Acta Math. Univ. Comenianae
Vol. XCIII, 3 (2024), pp. 129–136

LINE-GRACEFUL DESIGNS

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ABSTRACT. In [**3**], the authors adapted the edge-graceful graph labeling definition into block designs. In this article, we adapt the line-graceful graph labeling definition into block designs and define a block design $(V, \mathcal{B})$ with $|V| = v$ as line-graceful if there exists a function $f\colon \mathcal{B} \to \{0, 1, \dots, v-1\}$ such that the induced mapping $f^+\colon V \to \mathbb{Z}_v$ given by $f^+(x) = \sum_{A\in\mathcal{B}\,:\,x\in A} f(A) \pmod{v}$ is a bijection. In this article, the cases that are incomplete in terms of block-graceful labelings, are completed in terms of line-graceful labelings. Moreover, we prove that there exists a line-graceful Steiner quadruple system of order 2^n for all $n \geq 3$ by using a recursive construction.

1. INTRODUCTION

A *design* (or *block design*) is a pair $(V, \mathcal{B})$, where V is a finite set of points or symbols, and $\mathcal{B}$ is a collection (i.e., multiset) of nonempty subsets of V called blocks. Let t, v, k and λ be positive integers such that $t \leq k < v$. A t-(v, k, λ)-*design* is a design $(V, \mathcal{B})$ such that $|V| = v$, each block contains exactly k points, and every set of t distinct points is contained in exactly λ blocks. A 2-(v, k, λ)-design is also called a (v, k, λ)-*balanced incomplete block design* ($(v, k, \lambda)-$BIBD). In a t-(v, k, λ)-design, every point occurs in exactly $r = \lambda\binom{v-1}{t-1}/\binom{k-1}{t-1}$ blocks and there are exactly $b = \lambda\binom{v}{t}/\binom{k}{t}$ blocks. The number r is called the *repetition number* of the design. The *complement* $(V, \overline{\mathcal{B}})$ of a $(v, k, \lambda)-$BIBD $(V, \mathcal{B})$ with b blocks and repetition number r, where $\overline{\mathcal{B}} = \{V \setminus A : A \in \mathcal{B}\}$, is a $(v, v-k, b-2r+\lambda)-$BIBD.

Two designs $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are called *isomorphic* if there exists a bijection $\alpha\colon X \to Y$ such that $\{\{\alpha(x) : x \in S\} : S \in \mathcal{A}\} = \mathcal{B}$. Furthermore, α is called an *isomorphism*. An *automorphism* of a design is an isomorphism of this design with itself. An automorphism of a design partitions its blocks into classes called *orbits*. A t-(v, k, λ)-design is *cyclic* if it has an automorphism that is a permutation consisting of a single cycle of length v.

Let v, k and λ be positive integers such that $v > k \geq 2$, and G be a finite additive group of order v. A (v, k, λ)-*difference family* is a collection $[B_1, B_2, \dots, B_n]$ of

Received May 19, 2023; revised September 28, 2024.
2020 *Mathematics Subject Classification.* Primary 05B99, 05C78.
Key words and phrases. Line-graceful design; Steiner triple system; Steiner quadruple system; affine geometry; projective geometry.

k element subsets of G such that the multiset union $\bigcup_{i=1}^{n} [x-y : x,y \in B_i, x \neq y]$ contains every nonzero element of G exactly λ times. The difference family is called *cyclic* if G is cyclic. The development $\mathrm{Dev}(B_1, B_2, \dots, B_n) = [B_i + g : 1 \leq i \leq n \text{ and } g \in G]$, where $B_i + g = \{x + g : x \in B_i\}$, generates the block set of a $(v,k,\lambda)-$BIBD on the point set G [**1, 7**].

A *graph* is an ordered triple $G = (V(G), E(G), I_G)$, where $V(G)$ is a nonempty set, $E(G)$ is a set disjoint from $V(G)$, and I_G is an "incidence" relation that associates with each element of $E(G)$ an unordered pair of elements (same or distinct) of $V(G)$. Elements of $V(G)$ are called the vertices (or nodes or points) of G, and elements of $E(G)$ are called the edges (or lines) of G [**2**]. A graph labeling is an assignment of integers to the vertices, edges, or both of a graph subject to certain conditions. Since mid 1960s, over 200 graph labeling techniques have been studied in over 3000 papers. For a dynamic survey on graph labeling, see [**4**].

In [**3**], the authors adapted the edge-graceful graph labeling definition into block designs and defined a block design $(V,\mathcal{B})$ with $|V| = v$ and $|\mathcal{B}| = b$ as block-graceful if there exists a bijection $f\colon \mathcal{B} \to \{1,2,\dots,b\}$ such that the induced mapping $f^+\colon V \to \mathbb{Z}_v$ given by $f^+(x) = \sum_{A\in\mathcal{B}\,:\,x\in A} f(A) \pmod v$ is a bijection.

R. B. Gnanajothi defined a graph with n vertices *line-graceful* if it is possible to label its edges with $0,1,2,\dots,n$ such that when each vertex is assigned the sum modulo n of all the edge labels incident with that vertex the resulting vertex labels are $0,1,2,\dots,n-1$ [**4**]. In this article, we adapt this definition into block designs and define a block design $(V,\mathcal{B})$ with $|V| = v$ as *line-graceful* if there exists a function $f\colon \mathcal{B} \to \{0,1,\dots,v-1\}$ such that the induced mapping $f^+\colon V \to \mathbb{Z}_v$ given by $f^+(x) = \sum_{A\in\mathcal{B}\,:\,x\in A} f(A) \pmod v$ is a bijection. For a block $A \in \mathcal{B}$ and a point $x \in V$, the values $f(A)$ and $f^+(x)$ are called the weights of A and x, respectively.

Example 1.1. A line-graceful labeling of the 3-$(8,4,1)$-design with $V = \{0,1,3,4,5,6,7\}$ and $\mathcal{B} = \{\{0,1,3,7\},\{1,2,4,7\},\{2,3,5,7\},\{3,4,6,7\},$ $\{4,5,0,7\},\{5,6,1,7\},\{6,0,2,7\},\{2,4,5,6\},\{0,3,5,6\},\{0,1,4,6\},\{0,1,2,5\},$ $\{1,2,3,6\},\{0,2,3,4\},\{1,3,4,5\}\}$ is shown below:

$\{0,1,3,7\} \to 1$	$\{2,4,5,6\} \to 5$	$f^+(0) = 1+0+6+5+4+7+3 = 26 \equiv 2 \pmod 8$
$\{1,2,4,7\} \to 6$	$\{0,3,5,6\} \to 5$	$f^+(1) = 1+6+4+4+7+3+2 = 27 \equiv 3 \pmod 8$
$\{2,3,5,7\} \to 1$	$\{0,1,4,6\} \to 4$	$f^+(2) = 6+1+6+5+7+3+3 = 31 \equiv 7 \pmod 8$
$\{3,4,6,7\} \to 2$	$\{0,1,2,5\} \to 7$	$f^+(3) = 1+1+2+5+3+3+2 = 17 \equiv 1 \pmod 8$
$\{4,5,0,7\} \to 0$	$\{1,2,3,6\} \to 3$	$f^+(4) = 6+2+0+5+4+3+2 = 22 \equiv 6 \pmod 8$
$\{5,6,1,7\} \to 4$	$\{0,2,3,4\} \to 3$	$f^+(5) = 1+0+4+5+5+7+2 = 24 \equiv 0 \pmod 8$
$\{6,0,2,7\} \to 6$	$\{1,3,4,5\} \to 2$	$f^+(6) = 2+4+6+5+5+4+3 = 29 \equiv 5 \pmod 8$
		$f^+(7) = 1+6+1+2+0+4+6 = 20 \equiv 4 \pmod 8$

It is clear from the definitions that every block-graceful design is also line-graceful. In [**3**], the authors noted that one can easily obtain block-graceful Steiner triple systems of order v for all $v \equiv 1 \pmod 6$, and block-graceful projective geometries, i.e. $(\frac{q^{d+1}-1}{q-1}, \frac{q^d-1}{q-1}, \frac{q^{d-1}-1}{q-1})-$BIBDs for every prime power $q \geq 2$ and integer $d \geq 2$. Then the authors constructed infinite families of block-graceful Steiner triple systems of order v for $v \equiv 3 \pmod 6$, and by considering affine geometries, proved that for every integer $d \geq 2$ and $q \geq 3$, where q is an odd prime power or $q = 4$, there exists a block-graceful $(q^d, q, 1)-$BIBD.

In this article, we study the existence problem of line-graceful t-(v, k, λ)-designs. In Section 2, we give a necessary condition for the existence of a line-graceful t-(v, k, λ)-design and prove some basic results. The cases for Steiner triple systems and $(q^d, q, 1)-$BIBDs, that are incomplete in terms of block-graceful labelings, are completed in terms of line-graceful labelings in Sections 3 and 5, where we prove that there exists a line-graceful Steiner triple system of order v for all $v \equiv 1, 3 \pmod 6$, and a line-graceful $(q^d, q, 1)-$BIBD for every prime power $q \geq 2$ and integer $d \geq 2$. Moreover, in Section 4, we prove that there exists a line-graceful Steiner quadruple system of order 2^n for all $n \geq 3$ by using a recursive construction.

2. Basic results

In this section, we give a necessary condition for the existence of a line-graceful t-(v, k, λ)-design and prove some basic results.

Proposition 2.1. *If a t-(v, k, λ)-design is line-graceful, where $k = 2^\alpha s$ such that α is a nonnegative integer and s is odd, then either v is odd or $v \equiv 0 \pmod{2^{\alpha+1}}$.*

Proof. Let $(V, \mathcal{B})$ be a line-graceful t-(v, k, λ)-design, where its blocks are labeled as $a_1, a_2, \dots, a_b$. Then

$$\sum_{x \in V} f^+(x) = k(a_1 + a_2 + \cdots + a_b) \equiv \frac{(v-1)v}{2} \pmod v. \tag{1}$$

If v is odd, $\frac{(v-1)v}{2} \equiv 0 \pmod v$ and (1) can be satisfied by choosing the block labels in such a way that their sum is a multiple of v. On the other hand, if v is even, $\frac{(v-1)v}{2} \equiv \frac{v}{2} \pmod v$ and hence $2^\alpha s(a_1 + a_2 + \cdots + a_b)$ must be an odd multiple of $\frac{v}{2}$ which can be satisfied only if $v \equiv 0 \pmod{2^{\alpha+1}}$. □

In [**3**], the authors showed that every $(v, k, \lambda)-$BIBD with repetition number r that is generated from a cyclic difference family is block-graceful when $\gcd(v, r) = 1$. As a consequence of this result, all such BIBDs are also line-graceful. For completeness, we repeat the proof of this result in the line-graceful case here.

Proposition 2.2. *If $\gcd(v, r) = 1$, then every $(v, k, \lambda)-$BIBD with repetition number r that is generated from a cyclic difference family is line-graceful.*

Proof. Let $[B_1, \dots, B_n]$ be a cyclic (v, k, λ)-difference family on $\mathbb{Z}_v$ and let $\mathcal{B} = \mathrm{Dev}(B_1, \dots, B_n)$. Define a labeling $f\colon \mathcal{B} \to \{0, 1, \dots, v-1\}$ such that $f(B_i + g) = g$ for any $g \in \mathbb{Z}_v$ and $1 \leq i \leq n$. For any $a \in \mathbb{Z}_v$, we get $f^+(a) \equiv f^+(0) + ar \pmod v$. Since $\gcd(v, r) = 1$, the numbers $0, r, 2r, \dots, (v-1)r$ are all different mod v, and the result follows. □

Proposition 2.3. *If $(V, \mathcal{B})$ is a line-graceful $(v, k, \lambda)-$BIBD, then its complement $(V, \overline{\mathcal{B}})$ is also line-graceful.*

Proof. Suppose that $(V, \mathcal{B})$ has a line-graceful labeling $f\colon \mathcal{B} \to \{0, 1, \ldots, v-1\}$ with the induced mapping $f^+\colon V \to \mathbb{Z}_v$ on points, where the blocks are labeled as $a_1, a_2, \ldots, a_b$. Define $g\colon \overline{\mathcal{B}} \to \{0, 1, \ldots, v-1\}$ such that for every block $A \in \mathcal{B}$, $g(V \setminus A) = f(A)$. Then, the weight of any point $x \in V$ in the complement design will be $g^+(x) \equiv (a_1 + a_2 + \cdots + a_b) - f^+(x) \pmod v$. Since $f^+(x)$s are different mod v, $g^+(x)$s must also be different mod v, and the result follows. □

Proposition 2.4. *If there exists a line-graceful $(v, k, \lambda)-$BIBD and m is a positive integer, then there exists a line-graceful $(v, k, m\lambda)-$BIBD.*

Proof. Take m copies of each block in a line-graceful $(v, k, \lambda)-$BIBD, label the blocks in the first copy as they are labeled in the original design, and label the remaining blocks as 0. □

Note that, as a consequence of Propositions 2.3 and 2.4, to determine the set of parameters for which a line-graceful BIBD exists, it is sufficient to consider only the $(v, k, \lambda)-$BIBDs where $k \le v/2$ and λ is the minimum value satisfying the necessary conditions for the existence of a $(v, k, \lambda)-$BIBD and the condition in Proposition 2.1.

3. Steiner triple systems

A $(v, 3, 1)-$BIBD is called a *Steiner triple system of order* v and is denoted by STS(v). There exists an STS(v) if and only if $v \equiv 1$ or $3 \pmod 6$ [**7**].

In [**3**], the authors showed that there exists a block-graceful STS(v) for all $v \equiv 1 \pmod 6$, and proved the following result for the case $v \equiv 3 \pmod 6$.

Theorem 3.1 ([**3**]). *There exist block-graceful Steiner triple systems of order*
 (i) 3^t *for every integer* $t \ge 2$,
 (ii) $3^t 5^u$ *for all positive integers* t *and* u *with* $t \ge u$, *and*
 (iii) $3^t 7^u$ *for all positive integers* t *and* u.

The case $v \equiv 3 \pmod 6$ is far from complete for block-graceful STS(v). In this section we prove that there exists a line-graceful STS(v) for all $v \equiv 1, 3 \pmod 6$.

Theorem 3.2. *If there exists a line-graceful STS(v) where $v \equiv 3 \pmod 6$, then there exists a line-graceful STS($3v$).*

Proof. Let $(V, \mathcal{B})$ be an STS(v) with $v \equiv 3 \pmod 6$, where $V = \{0, 1, \ldots, v-1\}$, and $f\colon \mathcal{B} \to \{0, 1, 2, \ldots, v-1\}$ be a line-graceful labeling with the induced mapping $f^+\colon V \to \mathbb{Z}_v$ on points. Define $X = V \times \{0, 1, 2\}$, $\mathcal{C}_1 = \{\{(c, i), (d, i), (e, i)\} : \{c, d, e\} \in \mathcal{B},\ i \in \{0, 1, 2\}\}$, $\mathcal{C}_2 = \{\{(c, 0), (d, 1), (c + d \pmod v, 2)\} : c, d \in V\}$ and $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. Then, it is well known and can be easily seen that $(X, \mathcal{C})$ is an STS($3v$).

We define $g\colon \mathcal{C} \to \{0, 1, 2, \ldots, 3v-1\}$ as follows. Take $g(\{(c,i),(d,i),(e,i)\}) = 3f(\{c,d,e\}) + i$ for all $\{c,d,e\} \in B$ and $i \in \{0,1,2\}$, and $g(\{(c,0),(d,1),(c+d \pmod v), 2)\}) = 0$ for $c, d \in V$.

For any $a \in V$ and $i \in \{0,1,2\}$, we get $g^+(a,i) = 3f^+(a) + ri$, where $r = (v-1)/2$ is the repetition number of the STS(v). Since $v \equiv 3 \pmod 6$, we get $r \equiv 1 \pmod 3$ and hence $\gcd(r,3) = 1$. Since $f^+(a)s$ are all different mod v, we see that the values $3f^+(a) + ri$ are all different mod $3v$. □

A difference triple in $\mathbb{Z}_v$ is a 3-element subset of $\{1, 2, \ldots, (v-1)/2\}$ such that either the sum of these 3 elements is $0 \pmod v$ or one element is the sum of the other two mod v. For all $v \equiv 1, 3 \pmod 6, v \neq 9$, there exists a cyclic STS(v), and these designs can be constructed using difference triples that partition $\{1, 2, \ldots, (v-1)/2\}$ when $v \equiv 1 \pmod 6$, and $\{1, 2, \ldots, (v-1)/2\} \setminus \{v/3\}$ when $v \equiv 3 \pmod 6$. For each difference triple $\{x, y, z\}$ with $x + y \equiv \pm z \pmod v$, the base block $\{0, x, x+y\}$ is developed $\pmod v$. For $v \equiv 1 \pmod 6$, all orbits are of length v and the base blocks obtained from the difference triples form cyclic difference families, while for $v \equiv 3 \pmod 6$, in addition to the orbits of length v that are generated from the difference triples, one orbit of length $\frac{v}{3}$ is developed from the base block $\{0, v/3, 2v/3\}$. The difference triples that generate STS(v) for $v \equiv 3, 15 \pmod{18}$ are given in Table 1 [**7**].

Table 1. Difference triples of cyclic Steiner triple systems for $v \equiv 3, 15 \pmod{18}$ [**7**]

$v = 18s + 3,\ s \geq 1$	$\{3r+1, 8s-r+1, 8s+2r+2\}, 0 \leq r \leq s-1$ $\{3r+2, 4s-r, 4s+2r+2\}, 0 \leq r \leq s-1$ $\{3r+3, 6s-2r-1, 6s+r+2\}, 0 \leq r \leq s-1$
$v = 15$	$\{1,3,4\}, \{2,6,7\}$
$v = 18s + 15,\ s \geq 1$	$\{3r+1, 4s-r+3, 4s+2r+4\}, 0 \leq r \leq s$ $\{3r+2, 8s-r+6, 8s+2r+8\}, 0 \leq r \leq s$ $\{3r+3, 6s-2r+3, 6s+r+6\}, 0 \leq r \leq s-1$

Theorem 3.3. *There exists a line-graceful STS(v) for all $v \equiv 1, 3 \pmod 6$.*

Proof. The case $v \equiv 1 \pmod 6$ follows from Proposition 2.2. Let $v \equiv 3, 15 \pmod{18}$ and $(\mathbb{Z}_v, \mathcal{B})$ be a cyclic STS(v) constructed using the difference triples given in Table 1 as described above. We define a labeling $f\colon \mathcal{B} \to \{0, 1, \ldots, v-1\}$.

We call one of the orbits of length v as *grouped orbit* as follows. If $v = 18s+3$ for some integer $s \geq 1$ take the orbit with base block $\{0, 3, 6s+2\}$, if $v = 15$ take the orbit with base block $\{0,1,4\}$, if $v = 33$ take the orbit with base block $\{0,4,10\}$, and if $v = 18s+15$ for some integer $s \geq 2$ take the orbit with base block $\{0, 6, 6s+7\}$ and call this orbit as grouped orbit. Also call one of the remaining orbits of length v as *ordered orbit*.

Let A_1 and A_2 be the base blocks of the ordered orbit and grouped orbit, respectively. We label the blocks in these two orbits as follows. For any $x \in \mathbb{Z}_v$, define $f(A_1+x)=x$, $f(A_2+x)=1$ if $x\equiv 0 \pmod 3$, and $f(A_2+x)=0$ if $x\not\equiv 0 \pmod 3$. Moreover, we label all blocks in the remaining orbits as 0.

For any $x\in\mathbb{Z}_v$, the contribution of the block weights in the ordered orbit to the point weights gives $f^+(x)\equiv f^+(0)+3x \pmod v$ for all $x\in\mathbb{Z}_v$.

If $v=18s+3$ for some integer $s\geq 1$, the contribution of the block weights in the grouped orbit will be $1+1+0=2$ for $x\equiv 0 \pmod 3$, $0+0+0=0$ for $x\equiv 1 \pmod 3$, and $0+0+1=1$ for $x\equiv 2 \pmod 3$. If we add these to the point weights obtained from the ordered orbit, then $f^+(x)$s become all different mod v since $v/3=6s+1$ is relatively prime with 3. The case $v=18s+15$ for $s\geq 0$ follows similarly using the base blocks of the grouped orbits given above.

There exists a line-graceful STS(9) by Theorem 3.1. Then by induction and using the cases above, line-graceful STS(v) for $v=18s+9$ with $s\geq 1$ can be constructed using Theorem 3.2. □

4. Steiner quadruple systems

A 3-$(v,4,1)$-design is called a *Steiner quadruple system of order v* and is denoted by SQS(v). There exists an SQS(v) if and only if $v\equiv 2,4 \pmod 6$ [**8**].

By Proposition 2.1, a line-graceful *SQS*(v) must satisfy $v\equiv 0 \pmod 8$. Therefore, we get the following necessary condition for the existence of a line-graceful SQS(v).

Proposition 4.1. *If an SQS(v) is line-graceful, then $v\equiv 8,16 \pmod{24}$.*

We will now construct an infinite family of line-graceful Steiner quadruple systems using a well-known recursive construction, leaving 40 as the smallest order for which the existence of a line-graceful Steiner quadruple system is unknown.

Theorem 4.1. *If there exist a line-graceful SQS(v), then there exists a line graceful SQS($2v$).*

Proof. Let $(V,\mathcal{B})$ be an SQS(v) where $f\colon \mathcal{B}\to\{0,1,\dots,v-1\}$ is a line-graceful labeling with the induced mapping $f^+\colon V\to\mathbb{Z}_v$ on points. Define $X=\{0,1\}\times V$, $\mathcal{C}_1=\{\{(i,x),(j,y),(k,z),(l,t)\}:\{x,y,z,t\}\in\mathcal{B},\ i,j,k,l\in\{0,1\}, i+j+k+l\equiv 0 \pmod 2\}$, $\mathcal{C}_2=\{\{(0,a),(0,b),(1,a),(1,b)\}:a,b\in V,\ a\neq b\}$, and $\mathcal{C}=\mathcal{C}_1\cup\mathcal{C}_2$. It is known that $(X,\mathcal{C})$ is an SQS($2v$), see [**5**].

Define $g\colon \mathcal{C}\to\{0,1,\dots,2v-1\}$ as follows. Take $g(\{(i,x),(i,y),(i,z),(i,t)\})=2f(\{x,y,z,t\})+i$ for all $\{x,y,z,t\}\in\mathcal{B}$ and $i\in\{0,1\}$, and label the remaining blocks as 0. For all $x\in V$, the weight of the point $(0,x)$ is $2f^+(x)$, and the weight of the point $(1,x)$ is $2f^+(x)+r$ where $r=(v-1)(v-2)/6$ is the repetition number of $(V,\mathcal{B})$. Since $v\equiv 8,16 \pmod{24}$ by Proposition 4.1, r must be odd. Hence, the point weights are all different mod $2v$ since $f^+(x)s$ are all different mod v. □

Theorem 4.2. *There exists a line-graceful SQS(2^n) for all $n\geq 3$.*

Proof. Follows by Example 1.1 and Theorem 4.1. □

5. Affine and projective geometries

An $(n^2+n+1, n+1, 1)-$BIBD with $n \geq 2$ is called a *projective plane of order* n, and an $(n^2, n, 1)-$BIBD with $n \geq 2$ is called an *affine plane of order* n. It is well known that for every prime power $q \geq 2$, there exists a projective plane of order q and an affine plane of order q [**8**].

A generalization to higher dimensions shows that for any $d \geq 2$, the points and hyperplanes of the d-dimensional projective geometry $PG_d(q)$ form a $(\frac{q^{d+1}-1}{q-1}, \frac{q^d-1}{q-1}, \frac{q^{d-1}-1}{q-1})-$BIBD with repetition number $r = (q^d-1)/(q-1)$, and it is known that these BIBDs can be generated from cyclic difference families, see [**6, 8**]. The existence of line-graceful $(\frac{q^{d+1}-1}{q-1}, \frac{q^d-1}{q-1}, \frac{q^{d-1}-1}{q-1})-$BIBDs immediately follows from Proposition 2.2.

In [**3**], the authors also considered affine geometries and proved that for every integer $d \geq 2$ and $q \geq 3$, where q is an odd prime power or $q = 4$, there exists a block-graceful $(q^d, q, 1)-$BIBD. In this section, we prove that there exists a line-graceful $(q^d, q, 1)-$BIBD for every prime power $q \geq 2$ and integer $d \geq 2$.

Let $\mathbb{F}_q$ be the field with q elements and $X = \mathbb{F}_q \times \mathbb{F}_q$. For all $a, b \in \mathbb{F}_q$ define $C_{a,b} = \{(x, y) \in X : y = ax + b\}$ and $\mathcal{B}_1 = \{C_{a,b} : a, b \in \mathbb{F}_q\}$. Also, for all $c \in \mathbb{F}_q$ define $C_c = \{(x, y) \in X : x = c\}$ and $\mathcal{B}_2 = \{C_c : c \in \mathbb{F}_q\}$. Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. Then, $(X, \mathcal{B})$ is an affine plane of order q [**8**]. We will now make a line-graceful labeling of the blocks in this structure.

Theorem 5.1. *There exists a line-graceful affine plane of order q for every prime power $q \geq 2$.*

Proof. Let $(X, \mathcal{B})$ be an affine plane of order q as described above. Take any bijection $h\colon \mathbb{F}_q \to \{0, 1, 2, \ldots, q-1\}$ and define a labeling $f\colon \mathcal{B} \to \{0, 1, 2, \ldots, q^2 - 1\}$ as follows. For all $t \in \mathbb{F}_q$ define $f(C_{0,t}) = h(t)$, $f(C_t) = qh(t)$, and label the remaining blocks as 0. Then, $f^+(r, s) = qh(r) + h(s)$ for all $(r, s) \in X$ and hence the point weights are all different mod q^2. □

Theorem 5.2. *If there exists a line-graceful $(q^d, q, 1)-$BIBD, where q is a prime power and $d \geq 2$, then there exists a line-graceful $(q^{d+1}, q, 1)-$BIBD.*

Proof. Let $(V, \mathcal{B})$ be a $(q^d, q, 1)-$BIBD where $r = (q^d-1)/(q-1)$, $V = \mathbb{F}_{q^d}$, the finite field with q^d elements, and $f\colon \mathcal{B} \to \{0, 1, 2, \ldots, q^d-1\}$ is a line-graceful labeling with the induced mapping $f^+\colon V \to \mathbb{Z}_{q^d}$ on points. Let W be the subfield of order q in V and let $X = V \times W$.

For all $A \in \mathcal{B}$ and $y \in W$, let $C_{A,y} = A \times \{y\}$ and define $\mathcal{D}_1 = \{C_{A,y} : A \in \mathcal{B} \text{ and } y \in W\}$. For all $a, b \in V$, let $C_{a,b} = \{(ya + b, y) : y \in W\}$, define $\mathcal{D}_2 = \{C_{a,b} : a, b \in V\}$ and $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$. It can be easily seen that $(X, \mathcal{D})$ is a $(q^{d+1}, q, 1)$-BIBD.

Take any bijection $h\colon W \to \{0, 1, 2, \ldots, q-1\}$ and define $g\colon \mathcal{D} \to \{0, 1, 2, \ldots, q^{d+1}-1\}$ as follows. For any $A \in \mathcal{B}$ and $y \in W$ define $g(C_{A,y}) = qf(A) + h(y)$, and for any $a, b \in V$ define $g(C_{a,b}) = 0$. The weight of any point $(x, y) \in X$ is

$$\sum_{x\in A} g(C_{A,y}) = \sum_{x\in A} (qf(A) + h(y)) \equiv qf^+(x) + rh(y) \pmod{q^{d+1}}.$$

Since $f^+(x)$s are all different mod q^d, and $\gcd(q, r) = 1$, the values $qf^+(x) + rh(y)$ must be all different mod q^{d+1}. Therefore, g is a line-graceful labeling of $(X, \mathcal{D})$. □

As a consequence of Theorems 5.1 and 5.2, we get the following result.

Theorem 5.3. *There exists a line-graceful $(q^d, q, 1)-BIBD$ for every prime power $q \geq 2$ and integer $d \geq 2$.*

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