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# A NOTE ON THE SPECTRUM OF THE FOLDED HYPERCUBE

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ABSTRACT. The folded hypercube $FQ_n$ is the Cayley graph $\mathrm{Cay}(\mathbb{Z}_2^n, S)$, where $S = \{e_1, e_2, \dots, e_n\} \cup \{u = e_1 + e_2 + \cdots + e_n\}$, $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, with 1 at the $i$th position, $1 \le i \le n$. In this paper, the spectrum of this graph is determined by an elementary and self contained method. Then, some properties of this graph are studied.

## 1. Introduction

In this paper, a graph $\Gamma = (V, E)$ is considered as an undirected simple graph, where $V = V(\Gamma)$ is the vertex-set and $E = E(\Gamma)$ is the edge-set. For all the terminology and notation not defined here, we follow [**2, 6**].

Let $n \ge 3$ be an integer. The hypercube $Q_n$ of dimension $n$ is the graph with the vertex-set $\{(x_1, x_2, \dots, x_n) | x_i \in \{0, 1\}\}$, two vertices $(x_1, x_2, \dots, x_n)$, $(y_1, y_2, \dots, y_n)$ are adjacent if and only if $x_i = y_i$ for all but one $i$. This graph has been studied from various aspects by many authors. Some algebraic and topological aspects of this graph were studied in some recent works [**11, 13, 14, 15**]. As a variant of the hypercube, the $n$-dimensional folded hypercube $FQ_n$ was proposed first in [**4**]. The folded hypercube $FQ_n$ of dimension $n$, is the graph obtained from the hypercube $Q_n$ by adding edges, called complementary edges, between any two vertices $x = (x_1, x_2, \dots, x_n)$, $y = (\bar{x_1}, \bar{x_2}, \dots, \bar{x_n})$, where $\bar{1} = 0$ and $\bar{0} = 1$. The folded hypercube $FQ_n$ has some interesting properties. For example, although it is regular of degree $n + 1$ (while the hypercube $Q_n$ is regular of degree $n$), its diameter is almost half of the hypercube $Q_n$, that is, $\lceil \frac{n}{2} \rceil$ [**4**]. $FQ_n$ is highly symmetric, namely, it is arc-transitive [**9**], and hence its connectivity is maximum [**16**].

It can be shown that the hypercube $Q_n$ is the Cayley graph $\mathrm{Cay}(\mathbb{Z}_2^n, B)$, where $B = \{e_1, e_2, \dots, e_n\}$, $e_i$ is the element of $\mathbb{Z}_2^n$ with 1 in the $i$th position and 0 in the other positions for $1 \le i \le n$. Also, the folded hypercube $FQ_n$ is the Cayley graph $\mathrm{Cay}(\mathbb{Z}_2^n, S)$, where $S = B \cup \{u = e_1 + e_2 + \cdots + e_n\}$. Hence the hypercube $Q_n$ and the folded hypercube $FQ_n$ are vertex-transitive graphs. Since $Q_n$ is Hamiltonian

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[**17, 18**] and also it is a spanning subgraph of $FQ_n$, so $FQ_n$ is Hamiltonian. Some properties of the folded hypercube $FQ_n$ are discussed in [**5, 9, 15, 17, 18**].

The graphs shown in Figure 1 are the folded hypercubes $FQ_3$ and $FQ_4$.

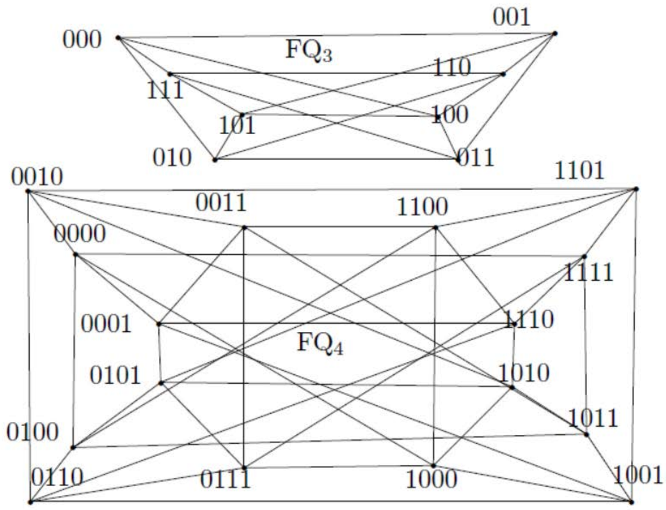


Fig. 1. The folded hypercubes $FQ_3$ and $FQ_4$.

In this paper, we determine the spectrum of the folded hypercube $FQ_n$. Up to our knowledge, there is a work in finding the spectrum of the folded hypercube in chinese language, available on Web, that uses linear algebraic methods [**3**]. Our method is completely different from that. We want to determine the spectrum of this graph by an elementary and self contained method. Our approach uses group theory. Next, we study some properties of this graph.

## 2. Preliminaries

Let $\Gamma = (V, E)$ be a finite simple graph with the vertex-set $V$ and the edge-set $E$. The adjacency matrix of $\Gamma$ is a (0-1) matrix indexed by the vertex set $V$, where $A_{vw} = 1$ when the vertices $v, w$ are adjacent and $A_{vw} = 0$ otherwise.

The eigenvalues of a graph are the eigenvalues of its adjacency matrix. The relationships between the algebraic properties of these eigenvalues and the usual (topological and geometric) properties of graphs have been studied quite intensively. The study of the relations between eigenvalues and structures in graphs is the heart of spectral graph theory. Thus someone interested in using spectral graph theory needs to be familiar with both the graph theory and the basic tools of linear algebra including eigenvalues, eigenvectors, determinants and so on. Most introductory linear algebra courses impart the belief that the way to compute the eigenvalues of a matrix is to find the zeros of its characteristic polynomial. For matrices with order greater than two, this is false. Generally, the best way to obtain eigenvalues lies in finding eigenvectors: If $Av = \lambda v, v \neq 0$, then $\lambda$ is an eigenvalue of A, and $v$ is an eigenvector of $A$ corresponding to $\lambda$ [**6**]. When we work with graphs, there is an additional refinement. We can suppose that an eigenvector is a real function $f$ defined on the vertex-set. Then if at any vertex $v$ you sum up the values of $f$ on its neighboring vertices, you should get $\lambda$ times the values of $f$ at $v$. Formally,

$$\sum_{w \in N(v)} f(w) = \lambda f(v).$$

In this paper, we think about eigenvectors and eigenvalues in this manner.

Let $G$ be a finite abelian group (written aditively) of order $|G|$ with identity element $0{=}0_G$. A character $\chi$ of $G$ is a homomorphism from $G$ into the multiplicative group $U$ of complex numbers of absolute value 1, that is, a mapping from $G$ into $U$ with $\chi(g_1 + g_2) = \chi(g_1)\chi(g_2)$ for all $g_1, g_2 \in G$. If $G$ is a finite abelian group, then there are integers $n_1, \ldots, n_k$, such that $G \cong \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$.

Now if $\omega_{ij} = \mathrm{e}^{\frac{2\pi i j}{n_i}}$, $0 \leq i \leq k$, $1 \leq j \leq n_i$, is an $n_i$th root of unity, then each homomorphism of $f\colon G \longrightarrow \mathbb{C}^*$, is of the form $f = f_{(\omega_1,\ldots,\omega_k)}$ with the rule $f_{(\omega_1,\ldots,\omega_k)}(x_1, \ldots, x_k) = \omega_1^{x_1}\omega_2^{x_2}\ldots\omega_k^{x_k}$ [**8**].

Let $G$ be any abstract finite group with identity 1, and suppose $\Omega$ is a subset of $G$, with the properties:

(i) $x \in \Omega \Longrightarrow x^{-1} \in \Omega$;
(ii) $1 \notin \Omega$.

The Cayley graph $\Gamma = \mathrm{Cay}(G, \Omega)$ is the (simple) graph whose vertex-set and edge-set are defined as follows:

$$V(\Gamma) = G, \qquad E(\Gamma) = \{\{g, h\} \mid g^{-1}h \in \Omega\}.$$

## 3. Main results

Let $G$ be a finite abelian group and $S = \{s_1, \ldots, s_n\}$ be a non-empty subset of $G$ such that $0 \notin S$ and $S = -S$. Let $\Gamma = \mathrm{Cay}(G, S)$. Assume $f\colon G \longrightarrow \mathbb{C}^*$ is a group homomorphism, where $\mathbb{C}^*$ is the multiplicative group of the complex numbers. If $v$ is a vertex of $\Gamma$, then we know that $N(v) = \{v + s_1, \ldots, v + s_n\}$ is the set of vertices that are adjacent to $v$. We now have

$$\sum_{w \in N(v)} f(w) = \sum_{i=1}^{n} f(v + s_i) = \sum_{i=1}^{n} f(v)f(s_i) = f(v)\Big(\sum_{i=1}^{n} f(s_i)\Big).$$

Therefore, if we let $\lambda = \lambda_f = \sum_{s \in S} f(s)$, then we have $\sum_{w \in N(v)} f(w) = \lambda_f f(v)$, and hence the mapping $f$ is an eigenvector for the Cayley graph $\Gamma$ with corresponding eigenvalue $\lambda = \lambda_f = \sum_{s \in S} f(s)$.

**Theorem 3.1.** *Let $G = \mathbb{Z}_2^n$, $S = \{e_1, \ldots, e_n, u\}$, where $u = e_1 + \cdots + e_n = \sum_{i=1}^{n} e_i$. If $\Gamma = FQ_n = \mathrm{Cay}(\mathbb{Z}_2^n, S)$, then each eigenvalue of $\Gamma$ is of the form*

$\lambda_i = (n-2i) + (-1)^i$, $0 \le i \le n$. *Moreover, if* $i \in \{1, 3, \dots, 2m-1\}$, $2m-1 \le n$, *then the multiplicity of the eigenvalue* $(n-2i)+(-1)^i$ *is* $\binom{n}{i} + \binom{n}{i+1,}$ *and* $\lambda_i = \lambda_{i+1}$.

*Proof.* According to what is stated above, every eigenvector of the graph $\Gamma = FQ_n = \mathrm{Cay}(\mathbb{Z}_2^n, S)$ is of the form $f = f_{(\omega_1,\dots,\omega_n)}$, where each $\omega_i$, $1 \le i \le n$, is a complex number such that $\omega_i^2 = 1$, namely, $\omega_i \in \{1, -1\}$. Let $x$ be a vertex in the folded hyper cube $FQ_n$. We now have

$$\begin{aligned} \sum_{w \in N(x)} f(w) &= \sum_{i=1}^{n} f(x+e_i) + f(x+u) \\ &= \sum_{i=1}^{n} f(x)f(e_i) + f(x)f(u) \\ &= \sum_{i=1}^{n} f(x)f(e_i) + f(x)f\Big(\sum_{i=1}^{n} e_i\Big) \\ &= f(x)\Big(\sum_{i=1}^{n} f(e_i) + \prod_{i=1}^{n} f(e_i)\Big) = \lambda_f f(x), \end{aligned}$$

where $\lambda_f = \sum_{i=1}^{n} f(e_i) + \prod_{i=1}^{n} f(e_i)$. Note that for every vertex $v = (x_1, \dots, x_n)$, $x_i \in \{0, 1\}$ in $\Gamma$, we have $f(x_1, \dots, x_n) = f_{(w_1,\dots,w_n)}(x_1 \dots, x_n) = w_1^{x_1} \dots w_n^{x_n}$. Since in computing the value of $w_1^{x_1} \dots w_n^{x_n}$ we can ignore $w_i$ when $w_i = 1$, thus for $e_k = (0, \dots, 0, 1, 0 \dots, 0)$, where 1 is the $k$th entry, we have

$$\begin{aligned} f(e_k) &= f_{(w_1,\dots,w_n)}(0, \dots, 0, 1, 0, \dots, 0) \\ &= w_1^0 \dots w_k^1 w_{k+1}^0 \dots w_n^0 \\ &= \begin{cases} -1 & \text{if } \ w_k = -1, \\ 1 & \text{if } \ w_k = 1. \end{cases} \end{aligned}$$

Hence, if in the sequence ($n$-tuple) $(w_1, \dots, w_n)$ the number of $-1s$ is $i$ (and therefore the number of $1s$ is $(n-i)$), then in the sum

$$\sum_{k=1}^{n} f(e_k) = \sum_{k=1}^{n} f_{(w_1,\dots,w_n)}(0, \dots, x_k, 0, \dots, 0), \qquad x_k = 1,$$

the contribution of $-1$ is $i$ and the contribution of 1 is $n-i$, and hence we have $\sum_{k=1}^{n} f(e_k) = -i + (n-i) = n-2i$. On the other hand, we have

$$f_{(w_1,\dots,w_n)}(u) = f_{(w_1,\dots,w_n)}\Big(\sum_{i=1}^{n} e_i\Big) = \prod_{i=1}^{n} f_{(w_1,\dots,w_n)}(e_i) = (-1)^i 1^{(n-i)} = (-1)^i,$$

and therefore,

$$\sum_{s \in S} f_{(w_1,\dots,w_n)}(s) = (n-2i) + (-1)^i = \lambda_i, \qquad 0 \le i \le n.$$

Note that the number of sequence $(w_1 \dots, w_n)$ such that in which $i$ entries are $-1$ is $\binom{n}{i}$, where $0 \le i \le n$. It is an easy task to show that if $i > j+1$, then

$(n-2i)+(-1)^i \neq (n-2j)+(-1)^j$. Now, note that for $i = 2t-1,\ t > 0$, we have $(n-2i)+(-1)^i = (n-2(2t-1))+(-1)^{(2t-1)} = n-4t+2-1 = n-4t+1$, also $(n-2(2t))+(-1)^{2t} = n-4t+1$. Hence we have $\lambda_i = \lambda_{i+1}$. This observation shows that it is sufficient to consider the eigenvalue $\lambda_i = (n-2i)+(-1)^i$ for $i \in \{1, 3, \ldots, 2m-1\}$, $2m-1 \leq n$.
Now by a simple calculation, we can see that the multiplicity of the eigenvalue $\lambda_i = (n-2i)+(-1)^i$, $i \in \{1, 3, \ldots, 2m-1\}$, is $\binom{n}{i} + \binom{n}{i+1}$. For $i = 0$, we have $\lambda_0 = n + (-1)^0 = n+1$, and the multiplicity of this eigenvalue is 1.

We can also see that if $n = 2m+1$ is an odd integer, then for $i = n = 2m+1$, we have $\lambda_i = (n-2i)+(-1)^i = -n-1$, and the multiplicity of this eigenvalue is 1. □

By a similar argument, we can obtain the following result.

**Corollary 3.2.** *Let $G = \mathbb{Z}_2^n$, $S_1 = \{e_1, \ldots, e_n\}$. If $\Gamma = Q_n = \mathrm{Cay}(\mathbb{Z}_2^n, S_1)$, then each eigenvalue of $\Gamma$ is of the form $\lambda_i = (n-2i)$, $0 \leq i \leq n$, with multiplicity $\binom{n}{i}$.*

**Remark 3.3.** Note that if $n = 2m+1$ is an odd integer, then for the odd integers $i$, $i \in \{1, 3, \ldots, 2m-1\}$, $n-i$ is an even integer, hence we have $\lambda_{n-i} = (n-2(n-i))+1 = -n+2i+1 = -\lambda_i = \lambda_{n-i-1}$. Note also that in this case, the multiplicity of $\lambda_i$ and $\lambda_{n-i}$ are the same. It is clear that $\lambda_{2m+1} = -n-1 = -\lambda_0$.

**Some results from Theorem 3.1**

A graph $\Gamma$ is called *integral* if all its eigenvalues are integers. The study of integral graphs was initiated by Harary and Schwenk in 1974 (see [**7**]). A survey of papers up to 2002 appeared in [**1**]. Recently, some classes of integral graphs were studied in [**10, 12**]. From Corollary 3.2, it follows that the hypercube $Q_n$ is an integral graph. It was also shown how new classes of integral graphs from the hypercube $Q_n$ can be constructed [**11, 14**]. From Theorem 3.1, we have the following result.

**Theorem 3.4.** *The folded hypercube $FQ_n$ is an integral graph.*

Let $\Gamma = (V, E)$ be a bipartite graph. Then we have the following result.

**Theorem 3.5** ([**2, 6**]). *A graph $\Gamma$ is bipartite if and only if its spectrum is symmetric about* 0.

From Theorem 3.1 and Remark 3.3, we can see that if $n = 2m$ is an even integer, then the spectrum of the folded hypercube $FQ_n$ is not symmetric about 0. Note that if $n = 2m$, then $\lambda_0 = n+1$ is the largest eigenvalue of $FQ_n$, while $\lambda_n = -n+1$ is the least eigenvalue of $FQ_n$. Hence $-n-1$ is not an eigenvalue of the graph $FQ_n$. On the other hand, if $n = 2m+1$ is an odd integer, then the spectrum of $FQ_n$ is symmetric about 0.

Now, from Theorem 3.1 and 3.4, we can conclude the following result.

**Theorem 3.6.** *The folded hypercube $FQ_n$ is a bipartite graph if and only if $n$ is an odd integer.*

The present result was obtained in [**17**] in a different manner.

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