Acta Math. Univ. Comenianae
Vol. XCII, 3 (2023), pp. 213–223

NEW JENSEN-TYPE INTEGRAL INEQUALITIES VIA MODIFIED (h,m)-CONVEXITY AND THEIR APPLICATIONS

B. BAYRAKTAR, P. KÓRUS and J. E. NÁPOLES VALDÉS

Abstract. This article presents new developments and applications of Jensen's inequality. We explore various variations of Jensen's inequality related to modified (h,m)-convex functions. The obtained results extend the applicability of the inequality to a broader class of functions and contexts. In addition to the fact that the results presented in the article provide additional results available in the literature, we also give examples of their application.

1. Introduction

The Jensen inequality is a significant result in mathematical analysis, establishing a relationship between the integral of a function over a certain interval and the average value of the function on that interval. This inequality is commonly used for convex functions and has substantial applications in various mathematical fields.

In recent years, research on different generalizations and variations of Jensen's inequality has garnered significant interest. These generalizations consider various types of integrals, function classes, and associated parameters, thereby expanding the applicability of the inequality to a broader class of functions and contexts, see e.g. [**2, 7, 8, 9, 10, 11**].

Let $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ and let w_k $(1 \leq k \leq n)$ be positive weights associated with these x_k and whose sum is unity. Then Jensen's inequality

$$f\left(\sum_{k=1}^{n} w_k x_k\right) \leq \sum_{k=1}^{n} w_k f\left(x_k\right) \tag{1}$$

holds (see [**12, 16**]).

The Jensen inequality and its refinement, the well-known Hermite–Hadamard inequality have been generalized by several authors through general convex functions of various type, see e.g. [**1, 5, 13, 14, 15**]. We recall some of the latest general convex notions.

Received May 30, 2023; revised September 13, 2023.
2020 *Mathematics Subject Classification.* Primary 26A51, 26D15.
Key words and phrases. Convex function; modified (h,m)-convexity; Jensen's inequality; Hermite–Hadamard inequality; Hölder's inequality; power mean inequality.

Definition 1.1 ([**13, 15**])**.** Let $h\colon [0,1] \to [0,\infty)$ and $f\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$. We say that f is an h-convex function on $[a,b]$ if for all $x, y \in [a,b]$ and $\lambda \in [0,1]$, we have

$$f(\lambda x + (1-\lambda)y) \leq h(\lambda) f(x) + h(1-\lambda) f(y).$$

Definition 1.2 ([**14**])**.** Let $h\colon [0,1] \to [0,\infty)$, $h \neq 0$ and $f\colon I = [0,\infty) \to \mathbb{R}$. If inequality

$$f\left(\lambda x + m(1-\lambda)y\right) \leq h(\lambda) f(x) + m h(1-\lambda) f(y) \tag{2}$$

is fulfilled for all $x, y \in I$ and $\lambda \in [0,1]$, where $m \in [0,1]$, then function f is called (h,m)-convex on I.

In [**3, 4**], the following definitions were presented.

Definition 1.3. Let $h\colon [0,1] \to (0,1]$ and $f\colon I = [0,\infty) \to \mathbb{R}$. If inequality

$$f\left(\lambda x + m(1-\lambda)y\right) \leq h^s(\lambda) f(x) + m(1 - h^s(\lambda)) f(y) \tag{3}$$

is fulfilled for all $x, y \in I$ and $\lambda \in [0,1]$, where $m \in [0,1]$, $s \in [-1,1]$, then function f is called (h,m)-convex modified of the first type on I.

Definition 1.4. Let $h\colon [0,1] \to (0,1]$ and $f\colon I = [0,\infty) \to \mathbb{R}$. If inequality

$$f\left(\lambda x + m(1-\lambda)y\right) \leq h^s(\lambda) f(x) + m(1 - h(\lambda))^s f(y) \tag{4}$$

is fulfilled for all $x, y \in I$ and $\lambda \in [0,1]$, where $m \in [0,1]$, $s \in [-1,1]$, then function f is called (h,m)-convex modified of the second type on I.

Remark. From the definitions above, the sets (h,m)-convex modified functions of the first and second types characterized by the triple $(h(t), m, s)$ are denoted by $N_{h,m}^{s,1}(I)$ and $N_{h,m}^{s,2}(I)$, respectively. In [**3, 4**] you can see the convex classes obtained from the special cases of this triple.

Throughout the article, we will introduce new results that enhance the understanding and applicability of Jensen's inequality. These findings may hold substantial significance in diverse fields of mathematics, economics, statistics, and other sciences where Jensen's inequality plays a vital role.

The purpose of this article is to present some new results in the field of Jensen's inequality and its applications. We will examine several variations of Jensen's inequality for modified (h,m)-convex functions.

2. Main results

The results obtained in the article are based on the following lemma:

Lemma 2.1. *Let $f\colon [a,b] \to \mathbb{R}$ be a twice differentiable function on (a,b). If $f'' \in L[a,b]$, then we have*

$$\sum_{k=1}^{n} \left[\frac{f(\xi_{k-1}) + f(\xi_k)}{2} - f\left(\frac{\xi_{k-1} + \xi_k}{2} \right) \right] = \frac{\Delta^2}{2} \sum_{k=1}^{n} \left(I_{1k} + I_{2k} \right), \tag{5}$$

where $n \in \mathbb{N}$, $\Delta = \frac{b-a}{n}$, $\xi_k = a + k\Delta$, $k = 0, 1, 2, \ldots, n$,

$$I_{1k} = \int_0^{\frac{1}{2}} t f''(t\xi_{k-1} + (1-t)\xi_k)\mathrm{d}t \quad \textit{and} \quad I_{2k} = \int_0^{\frac{1}{2}} t f''((1-t)\,\xi_{k-1} + t\xi_k)\mathrm{d}t.$$

Proof. Integrating the first integral by parts twice, we have

$$\begin{aligned}
I_{1k} &= \frac{t}{\xi_{k-1} - \xi_k} f'(\xi_{k-1}t + (1-t)\xi_k)\Big|_0^{\frac{1}{2}} - \frac{1}{\xi_{k-1} - \xi_k}\int_0^{\frac{1}{2}} f'(\xi_{k-1}t + (1-t)\xi_k)\mathrm{d}t \\
&= \frac{1}{2\,(\xi_{k-1} - \xi_k)} f'\left(\frac{\xi_{k-1} + \xi_k}{2}\right) - \frac{1}{(\xi_{k-1} - \xi_k)^2} f(\xi_{k-1}t + (1-t)\xi_k)\Big|_0^{\frac{1}{2}} \\
&= \frac{1}{2\,(\xi_{k-1} - \xi_k)} f'\left(\frac{\xi_{k-1} + \xi_k}{2}\right) - \frac{1}{(\xi_{k-1} - \xi_k)^2}\left[f\left(\frac{\xi_{k-1} + \xi_k}{2}\right) - f(\xi_k)\right].
\end{aligned}$$

Similarly, for the integral I_2, we can write

$$I_{2k} = \frac{1}{2\,(\xi_k - \xi_{k-1})} f'\left(\frac{\xi_{k-1} + \xi_k}{2}\right) - \frac{1}{(\xi_k - \xi_{k-1})^2}\left[f\left(\frac{\xi_{k-1} + \xi_k}{2}\right) - f(\xi_{k-1})\right].$$

Adding integrals I_{1k} and I_{2k} yields

$$I_{1k} + I_{2k} = \frac{f(\xi_{k-1}) + f(\xi_k)}{(\xi_k - \xi_{k-1})^2} - \frac{2}{(\xi_k - \xi_{k-1})^2} f\left(\frac{\xi_{k-1} + \xi_k}{2}\right),$$

that is equivalent to

$$\frac{f(\xi_{k-1}) + f(\xi_k)}{2} - f\left(\frac{\xi_{k-1} + \xi_k}{2}\right) = \frac{(\xi_k - \xi_{k-1})^2}{2}\left(I_{1k} + I_{2k}\right).$$

By summing over the variable k, we obtain (5). □

Remark. For $n = 1$, from (5), we get the identity in Corollary 2.1 of [**2**].

2.1. Results for (h, m)-convexity modified in the first sense

Theorem 2.2. *Let* $f\colon \left[a, \frac{b}{m}\right] \to \mathbb{R}$ *be a twice differentiable function on* $\left(a, \frac{b}{m}\right)$. *If* $f'' \in L\left[a, \frac{b}{m}\right]$ *and* $|f''| \in N_{h,m}^{s,1}\left[a, \frac{b}{m}\right]$, *then the following inequality holds:*

$$\begin{aligned}
(6)\qquad &\left|\sum_{k=1}^{n}\left[\frac{f(\xi_{k-1}) + f(\xi_k)}{2} - f\left(\frac{\xi_{k-1} + \xi_k}{2}\right)\right]\right| \\
&\le \frac{\Delta^2 m}{2}\left(\frac{1}{8} - \int_0^{\frac{1}{2}} t h^s(t)\mathrm{d}t\right)\sum_{k=1}^{n}\left[\left|f''\left(\frac{\xi_{k-1}}{m}\right)\right| + \left|f''\left(\frac{\xi_k}{m}\right)\right|\right] \\
&\quad + \frac{\Delta^2}{2}\left(\int_0^{\frac{1}{2}} t h^s(t)\mathrm{d}t\right)\sum_{k=1}^{n}\left[|f''(\xi_{k-1})| + |f''(\xi_k)|\right],
\end{aligned}$$

where Δ *and* ξ_k *is from Lemma 2.1.*

Proof. From Lemma 2.1 and modulus properties, we can write

$$\left|\sum_{k=1}^{n}\left[\frac{f(\xi_{k-1})+f(\xi_k)}{2}-f\left(\frac{\xi_{k-1}+\xi_k}{2}\right)\right]\right| \le \frac{\Delta^2}{2}\sum_{k=1}^{n}\left(|I_{1k}|+|I_{2k}|\right). \tag{7}$$

By using (h,m)-convexity modified in the first sense of function $|f''|$, for integral I_{1k}, we get

$$\begin{aligned}
|I_{1k}| &\le \int_0^{\frac{1}{2}} t\,|f''(t\xi_{k-1}+(1-t)\xi_k)|\,\mathrm{d}t\\
&\le \int_0^{\frac{1}{2}} t\left[h^s(t)\,|f''(\xi_{k-1})|+m(1-h^s(t))\left|f''\left(\frac{\xi_k}{m}\right)\right|\right]\mathrm{d}t\\
&= |f''(\xi_{k-1})|\int_0^{\frac{1}{2}} th^s(t)\mathrm{d}t + m\left|f''\left(\frac{\xi_k}{m}\right)\right|\int_0^{\frac{1}{2}} t\left[1-h^s(t)\right]\mathrm{d}t\\
&= |f''(\xi_{k-1})|\int_0^{\frac{1}{2}} th^s(t)\mathrm{d}t + m\left|f''\left(\frac{\xi_k}{m}\right)\right|\left(\frac{1}{8}-\int_0^{\frac{1}{2}} th^s(t)\mathrm{d}t\right).
\end{aligned}$$

Similarly, for the second integral I_{2k}, one can write

$$\begin{aligned}
|I_{2k}| &\le \int_0^{\frac{1}{2}} t\,|f''((1-t)\,\xi_{k-1}+t\xi_k)|\,\mathrm{d}t\\
&\le \int_0^{\frac{1}{2}} t\left[m(1-h^s(t))\left|f''\left(\frac{\xi_{k-1}}{m}\right)\right|+h^s(t)\,|f''(\xi_k)|\right]\mathrm{d}t\\
&= |f''(\xi_k)|\int_0^{\frac{1}{2}} th^s(t)\mathrm{d}t + m\left|f''\left(\frac{\xi_{k-1}}{m}\right)\right|\left(\frac{1}{8}-\int_0^{\frac{1}{2}} th^s(t)\mathrm{d}t\right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
|I_{1k}|+|I_{2k}| \le{}& m\left[\left|f''\left(\frac{\xi_{k-1}}{m}\right)\right|+\left|f''\left(\frac{\xi_k}{m}\right)\right|\right]\left[\frac{1}{8}-\int_0^{\frac{1}{2}} th^s(t)\mathrm{d}t\right]\\
&+\left[|f''(\xi_{k-1})|+|f''(\xi_k)|\right]\int_0^{\frac{1}{2}} th^s(t)\mathrm{d}t.
\end{aligned}$$

By multiplying the last inequality by $\frac{\Delta^2}{2}$, and taking into account (7), we obtain (6). □

Corollary 2.3. *If in Theorem 2.2, we choose $n=s=m=1$ and $h(t)=t$, then we obtain Corollary 2.2 of* [**2**].

Theorem 2.4. *Let $f\colon\left[a,\frac{b}{m}\right]\to\mathbb{R}$ be a twice differentiable function on $\left(a,\frac{b}{m}\right)$. If $f''\in L\left[a,\frac{b}{m}\right]$ and $|f''|^q\in N_{h,m}^{s,1}\left[a,\frac{b}{m}\right]$, then for all $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, the following inequality holds:*

$$\left|\sum_{k=1}^{n}\left[\frac{f(\xi_{k-1})+f(\xi_k)}{2}-f\left(\frac{\xi_{k-1}+\xi_k}{2}\right)\right]\right| \le \frac{\Delta^2}{2^{2+\frac{1}{p}}\,(p+1)^{\frac{1}{p}}}\sum_{k=1}^{n}\left(\mathbf{C}_{1k}^{\frac{1}{q}}+\mathbf{D}_{1k}^{\frac{1}{q}}\right), \tag{8}$$

where Δ *and* ξ_k *is from Lemma 2.1,*

$$\mathbf{C}_{1k} = |f''(\xi_{k-1})|^q \int_0^{\frac{1}{2}} h^s(t)\mathrm{d}t + m\left|f''\left(\frac{\xi_k}{m}\right)\right|^q \int_0^{\frac{1}{2}} (1-h^s(t))\mathrm{d}t,$$
$$\mathbf{D}_{1k} = m\left|f''\left(\frac{\xi_{k-1}}{m}\right)\right|^q \int_0^{\frac{1}{2}} (1-h^s(t))\mathrm{d}t + |f''(\xi_k)|^q \int_0^{\frac{1}{2}} h^s(t)\mathrm{d}t.$$

Proof. By using the well-known Hölder integral inequality and since $|f''|^q$ is an (h,m)-convex function modified in the first sense, we get

$$(9)\quad |I_{1k}| \le \int_0^{\frac{1}{2}} t\left|f''(t\xi_{k-1}+(1-t)\xi_k)\right|\mathrm{d}t$$
$$\le \left(\int_0^{\frac{1}{2}} t^p \mathrm{d}t\right)^{\frac{1}{p}} \left\{\int_0^{\frac{1}{2}} \left[h^s(t)\left|f''(\xi_{k-1})\right|^q + m(1-h^s(t))\left|f''\left(\frac{\xi_k}{m}\right)\right|^q\right]\mathrm{d}t\right\}^{\frac{1}{q}}$$
$$= \left(\frac{2^{-p-1}}{p+1}\right)^{\frac{1}{p}} \left\{\left|f''(\xi_{k-1})\right|^q \int_0^{\frac{1}{2}} h^s(t)\mathrm{d}t + m\left|f''\left(\frac{\xi_k}{m}\right)\right|^q \int_0^{\frac{1}{2}} (1-h^s(t))\mathrm{d}t\right\}^{\frac{1}{q}}.$$

Similarly, for the second integral, we can write
(10)
$$|I_{2k}| \le \left(\frac{2^{-p-1}}{p+1}\right)^{\frac{1}{p}} \left\{m\left|f''\left(\frac{\xi_{k-1}}{m}\right)\right|^q \int_0^{\frac{1}{2}} (1-h^s(t))\mathrm{d}t + |f''(\xi_k)|^q \int_0^{\frac{1}{2}} h^s(t)\mathrm{d}t\right\}^{\frac{1}{q}}.$$

By adding inequalities (9) and (10), we obtain

$$|I_1| + |I_2| \le \left(\frac{2^{-p-1}}{p+1}\right)^{\frac{1}{p}} \left(\mathbf{C}_{1k}^{\frac{1}{q}} + \mathbf{D}_{1k}^{\frac{1}{q}}\right).$$

Multiplying both sides of the last inequality by $\frac{\Delta^2}{2}$ and taking into account (7) yield (8). The proof is complete. □

Corollary 2.5.

(i1) *If we choose* $n = m = 1$ *and* $h(t) = t$*, then* (8) *becomes*

$$\left|\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^2}{2^{2+\frac{1}{p}}(p+1)^{\frac{1}{p}}} \left(\mathbf{C}_{11}^{\frac{1}{q}} + \mathbf{D}_{11}^{\frac{1}{q}}\right)$$

with

$$\mathbf{C}_{11} = \frac{|f''(a)|^q}{(s+1)\,2^{s+1}} + |f''(b)|^q \left(\frac{1}{2} - \frac{1}{(s+1)\,2^{s+1}}\right),$$
$$\mathbf{D}_{11} = \frac{|f''(b)|^q}{(s+1)\,2^{s+1}} + |f''(a)|^q \left(\frac{1}{2} - \frac{1}{(s+1)\,2^{s+1}}\right).$$

(i2) *If we also choose* $s = 1$*, then we have*

$$\mathbf{C}_{11} = \frac{|f''(a)|^q}{8} + \frac{3\,|f''(b)|^q}{8} \quad \textit{and} \quad \mathbf{D}_{11} = \frac{3\,|f''(a)|^q}{8} + \frac{|f''(b)|^q}{8}.$$

(i3) *Moreover, if* $p = q = 2$*, then we get Corollary 2.4 of* [**2**].

Theorem 2.6. *Let* $f\colon \left[a, \frac{b}{m}\right] \to \mathbb{R}$ *be a twice differentiable function on* $\left(a, \frac{b}{m}\right)$. *If* $f'' \in L\left[a, \frac{b}{m}\right]$ *and* $|f''|^q \in N^{s,1}_{h,m}\left[a, \frac{b}{m}\right]$, *then for all* $q \geq 1$, *the following inequality holds:*

$$\left|\sum_{k=1}^{n}\left[\frac{f(\xi_{k-1})+f(\xi_k)}{2} - f\left(\frac{\xi_{k-1}+\xi_k}{2}\right)\right]\right| \leq \frac{\Delta^2}{2^{4-\frac{3}{q}}}\sum_{k=1}^{n}\left(\mathbf{E}_{1k}^{\frac{1}{q}} + \mathbf{F}_{1k}^{\frac{1}{q}}\right), \tag{11}$$

where Δ *and* ξ_k *is from Lemma 2.1,*

$$\mathbf{E}_{1k} = |f''(\xi_{k-1})|^q \int_0^{\frac{1}{2}} t h^s(t)\mathrm{d}t + m\left|f''\left(\frac{\xi_k}{m}\right)\right|^q \int_0^{\frac{1}{2}} t(1-h^s(t))\mathrm{d}t,$$

$$\mathbf{F}_{1k} = m\left|f''\left(\frac{\xi_{k-1}}{m}\right)\right|^q \int_0^{\frac{1}{2}} t(1-h^s(t))\mathrm{d}t + |f''(\xi_k)|^q \int_0^{\frac{1}{2}} t h^s(t)\mathrm{d}t.$$

Proof. By using the well-known power mean integral inequality and since $|f''|^q$ is an (h,m)-convex function modified in the first sense, we get

(12)

$$\begin{aligned}|I_{1k}| &\leq \int_0^{\frac{1}{2}} t\,|f''(t\xi_{k-1}+(1-t)\xi_k)|\,\mathrm{d}t \\ &\leq \left(\int_0^{\frac{1}{2}} t\mathrm{d}t\right)^{1-\frac{1}{q}}\left\{\int_0^{\frac{1}{2}} t\left[\left(h^s(t)\,|f''(\xi_{k-1})|^q + m(1-h^s(t))\left|f''\left(\frac{\xi_k}{m}\right)\right|^q\right)\right]\mathrm{d}t\right\} \\ &= \left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left\{|f''(\xi_{k-1})|^q\int_0^{\frac{1}{2}} t h^s(t)\mathrm{d}t + m\left|f''\left(\frac{\xi_k}{m}\right)\right|^q\int_0^{\frac{1}{2}} t(1-h^s(t))\mathrm{d}t\right\}^{\frac{1}{q}}.\end{aligned}$$

Similarly, for I_{2k}, we can write

(13)

$$|I_{2k}| \leq \left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left\{m\left|f''\left(\frac{\xi_{k-1}}{m}\right)\right|^q\int_0^{\frac{1}{2}} t(1-h^s(t))\mathrm{d}t + |f''(\xi_k)|^q\int_0^{\frac{1}{2}} t h^s(t)\mathrm{d}t\right\}^{\frac{1}{q}}.$$

By adding inequalities (12) and (13), we get

$$|I_1| + |I_2| \leq \left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left(\mathbf{E}_{1k}^{\frac{1}{q}} + \mathbf{F}_{1k}^{\frac{1}{q}}\right).$$

By multiplying both sides of the last inequality by $\frac{\Delta^2}{2}$ and taking into account (7), we obtain (11). The proof is complete. □

Corollary 2.7. *If we choose* $n = m = 1$ *and* $h(t) = t$, *then from* (11), *we get*

$$\left|\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{2^{4-\frac{3}{q}}}\left(\mathbf{E}_{11}^{\frac{1}{q}} + \mathbf{F}_{11}^{\frac{1}{q}}\right)$$

with

$$\mathbf{E}_{11} = \frac{|f''(a)|^q}{(s+2)\,2^{s+2}} + |f''(b)|^q \left(\frac{1}{8} - \frac{1}{(s+2)\,2^{s+2}}\right),$$

$$\mathbf{F}_{11} = \frac{|f''(b)|^q}{(s+2)\,2^{s+2}} + |f''(a)|^q \left(\frac{1}{8} - \frac{1}{(s+2)\,2^{s+2}}\right),$$

and for $s = q = 1$, we have

$$\left| \left[\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{(b-a)^2}{16}\left[|f''(a)| + |f''(b)|\right]. \tag{14}$$

2.2. Results for (h,m)-convexity modified in the second sense

Theorem 2.8. *Let $f\colon \left[a, \frac{b}{m}\right] \to \mathbb{R}$ be a twice differentiable function on $\left(a, \frac{b}{m}\right)$. If $f'' \in L\left[a, \frac{b}{m}\right]$ and $|f''| \in N_{h,m}^{s,2}\left[a, \frac{b}{m}\right]$, then the following inequality holds:*

$$\begin{aligned} &\left|\sum_{k=1}^{n}\left[\frac{f(\xi_{k-1})+f(\xi_k)}{2} - f\left(\frac{\xi_{k-1}+\xi_k}{2}\right)\right]\right| \\ &\leq \frac{\Delta^2 m}{2}\left(\int_0^{\frac{1}{2}} t(1-h(t))^s \mathrm{d}t\right)\sum_{k=1}^{n}\left[\left|f''\left(\frac{\xi_{k-1}}{m}\right)\right| + \left|f''\left(\frac{\xi_k}{m}\right)\right|\right] \\ &\quad + \frac{\Delta^2}{2}\left(\int_0^{\frac{1}{2}} t h^s(t)\mathrm{d}t\right)\sum_{k=1}^{n}\left[|f''(\xi_{k-1})| + |f''(\xi_k)|\right], \end{aligned} \tag{15}$$

where Δ and ξ_k is from Lemma 2.1.

Proof. The proof is analogous to that of Theorem 2.2. □

Corollary 2.9. *If in Theorem 2.8, we choose $n = m = 1$ and $h(t) = t$, then we have*

$$\left| \left[\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{(b-a)^2\left(1-\frac{1}{2^{s+1}}\right)}{2\,(s+1)\,(s+2)}\left[|f''(a)| + |f''(b)|\right].$$

Remark. If we choose $n = m = s = 1$ and $h(t) = t$, then (15) becomes (14).

Theorem 2.10. *Let $f\colon \left[a, \frac{b}{m}\right] \to \mathbb{R}$ be a twice differentiable function on $\left(a, \frac{b}{m}\right)$. If $f'' \in L\left[a, \frac{b}{m}\right]$ and $|f''|^q \in N_{h,m}^{s,2}\left[a, \frac{b}{m}\right]$, then for all $q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$, the following inequality holds:*

$$\left|\sum_{k=1}^{n}\left[\frac{f(\xi_{k-1})+f(\xi_k)}{2} - f\left(\frac{\xi_{k-1}+\xi_k}{2}\right)\right]\right| \leq \frac{\Delta^2}{2^{2+\frac{1}{p}}\,(p+1)^{\frac{1}{p}}}\sum_{k=1}^{n}\left(\mathbf{C}_{2k}^{\frac{1}{q}} + \mathbf{D}_{2k}^{\frac{1}{q}}\right), \tag{16}$$

where Δ and ξ_k is from Lemma 2.1,

$$\mathbf{C}_{2k} = |f''(\xi_{k-1})|^q \int_0^{\frac{1}{2}} h^s(t)\mathrm{d}t + m\left|f''\left(\frac{\xi_k}{m}\right)\right|^q \int_0^{\frac{1}{2}} (1-h(t))^s\mathrm{d}t,$$

$$\mathbf{D}_{2k} = m\left|f''\left(\frac{\xi_{k-1}}{m}\right)\right|^q \int_0^{\frac{1}{2}} (1-h(t))^s\mathrm{d}t + |f''(\xi_k)|^q \int_0^{\frac{1}{2}} h^s(t)\mathrm{d}t.$$

Proof. The proof is analogous to that of Theorem 2.4. □

Corollary 2.11. *If we choose $n=m=1$ and $h(t)=t$, then* (16) *becomes*

$$\left|\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{2^{2+\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left(\mathbf{C}_{21}^{\frac{1}{q}}+\mathbf{D}_{21}^{\frac{1}{q}}\right)$$

with

$$\mathbf{C}_{21}=\frac{|f''(a)|^q}{(s+1)\,2^{s+1}}+\frac{(2^{s+1}-1)\,|f''(b)|^q}{(s+1)\,2^{s+1}},$$
$$\mathbf{D}_{21}=\frac{(2^{s+1}-1)\,|f''(a)|^q}{(s+1)\,2^{s+1}}+\frac{|f''(b)|^q}{(s+1)\,2^{s+1}}.$$

Moreover, for $s=1$, (16) *yields* (i2)–(i3) *of Corollary 2.5.*

Theorem 2.12. *Let $f\colon \left[a,\frac{b}{m}\right]\to\mathbb{R}$ be a twice differentiable function on $\left(a,\frac{b}{m}\right)$. If $f''\in L\left[a,\frac{b}{m}\right]$ and $|f''|^q\in N_{h,m}^{s,2}\left[a,\frac{b}{m}\right]$, then for all $q\geq 1$, the following inequality holds:*

$$\left|\sum_{k=1}^{n}\left[\frac{f(\xi_{k-1})+f(\xi_k)}{2}-f\left(\frac{\xi_{k-1}+\xi_k}{2}\right)\right]\right| \leq \frac{\Delta^2}{2^{4-\frac{3}{q}}}\sum_{k=1}^{n}\left(\mathbf{E}_{2k}^{\frac{1}{q}}+\mathbf{F}_{2k}^{\frac{1}{q}}\right), \tag{17}$$

where Δ and ξ_k is from Lemma 2.1,

$$\mathbf{E}_{2k}=|f''(\xi_{k-1})|^q\int_0^{\frac{1}{2}} th^s(t)\mathrm{d}t+m\left|f''\left(\frac{\xi_k}{m}\right)\right|^q\int_0^{\frac{1}{2}} t(1-h(t))^s\mathrm{d}t,$$
$$\mathbf{F}_{2k}=m\left|f''\left(\frac{\xi_{k-1}}{m}\right)\right|^q\int_0^{\frac{1}{2}} t(1-h(t))^s\mathrm{d}t+|f''(\xi_k)|^q\int_0^{\frac{1}{2}} th^s(t)\mathrm{d}t.$$

Proof. The proof is analogous to that of Theorem 2.6. □

Corollary 2.13. *If we choose $n=m=s=1$ and $h(t)=t$, then* (17) *yields*

$$\left|\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{2^{4-\frac{3}{q}}}\left(\mathbf{E}_{21}^{\frac{1}{q}}+\mathbf{F}_{21}^{\frac{1}{q}}\right),$$

where

$$\mathbf{E}_{21}=\frac{|f''a)|^q}{24}+\frac{|f''\,(b)|^q}{12}\quad\text{and}\quad \mathbf{F}_{21}=\frac{|f''a)|^q}{12}+\frac{|f''\,(b)|^q}{24},$$

and for $q=1$, we have (14).

3. Applications

As we remarked it previously, our main results extend previously known inequalities from the literature. Moreover, they can be applied to functions of various type.

First, let us consider $f\colon [a,b]\subset[0,\infty)\to\mathbb{R}$, $f(x)=x^\alpha$, $\alpha\ge 2$. Then, by Theorem 2.2 or Theorem 2.8, taking $n=2$, we have

$$\frac{1}{2}a^\alpha-\left(\frac{3a+b}{4}\right)^\alpha+\left(\frac{a+b}{2}\right)^\alpha-\left(\frac{a+3b}{4}\right)^\alpha+\frac{1}{2}b^\alpha$$
$$\le\frac{(b-a)^2}{32}\alpha(\alpha-1)\left(\frac{1}{2}a^{\alpha-2}+\left(\frac{a+b}{2}\right)^{\alpha-2}+\frac{1}{2}b^{\alpha-2}\right).$$

Second, let us consider $f\colon \left[0,\frac{\pi}{4}\right]\to\mathbb{R}$, $f(x)=\sin x$, for which $|f|''^2=\sin^2 x$ is convex. Then, by Theorem 2.4 or Theorem 2.10, taking $n=3$ and $q=2$, we have

$$\left|\frac{1}{2}\sin a-\sin\left(\frac{5a+b}{6}\right)+\sin\left(\frac{2a+b}{3}\right)-\sin\left(\frac{a+b}{2}\right)+\sin\left(\frac{a+2b}{3}\right)-\sin\left(\frac{a+5b}{6}\right)+\frac{1}{2}\sin b\right|$$
$$\le\frac{(b-a)^2}{144\sqrt{3}}\left(\sqrt{\sin^2 a+3\sin^2\left(\frac{2a+b}{3}\right)}+\sqrt{3\sin^2 a+\sin^2\left(\frac{2a+b}{3}\right)}\right.$$
$$+\sqrt{\sin^2\left(\frac{2a+b}{3}\right)+3\sin^2\left(\frac{a+2b}{3}\right)}+\sqrt{3\sin^2\left(\frac{2a+b}{3}\right)+\sin^2\left(\frac{a+2b}{3}\right)}$$
$$\left.+\sqrt{\sin^2\left(\frac{a+2b}{3}\right)+3\sin^2 b}+\sqrt{3\sin^2\left(\frac{a+2b}{3}\right)+\sin^2 b}\right),$$

for any $0<a<b<\frac{\pi}{4}$, and similarly, by Theorem 2.6 or Theorem 2.12, we get

$$\left|\frac{1}{2}\sin a-\sin\left(\frac{5a+b}{6}\right)+\sin\left(\frac{2a+b}{3}\right)-\sin\left(\frac{a+b}{2}\right)+\sin\left(\frac{a+2b}{3}\right)-\sin\left(\frac{a+5b}{6}\right)+\frac{1}{2}\sin b\right|$$
$$\le\frac{(b-a)^2}{144\sqrt{3}}\left(\sqrt{\sin^2 a+2\sin^2\left(\frac{2a+b}{3}\right)}+\sqrt{2\sin^2 a+\sin^2\left(\frac{2a+b}{3}\right)}\right.$$
$$+\sqrt{\sin^2\left(\frac{2a+b}{3}\right)+2\sin^2\left(\frac{a+2b}{3}\right)}+\sqrt{2\sin^2\left(\frac{2a+b}{3}\right)+\sin^2\left(\frac{a+2b}{3}\right)}$$
$$\left.+\sqrt{\sin^2\left(\frac{a+2b}{3}\right)+2\sin^2 b}+\sqrt{2\sin^2\left(\frac{a+2b}{3}\right)+\sin^2 b}\right),$$

which shows that Theorem 2.6 or Theorem 2.12 gives a better estimation in this case.

Finally, let $f\colon \left[0,\arccos\left(\sqrt{\frac{2}{3}}\right)\right]\to\mathbb{R}$, $f(x)=\sin x$, for which $|f|''^3=\sin^3 x$ is convex. Then, by Theorem 2.4 or Theorem 2.10, taking $n=2$ and $q=3$, we have

$$\left|\frac{1}{2}\sin a-\sin\left(\frac{3a+b}{4}\right)+\sin\left(\frac{a+b}{2}\right)-\sin\left(\frac{a+3b}{4}\right)+\frac{1}{2}\sin b\right|$$
$$\le\frac{(b-a)^2}{8\cdot\sqrt[3]{25}}\left(\sqrt[3]{\sin^3 a+3\sin^3\left(\frac{a+b}{2}\right)}+\sqrt[3]{3\sin^3 a+\sin^3\left(\frac{a+b}{2}\right)}\right.$$
$$\left.+\sqrt[3]{\sin^3\left(\frac{a+b}{2}\right)+3\sin^3 b}+\sqrt[3]{3\sin^3\left(\frac{a+b}{2}\right)+\sin^3 b}\right),$$

for any $0 < a < b < \arccos\left(\sqrt{\frac{2}{3}}\right)$, and similarly, by Theorem 2.6 or Theorem 2.12, we get

$$\begin{aligned}
&\left|\frac{1}{2}\sin a - \sin\left(\frac{3a+b}{4}\right) + \sin\left(\frac{a+b}{2}\right) - \sin\left(\frac{a+3b}{4}\right) + \frac{1}{2}\sin b\right| \\
&\leq \frac{(b-a)^2}{16\cdot\sqrt[3]{3}}\left(\sqrt[3]{\sin^3 a + 2\sin^3\left(\frac{a+b}{2}\right)} + \sqrt[3]{2\sin^3 a + \sin^3\left(\frac{a+b}{2}\right)}\right. \\
&\qquad\left. + \sqrt[3]{\sin^3\left(\frac{a+b}{2}\right) + 2\sin^3 b} + \sqrt[3]{2\sin^3\left(\frac{a+b}{2}\right) + \sin^3 b}\right).
\end{aligned}$$

4. Conclusions

In this article, we have examined the Jensen inequality for modified (h, m)-convex functions. We have presented a generalized formulation of Jensen's inequality and explored its application in the context of these functions. In conclusion, our research contributes to a better understanding and application of Jensen's inequality for modified (h, m)-convex functions. We hope that the findings of this study will be valuable for further research in this area and find practical applications in various real-life situations.

References

1. Andrić M., *Jensen-type inequalities for $(h, g; m)$-convex functions*, Mathematics **9** (2021), 3312.

2. Bayraktar B. and Gürbüz M., *On some integral inequalities for (s, m)-convex functions*, TWMS J. Appl. Eng. Math. **10** (2020), 288–295.

3. Bayraktar B. and Nápoles Valdés J. E., *New generalized integral inequalities via (h, m)-convex modified functions*, Izv. Inst. Mat. Inform. **60** (2022), 3–15.

4. Bayraktar B. and Nápoles Valdés J. E., *Integral inequalities for mappings whose derivatives are (h, m, s)-convex modified of second type via Katugampola integrals*, Univ. Craiova Ser. Mat. Inform. **49** (2022), 371–383.

5. Bosch P., Quintana Y., Rodríguez J. M. and Sigarreta J. M., *Jensen-type inequalities for m-convex functions*, Open Math. **20** (2022), 946–958.

6. Butt S. I., Nasir J., Qaisar S. and Abualnaja K. M., *k-Fractional variants of Hermite-Mercer-type inequalities via s-convexity with applications*, J. Funct. Spaces **2021** (2021), Art. ID 5566360.

7. Dragomir S. S., *Some reverses of the Jensen inequality with applications*, Bull. Aust. Math. Soc. **87** (2013), 177–194.

8. Duc D. T. and Hue N. N., *Jensen-type inequalities and their applications*, J. Math. Inequal. **14** (2020), 319–327.

9. Hot J. M. and Seo Y., *An interpolation of Jensen's inequality and its applications to mean inequalities*, J. Math. Inequal. **12** (2018), 303–313.

10. Khan M. A., Khan G. A., Ali T. and Kilicman A., *On the refinement of Jensen's inequality*, Appl. Math. Comput. **262** (2015), 128–135.

11. Lu G., *New refinements of Jensen's inequality and entropy upper bounds*, J. Math. Inequal. **12** (2018), 403–421.
12. Mitrinović D. S., Pečarić J. E. and Fink A. M., *Classical and New Inequalities in Analysis*, Springer Science+Business Media, Dordrecht, 1993.
13. Noor M. A., Noor K. I. and Awan M. U., *A new Hermite–Hadamard type inequality for h-convex functions*, Creat. Math. Inform. **24** (2015), 191–197.
14. Özdemir M. E., Akdemir A. O. and Set E., *On $(h$-$m)$-convexity and Hadamard-type inequalities*, Transylv. J. Math. Mech. **8** (2016), 51–58.
15. Varošanec S., *On h-convexity*, J. Math. Anal. Appl. **326** (2007), 303–311.
16. Vasić P. M. and Pečarić J. E., *On the Jensen inequality*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **634–677** (1979), 50–54.

B. Bayraktar, Faculty of Education, Bursa Uludag University, Gorukle Campus, 16059, Bursa, Turkey,
e-mail: `bbayraktar@uludag.edu.tr`

P. Kórus, Institute of Applied Pedagogy, Juhász Gyula Faculty of Education, University of Szeged, Hattyas utca 10, H-6725 Szeged, Hungary,
e-mail: `korus.peter@szte.hu`

J. E. Nápoles Valdés, UNNE, FaCENA, Ave. Libertad 5450, Corrientes 3400, Argentina and UTN-FRRE, French 414, Resistencia, Chaco 3500, Argentina,
e-mail: `jnapoles@exa.unne.edu.ar`