ON SOLUTIONS OF THE VOLterra EQUATION
IN THE SPACE OF FUNCTIONS
OF BOUNDED VARIATION

J. MATUTE

Abstract. In this paper, we use a Leray-Schauder alternative in order to prove the existence and uniqueness of solutions for the Volterra equation in the Banach space of functions of bounded variation.

1. Introduction

In the last decades, the study integral equations has considered diverse spaces of functions endowed with some sorts of bounded variation. Examples of this class of study are the papers [1], [2], [3], [4], [5] and the book [8]. In particular, we find in [4] the following statement.

Theorem. [4, Theorem 4.1] If

1. \( g : [0, a] \rightarrow \mathbb{R} \) is a function of bounded variation,
2. \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a locally Lipschitz function and
3. \( K : \{(t, s) : 0 \leq t \leq a, 0 \leq s \leq a\} \rightarrow \mathbb{R} \) is a function such that \( K(t, \cdot) \) is Lebesgue integrable on \([0, t]\) for every \( t \in [0, a] \) and \(|K(s, s)| + \text{var}_s^a K(\cdot, s) \leq m(s)\) for a.e \( s \in [0, a] \), where \( m : [0, a] \rightarrow \mathbb{R}_+ \) is Lebesgue integrable and \( \text{var}_s^a K(\cdot, s) \) is the total variation of \( K(\cdot, s) : [s, a] \rightarrow \mathbb{R} \), then there exists an interval \( J := [0, d] \subset [0, a] \) such that the Volterra equation

\[
\begin{align*}
    x(t) = g(t) + \int_0^t K(t, s)f(x(s))ds
\end{align*}
\]

(1)

has a unique bounded variation solution defined on \( J \), where the integral sign stands for the Lebesgue integral.

Motivated by the previous Theorem, we study the global existence of solutions instead of the local existence near \( t = a \) for the equation

\[
\begin{align*}
    x(t) = g(t) + \int_0^t K(t, s)f(x(s))ds
\end{align*}
\]

(2)

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in the space of functions of bounded variation defined on the interval \([a, b]\), where the integral sign stands for the Lebesgue integral.

This paper consists of four sections. In Section 2, we give the necessary preliminaries. In Section 3, we prove two theorems concerning with the existence and uniqueness of solutions for the integral equation (2) and finally, in Section 4, we provide two examples illustrating our main results.

2. Preliminaries

Let us begin by recalling the following definitions.

**Definition 2.1.** A function \(f : [a, b] \to \mathbb{R}\) is said to be of bounded variation on \([a, b]\) if there exists a real number \(M > 0\) such that
\[
\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq M
\]
for all partition \(\prod : \{a = t_0 < t_1 < \cdots < t_n = b\}\) of interval \([a, b]\).

**Definition 2.2.** Let \(f\) be of bounded variation on \([a, b]\). The real number
\[
\sup \left\{ \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| : \{a = t_0 < t_1 < \cdots < t_n = b\} \right\}
\]
is called the total variation of \(f\) on the interval \([a, b]\). The total variation of \(f\) on \([a, b]\) is denoted by \(\text{var}_b^a f\).

Throughout the paper, we keep the following assumptions:

**Assumption 2.1.** The function \(f : \mathbb{R} \to \mathbb{R}\) is globally Lipschitz with the Lipschitz constant \(L > 0\). Furthermore, there is a real number \(\alpha \geq 0\) such that
\[
\max_{s \in [-r, r]} |f(s)| \leq r + \alpha \text{ for each } r \geq 0.
\]

**Assumption 2.2.** The function \(K : \{(t, s) \in [a, b] \times [a, b] : s \leq t\} \to \mathbb{R}\) is such that
\[
(1) \text{ } K(t, \cdot) \text{ is Lebesgue integrable on } [a, b] \text{ for each } t \in [a, b],
\]
\[
(2) \text{ there exists a bounded Lebesgue integrable function } h : [a, b] \to [0, \infty) \text{ such that } \text{var}_s^b K(\cdot, s) \leq h(s) \text{ for a.e. } s \in [a, b].
\]

**Example 2.1.** An example of a function \(f : \mathbb{R} \to \mathbb{R}\) satisfying Assumption 2.1 is \(f(s) := \beta s + \gamma \sin s\), where \(|\beta| \leq 1\) and \(\gamma\) is an arbitrary real number.

**Lemma 2.1.** If there is a real number \(\alpha \geq 0\) such that \(|f(s)| \leq \alpha\) for all \(s \in \mathbb{R}\), then
\[
\sup_{s \in [-r, r]} |f(s)| \leq r + \alpha \text{ for each } r \geq 0.
\]

**Example 2.2 ([4]).** An example of the function \(K : \{(t, s) \in [a, b] \times [a, b] : s \leq t\} \to \mathbb{R}\) satisfying Assumption 2.2 is \(K(t, s) := \varphi(t)\psi(s)\), where \(\varphi : [a, b] \to \mathbb{R}\) is a function of bounded variation and \(\psi : [a, b] \to \mathbb{R}\) is a bounded Lebesgue integrable function.
Let us recall the following Leray-Schauder alternative, which statement was taken from [3].

**Theorem 2.1.** Let $U$ be an open subset of a Banach space $(X, \| \cdot \|)$ with $0 \in U$. Suppose $H : \overline{U} \to X$ and assume there exists a continuous nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that for $x, y \in \overline{U}$, we have $\| H(x) - H(y) \| \leq \phi(\|x - y\|)$; here $\overline{U}$ denotes the closure of $U$ in $X$. In addition, assume $H(\overline{U})$ is bounded and $x \neq \lambda H(x)$ for $x \in \partial U$ and $\lambda \in (0, 1]$; here $\partial U$ denotes the boundary of $U$ in $X$. Then $H$ has a fixed point in $U$.

3. Existence and Uniqueness of Solutions

We denote by $BV[a, b]$ the Banach space of functions of bounded variation defined on $[a, b]$ endowed with the bounded variation norm $\|x\| := |x(a)| + \text{var}_a^b x$, where $\text{var}_a^b x$ denotes the total variation for $x \in BV[a, b]$. In this section, we prove the existence and uniqueness of solutions in $BV[a, b]$ for the equation (2).

**Remark 3.1.** Let us fix an arbitrary $t \in [a, b]$. If $x \in BV[a, b]$, then the function $[a, t] \ni s \mapsto K(t, s)f(x(s))$ is Lebesgue integrable; which implies that the function

$$t \mapsto \int_a^t K(t, s)f(x(s))ds$$

is well defined.

**Definition 3.1.** Given $x \in BV[a, b]$, we define $F(x) : [a, b] \to \mathbb{R}$ by

$$F(x)(t) := \int_a^t K(t, s)f(x(s))ds.$$ 

**Lemma 3.1.** Let us fix an arbitrary $r > 0$. If $x \in \overline{B}_r := \{x \in BV[a, b] : \|x\| < r\}$, then

$$\|F(x)\| \leq (r + \alpha) \int_a^b h(s)ds,$$

where $\alpha$ is the real number indicated in Assumption 2.1.

**Proof.** Now let us define $\hat{K} : [a, b] \times [a, b] \to \mathbb{R}$ by

$$\hat{K}(t, s) = \begin{cases} K(t, s) & \text{if } a \leq s \leq t, \\ 0 & \text{if } t < s \leq b. \end{cases}$$

Let $\{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of $[a, b]$. If $x \in \overline{B}_r$, then

$$\sum_{i=1}^n \left| F(x)(t_i) - F(x)(t_{i-1}) \right| = \sum_{i=1}^n \left| \int_{t_{i-1}}^{t_i} \hat{K}(t, s)f(x(s))ds - \int_a^b \hat{K}(t_{i-1}, s)f(x(s))ds \right|$$

$$\leq \max_{s \in [-r, r]} |f(s)| \cdot \int_a^b \sum_{i=1}^n |\hat{K}(t_{i-1}, s) - \hat{K}(t_i, s)| ds$$

$$\leq \max_{s \in [-r, r]} |f(s)| \cdot \int_a^b \text{var}_a^b \hat{K}(\cdot, s) ds.$$
By the above inequality, Definition 2.2 and Assumption 2.2, we have that
\[
\begin{align*}
\text{var}_a^b F(x) &\leq \max_{s \in [-r, r]} |f(s)| \cdot \int_a^b \text{var}_a^b \hat{K}(\cdot, s) \, ds \\
&\leq \max_{s \in [-r, r]} |f(s)| \cdot \int_a^b \text{var}_a^b K(\cdot, s) \, ds \\
&\leq \max_{s \in [-r, r]} |f(s)| \cdot \int_a^b h(s) \, ds.
\end{align*}
\]
Finally, combining Assumption 2.1 and the above inequality, we conclude that
\[
\|F(x)\| = |F(a)| + \text{var}_a^b F(x) = \text{var}_a^b F(x) \leq (r + \alpha) \int_a^b h(s) \, ds.
\]

\[\square\]

**Corollary 3.1.** If \(x \in BV[a, b]\), then \(F(x) \in BV[a, b]\).

Let us define a function \(G_g\).

**Definition 3.2.** Given \(g \in BV[a, b]\), we define \(G_g : BV[a, b] \to BV[a, b]\) by
\[
G_g(x)(t) := g(t) + \int_a^t K(t, s)f(x(s)) \, ds = g(t) + F(x)(t).
\]

**Lemma 3.2.** If \(x \in BV[a, b]\), then \(G_g(x) \in BV[a, b]\).

Now we have all necessary knowledge in order to prove the main results of this paper.

**Theorem 3.1.** If \(\int_a^b h(s) \, ds < 1\) and \(L \cdot \int_a^b h(s) \, ds < 1\), then for each function \(g \in BV[a, b]\), there exists a solution \(x_g \in BV[a, b]\) for the equation
\[
x(t) = g(t) + \int_a^t K(t, s)f(x(s)) \, ds.
\]

**Proof.** Let \(g \in BV[a, b]\) be given. Define \(\hat{K} : [a, b] \times [a, b] \to \mathbb{R}\) by
\[
\hat{K}(t, s) = \begin{cases} 
K(t, s) & \text{if } a \leq s \leq t, \\
0 & \text{if } t < s \leq b.
\end{cases}
\]

Let \(r > 0\) be such that
\[
(3) \quad \|g\| + \alpha \int_a^b h(s) \, ds < r \left(1 - \int_a^b h(s) \, ds\right).
\]
Now we define \(H : \mathbb{B}_r \to BV[a, b]\) by \(H(x) := G_g(x)\).
Let \( \Pi := \{a = t_0 < t_1 < \cdots < t_n = b\} \) be a partition of \([a, b]\). Given two arbitrary functions \(x, y \in BV_r\), we have that
\[
\sum_{i=1}^{n} |(H(x) - H(y))(t_i) - (H(x) - H(y))(t_{i-1})|
\]
\[
= \sum_{i=1}^{n} \left| (F(x) - F(y))(t_i) - (F(x) - F(y))(t_{i-1}) \right|
\]
\[
= \sum_{i=1}^{n} \left| \int_{t_{i-1}}^{t_i} \tilde{K}(t_i, s)(f(x(s)) - f(y(s)))ds - \int_{t_{i-1}}^{t_i} \tilde{K}(t_{i-1}, s)(f(x(s)) - f(y(s)))ds \right|
\]
\[
\leq L \cdot \sup_{s \in [a, b]} |x(s) - y(s)| \cdot \int_{a}^{b} \sum_{i=1}^{n} |\tilde{K}(t_i, s) - \tilde{K}(t_{i-1}, s)| ds
\]
\[
\leq L \cdot \int_{a}^{b} \var_{a}^{b} \tilde{K}(\cdot, s) ds \cdot \|x - y\| \leq L \cdot \int_{a}^{b} \var_{a}^{b} K(\cdot, s) ds \cdot \|x - y\|
\]
\[
\leq L \cdot \int_{a}^{b} h(s) ds \cdot \|x - y\|.
\]

Therefore,
\[
\|H(x) - H(y)\| \leq L \cdot \int_{a}^{b} h(s) ds \cdot \|x - y\|.
\]

Now, let us assume that there exists \(x \in BV_r\) such that \(x = \lambda H(x)\) for some \(\lambda \in (0, 1]\). By Lemma 3.1 and the above inequality (3), we have that
\[
\|x\| = \|\lambda H(x)\| \leq \|H(x)\| \leq \|g\| + \|F(x)\|
\]
\[
\leq \|g\| + (r + \alpha) \int_{a}^{b} h(s) ds < r.
\]

The above inequality implies that if \(x \in BV_r\) is such that \(x = \lambda H(x)\) for some \(\lambda \in (0, 1]\), then \(x\) does not belong to \(\partial B_r\).

If we define \(\phi: [0, \infty) \rightarrow [0, \infty]\) by \(\phi(z) := (L \int_{a}^{b} h(s) ds)z\), then the Leray-Schauder alternative implies that there exists a solution \(x_g\) belonging to \(B_r \subseteq BV[a, b]\) for the equation
\[
x(t) = g(t) + \int_{a}^{t} K(t, s)f(x(s)) ds.
\]

\[\Box\]

**Theorem 3.2.** If \(\sup_{s \in [a, b]} |K(s, s)| < \infty, \quad L(b - a) \sup_{s \in [a, b]} |K(s, s)| + \int_{a}^{b} h(s) ds < 1, \) and \(\int_{a}^{b} h(s) ds < 1, \) then for each function \(g \in BV[a, b],\) there exists a unique solution \(x_g \in BV[a, b]\) for the equation
\[
x(t) = g(t) + \int_{a}^{t} K(t, s)f(x(s)) ds.
\]
Proof. Observe that \( L \left( (b-a) \sup_{s \in [a,b]} |K(s, s)| + \int_a^b h(s) ds \right) < 1 \) implies that \( L \int_a^b h(s) ds < 1 \). By Theorem 3.1, there exists a solution in \( BV[a, b] \) for the equation
\[
x(t) = g(t) + \int_a^t K(t, s)f(x(s))ds.
\]
Let us prove uniqueness of solutions by contradiction. If there exist two different solutions \( x \) and \( y \) for the integral equation (2), then
\[
|y(t) - x(t)| = \left| \int_a^t K(t, s)f(y(s))ds - \int_a^t K(t, s)f(x(s))ds \right|
\]
\[
\leq \int_a^t |K(t, s)| \cdot |f(y(s)) - f(x(s))|ds
\]
\[
\leq \int_a^t |K(t, s)| L |y(s) - x(s)|ds.
\]
Hence
\[
\sup_{s \in [a,b]} |y(s) - x(s)| \leq L \cdot \sup_{s \in [a,b]} |y(s) - x(s)| \cdot \sup_{t \in [a,b]} \int_a^t |K(t, s)| ds.
\]
Now, observe that
\[
\int_a^t |K(t, s)| ds \leq \int_a^t \sup_{\tau \in [s,b]} |K(\tau, s)| ds \leq \int_a^t \left( \sup_{s \in [a,b]} |K(s, s)| + \text{var}_s^b K(\cdot, s) \right) ds
\]
\[
\leq \int_a^t \left( \sup_{s \in [a,b]} |K(s, s)| + h(s) \right) ds \leq \int_a^b \left( \sup_{s \in [a,b]} |K(s, s)| + h(s) \right) ds
\]
\[
= (b-a) \sup_{s \in [a,b]} |K(s, s)| + \int_a^b h(s) ds.
\]
Therefore,
\[
1 \leq L \left( (b-a) \sup_{s \in [a,b]} |K(s, s)| + \int_a^b h(s) ds \right),
\]
which contradicts the hypothesis. \( \Box \)

4. Examples

In this section, we give two applications of Theorems 3.1 and 3.2, respectively.

Definition 4.1 ([7]). A function \( g: I \to I \) satisfies a Lipschitz condition on \( I := [0, 1] \) if there exists a constant \( M > 0 \) such that \( |g(t) - g(\tau)| \leq M|t - \tau| \) for all \( t, \tau \in I \).

Lemma 4.1. [7, Theorem 4] Let \( BV \) denote the set of functions \( f: I \to I \) of bounded variation, where \( I := [0, 1] \). For \( g: I \to I \), the composition \( g \circ f \) belongs to \( BV \) for all \( f \) in \( BV \) if and only if \( g \) satisfies a Lipschitz condition on \( I \).
Example 4.1. Observe that the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(s) = e^{-|s|}$ is globally Lipschitz with the Lipschitz constant $L = 2$. Since $|f(s)| \leq 1$ for all $s \in \mathbb{R}$, by Lemma 2.1, we have that $\sup_{s \in [-r, r]} |f(s)| \leq r + 1$ for each $r \geq 0$.

Now, we define $K: \{(t, s) \in [0, 1] \times [0, 1] : s \leq t\} \rightarrow \mathbb{R}$ by

$$K(t, s) = \begin{cases} 
1 & \text{if } 0 \leq t < \frac{1}{2}, \\
\frac{5}{4} & \text{if } \frac{1}{2} \leq t \leq 1 \text{ and } 0 \leq s \leq \frac{1}{2}, \\
-50 & \text{if } \frac{1}{2} < t \leq 1 \text{ and } \frac{1}{2} < s \leq 1.
\end{cases}$$

Observe that

$$\text{var}_t^1 K(\cdot, s) = \begin{cases} 
\frac{1}{4} & \text{if } 0 \leq s < \frac{1}{2}, \\
0 & \text{if } \frac{1}{2} \leq s \leq 1.
\end{cases}$$

When we define $h: [0, 1] \rightarrow [0, 1)$ by $h(s) := \text{var}_t^1 K(\cdot, s)$, we have that $\int_0^1 h(s) ds = \frac{1}{8}$.

Let us consider the equation

$$x(t) = \sin(\omega(t)) + \int_0^t K(t, s) e^{-|x(s)|} \, ds,$$

where $\omega: [0, 1] \rightarrow \mathbb{R}$ is defined by

$$\omega(t) = \begin{cases} 
0 & \text{if } t = 0, \\
1 & \text{if } 0 < t \leq \frac{1}{3}, \\
t^3 & \text{if } \frac{1}{3} < t \leq \frac{2}{3}, \\
t & \text{if } \frac{2}{3} \leq t \leq 1.
\end{cases}$$

By Lemma 4.1, the function $[0, 1] \ni t \mapsto \sin(\omega(t))$ belongs to $BV[0, 1]$ and due to Theorem 3.1, there exists a solution belonging to $BV[0, 1]$ for the equation (4).

**Lemma 4.2 (6).** If the function $g: [a, b] \rightarrow \mathbb{R}$ is Lebesgue integrable, then $W: [a, b] \rightarrow \mathbb{R}; W(t) := \int_a^t g(s) ds$ is an absolutely continuous function and therefore $W$ is of bounded variation.

**Example 4.2.** Observe that the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(s) := \frac{s}{1+|s|}$ is globally Lipschitz with the Lipschitz constant $L = 2$. Since $|f(s)| \leq 1$ for all $s \in \mathbb{R}$, by Lemma 2.1 we have that $\sup_{s \in [-r, r]} |f(s)| \leq r + 1$ for each $r \geq 0$.

Now we define $\omega: [0, 1] \rightarrow \mathbb{R}$ by

$$\omega(t) = \begin{cases} 
0 & \text{if } t = 0, \\
\frac{1}{k} & \text{if } \frac{1}{k+1} < t \leq \frac{1}{k}.
\end{cases}$$
such that $k$ is a positive integer. Let us consider the equation
\begin{equation}
  x(t) = \int_0^t \omega(s)ds + \int_0^t (t \cdot s - s^2) \frac{x(t)}{1 + |x(t)|} ds
\end{equation}
on the interval $[0, 1]$. By Lemma 4.2, the function $[0, 1] \ni t \mapsto \int_0^t \omega(s)ds$ belongs to $BV[0, 1]$. Observe that $h(s) := \text{var}_s^t(K(\cdot, s)) = \text{var}_s^t(t \cdot s - s^2) = s - s^2$ and $\int_0^1 h(s)ds = \frac{1}{6}$. Since $L[(1 - 0) \sup_{s \in [0, 1]} |K(s, s)| + \int_0^1 h(s)ds] = L \cdot \int_0^1 h(s)ds = \frac{1}{3} < 1$, by Theorem 3.2, there exists a unique solution belonging to $BV[0, 1]$ for the equation (5).

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References


J. Matute, Departamento de Matemáticas, Universidad de Los Andes, La Hechicera, Mérida 5101, Venezuela, e-mail: jmatus@ula.ve