_{(i,j)}(H), \bullet)$.*

Proof. Obviously, (H, o) is a Menger hypercompositional subalgebra of (G, o). For each element $h \in H$, define a mapping $\varphi\colon (H, o) \to (C_H, \bullet)$ by $\varphi(h) = f_h$. To show that φ is a homomorphism, let $h_0, h_1, \ldots, h_n \in H$. Then we have $\varphi(o(h_0, h_1, \ldots, h_n)) = f_{\circ(h_0, h_1, \ldots, h_n)} = f_{\{h_0\}} = \bullet(\varphi(h_0), \varphi(h_1), \ldots, \varphi(h_n))$. In fact, we have $f_{\{h_0\}}(h_1, \ldots, h_n) = \{h_0\} = \bullet(f_{h_0}, f_{h_1}, \ldots, f_{h_n})(h_1, \ldots, h_n)$. Clearly, φ is a surjective mapping. Moreover, φ is injective, since for $h_1, h_2 \in H$, if

$\varphi(h_1) = \varphi(h_2)$, then $f_{h_1} = f_{h_2}$, which means $f_{h_1}(h_1, \ldots, h_n) = f_{h_2}(h_1, \ldots, h_n)$. This implies that $\{h_1\} = \{h_2\}$, subsequently, $h_1 = h_2$. Therefore, φ is an isomorphism. $\square$

3. Discussion

In this paper, the concepts of diagonal semigroup derived from a Menger algebra and other special elements with respect to a binary operation $\cdot$ are generalized to hyperstructures. In particular, the notions of diagonal semihypergroup induced by a Menger hypercompositional algebra and a diagonal scalar identity are defined. The characterization of any semihypergroup to be a diagonal semihypergroup is provided. Besides, based on the paper [**12**], we extend k-commutative operation defined on any nonempty set A to a k-commutative hyperoperation. It is proved that there is a relationship between these concepts via a monomorphism. We also give necessary and sufficient conditions under which each Menger hypercompositional algebra of rank $n > 1$ can be isomorphically embedded into an algebra of all k-commutative hyperoperations defined on some set. During our study, we have also found several questions that need further investigation. For example: Can we describe an automorphism on an algebra all k-commutative hyperoperations? and if any, What is a general form of such automorphism?

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