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CONFORMALLY CLOSED WEAKLY LANDSBERG METRICS

A. TAYEBI AND B. NAJAFI

ABSTRACT. In this paper, we define the notion of conformally closed weakly Landsberg metric. Then, we prove that a Finsler metric is a conformally closed weakly Landsberg metric if and only if it is a Riemannian metric.

1. Introduction

Let $F = F(x, y)$ and $\tilde{F} = \tilde{F}(x, y)$ be two Finsler metrics on a manifold M. Then, F is said to be conformal with $\tilde{F}$ if there is a scalar function $\sigma = \sigma(x)$, called the conformal factor of the conformal transformation such that $\tilde{F}(x, y) = \exp(\sigma)F(x, y)$. The conformal transformations of Finsler metrics and their curvatures have been studied by many geometers [**5, 6, 7, 10, 11, 17**]. The celebrated Weyl theorem shows that the conformal and projective properties of a Finsler metric characterize the metric properties uniquely. Therefore, studying the conformal transformations of a Finsler metric deserves an extra consideration.

In [**6**], Knebelman found out that the conformal factor of the conformal transformation of Finsler metrics is a function of position only. Then, Hashiguchi studied the basic and interesting properties of Finsler conformal transformations and obtained some relations between the Riemannian curvature and some non-Riemannian curvatures of conformal Finsler metrics. Also, he obtained some conformal invariants. Their investigations show that the conformal transformations do not preserve the Riemannian and non-Riemannian curvatures in Finsler geometry [**1, 2, 15**].

Among the non-Riemannian curvatures in Finsler geometry, the mean Landsberg curvature has distinguished place. Geometrically, the mean Landsberg curvature $\mathbf{J}$ measures the rate of changes of the mean Cartan torsion $\mathbf{I}$ along the geodesics in a general Finsler space. Indeed, $\mathbf{J} = \nabla_0\mathbf{I}$, where ∇_0 denotes the horizontal derivation along Finslerian geodesics. A Finsler metric F is called weakly Landsberg metric if it has vanishing mean Landsberg curvature $\mathbf{J} = 0$. This non-Riemannian curvature has been observed in many situations, including the case when working with the Gauss–Bonnet theorem in the Finslerian setting. Consider

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the Riemannian metric $\hat{\mathbf{g}}_x := g_{ij}(x,y)\delta y^i \otimes \delta y^j$ on T_xM_0, where $g_{ij} := 1/2[F^2]_{y^i y^j}$ and $\{\delta y^i := dy^i + N^i_j dx^j\}$ is the natural coframe on T_xM associated with the natural basis $\{\partial/\partial x^i|_x\}$ for T_xM. The constancy of the volume function of the unit tangent sphere $S_xM \subset (T_xM, \hat{\mathbf{g}}_x)$ is required to establish a Gauss–Bonnet theorem for Finsler manifolds. In [**4**], it is proved that the volume function is a constant for every weakly Landsberg metric.

In conformal Finsler geometry, the class of conformally closed Finsler metrics has an interesting position and many investigations have been done for this class of metrics. Let us denote by $\mathcal{F}^n$ the set of a special kind of n-dimensional Finsler metrics, e.g., Berwald metrics or weakly Landsberg metrics. If $F \in \mathcal{F}^n$ remains to belong to $\mathcal{F}^n$ by any conformal transformation, then $\mathcal{F}^n$ is called conformally closed. It is well known that every Riemannian metric is conformally closed. Also, if a Finsler metric F is conformal to a locally Minkowskian metric, then the Finsler metric $\tilde{F} = \exp(\sigma)F$ is also conformal to a locally Minkowskian metric [**14, 16**]. Then, any conformally flat Finsler metric is conformally closed. In [**7**], Matsumoto studied conformally closed Berwald metrics and found the necessary and sufficient conditions under which a Berwald metric is conformally closed. More precisely, he proved that a Berwald metric is conformally closed if and only if $F^2 g^{ij}$ are homogeneous polynomials of degree 2 in $y = (y^i)$. Also, he considered conformally closed Douglas metrics. In [**12**], Shen studied S-closed conformal transformations in Finsler geometry and proved that such a transformation must be a homothety unless the Finsler manifold is Riemannian. Recently, it was proved that two-dimensional conformally related Douglas metrics are Randers metrics [**9**]. In [**8**], Matsumoto showed that a Landsberg metric is conformally closed if and only if its T-tensor is vanishing. By definition $\mathbf{J} := \mathrm{trace}(\mathbf{L})$, every Landsberg metric is a weakly Landsberg metric. In this paper, we generalize and complete the Matsumoto theorem. More precisely, we prove the following.

Theorem 1.1. *A weakly Landsberg metric remains weakly Landsberg metric under any conformal transformation if and only if it is a Riemannian metric.*

2. Preliminaries

Let M be an n-dimensional C^∞ manifold, $TM = \bigcup_{x\in M} T_xM$ the tangent bundle, and $TM_0 := TM - \{0\}$ the slit tangent bundle. A Finsler structure on M is a function $F\colon TM \to [0,\infty)$ with the following properties:

(i) F is C^∞ on TM_0;
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, i.e., $F(x,\lambda y) = \lambda F(x,y)$ for all $\lambda > 0$;
(iii) The following quadratic form $\mathbf{g}_y\colon T_xM \times T_xM \to \mathbb{R}$ is positive definite on TM_0,

$$\mathbf{g}_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y+su+tv)\Big]_{s=t=0}, \qquad u,v \in T_xM.$$

Then the pair (M,F) is called a Finsler manifold.

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, one can define $\mathbf{C}_y\colon T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\mathbf{g}_{y+tw}(u,v)\Big]_{t=0}, \qquad u,v,w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian.

For $y \in T_xM_0$, define $\mathbf{I}_y\colon T_xM \to \mathbb{R}$ by

$$\mathbf{I}_y(u) := \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u,\partial_i,\partial_j),$$

where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y\in TM_0}$ is called the mean Cartan torsion. By definition, $\mathbf{I}_y(y) = 0$ and $\mathbf{I}_{\lambda y} = \lambda^{-1}\mathbf{I}_y$, $\lambda > 0$. Therefore, $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$.

For a Finsler manifold (M,F), a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i,y^i) for TM_0, is given by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i},$$

where $G^i = G^i(x,y)$ are local functions on TM given by

$$G^i := \frac{1}{4}g^{il}\Big\{\frac{\partial^2[F^2]}{\partial x^k\partial y^l}y^k - \frac{\partial[F^2]}{\partial x^l}\Big\}, \qquad y \in T_xM.$$

$\mathbf{G}$ is called the associated spray to (M,F).

Define $\mathbf{B}_y\colon T_xM\times T_xM\times T_xM \to T_xM$ by $\mathbf{B}_y(u,v,w) := B^i{}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}|_x$, where

$$B^i{}_{jkl} := \frac{\partial^3G^i}{\partial y^j\partial y^k\partial y^l}.$$

$\mathbf{B}$ is called the Berwald curvature and F is called a Berwald metric if $\mathbf{B} = 0$.

For $y \in T_xM$, define the Landsberg curvature $\mathbf{L}_y\colon T_xM\times T_xM\times T_xM \to \mathbb{R}$ by

$$\mathbf{L}_y(u,v,w) := -\frac{1}{2}\mathbf{g}_y\big(\mathbf{B}_y(u,v,w),y\big).$$

See page [**13**, p. 84]. In local coordinates, $\mathbf{L}_y(u,v,w) := L_{\ ijk}(y)u^iv^jw^k$, where

$$L_{ijk} := -\frac{1}{2}y_lB^l{}_{ijk}.$$

$\mathbf{L}_y(u,v,w)$ is symmetric in u, v and w, and $\mathbf{L}_y(y,v,w) = 0$. $\mathbf{L}$ is called the Landsberg curvature. A Finsler metric F is called a Landsberg metric if $\mathbf{L} = 0$. Also, the Landsberg curvature of F can be defined by following

$$\mathbf{L}_y(u,v,w) := \frac{\mathrm{d}}{\mathrm{d}t}\Big[\mathbf{C}_{\dot\sigma(t)}\big(U(t),V(t),W(t)\big)\Big]_{t=0},$$

where $y \in T_xM$, $\sigma = \sigma(t)$ is the geodesic with $\sigma(0) = x$, $\dot\sigma(0) = y$, and $U(t)$, $V(t$, $W(t)$ are linearly parallel vector fields along σ with $U(0) = u$, $V(0) = v$, $W(0) = w$. Then the Landsberg curvature $\mathbf{L}_y$ is the rate of change of $\mathbf{C}_y$ along geodesics for any $y \in T_xM_0$. For more details, see [**13**].

For $y \in T_xM$, define $\mathbf{J}_y : T_xM \to \mathbb{R}$ by $\mathbf{J}_y(u) := J_i(y)u^i$, where

$$\mathbf{J}_y(u) = \sum_{i=1}^{n} g^{ij}(y)\mathbf{L}_y(u, \partial_i, \partial_j).$$

The non-Riemannian quantity $\mathbf{J}$ is called the *mean Landsberg curvature* or $\mathbf{J}$-curvature of F. We say that F is a weakly Landsberg metric if $\mathbf{J} = 0$. It is easy to see that the mean Landsberg curvature of F is also given by

$$\mathbf{J}_y(u) := \frac{\mathrm{d}}{\mathrm{d}t}\Big[\mathbf{I}_{\dot\sigma(t)}\big(U(t)\big)\Big]_{t=0},$$

where $y \in T_xM$, σ is the geodesic with $\sigma(0) = x$, $\dot\sigma(0) = y$ and U is linearly parallel vector field along σ with $U(0) = u$. Thus the mean Landsberg curvature $\mathbf{J}_y$ is the rate of change of $\mathbf{I}_y$ along geodesics for any $y \in T_xM_0$ ([**13**]).

Throughout this paper, we use the Berwald connection on Finsler manifolds. The h- and v-covariant derivatives of a Finsler tensor field are denoted by " | " and " . ", respectively.

3. Proof of Theorem 1.1

Lemma 3.1. *A Finsler metric F on a manifold M is conformally closed Berwald metric if and only if $A^{ij} := F^2 g^{ij}$ are homogeneous polynomials of degree 2 in $y = (y^i)$.*

Proof. Let F and $\bar F$ be two Finsler metrics on a manifold M. By using the Rapcsáks identity, the following relationship between G^i and $\bar G^i$ holds

$$\bar G^i = G^i + \frac{1}{2\bar F}\bar F_{|m} y^m y^i + \frac{\bar g^{il}}{2}\Big\{\bar F(\bar F_{|k})_{.l} y^k - \bar F \bar F_{|l}\Big\}, \tag{1}$$

where " | " and " . " denote the horizontal and vertical derivations with respect to the Berwald connection of F (see [**13**, p. 180]). Suppose that F is conformally related to $\bar F$, namely, $\bar F = \mathrm{e}^{\sigma} F$, where $\sigma = \sigma(x)$ is a scalar function on M. Since $F_{|m} = 0$, then the following hold:

$$\begin{aligned} &\bar F_{|m} = \sigma_m \mathrm{e}^{\sigma} F, \quad \bar F_{.i} = \mathrm{e}^{\sigma} F_{.i}, \quad (\bar F_{|m})_{.l} = \sigma_m \mathrm{e}^{\sigma} F_{.l}, \\ &\bar g_{ij} = \mathrm{e}^{2\sigma} g_{ij}, \qquad \bar g^{ij} = \mathrm{e}^{-2\sigma} g^{ij}, \end{aligned} \tag{2}$$

where we put

$$\sigma_m := \frac{\partial \sigma}{\partial x^m}.$$

By putting (2) in (1), we get

$$\bar G^i = G^i + \sigma_0 y^i - \frac{1}{2} F^2 \sigma^i, \tag{3}$$

where we define

$$\sigma_0 := \sigma_i y^i, \qquad \sigma^i := g^{im}\sigma_m.$$

Then, (3) can be written as follows

$$\bar{G}^i = G^i + Py^i - Q^i, \tag{4}$$

where

$$P := \sigma_k y^k, \qquad Q^i := \frac{1}{2}F^2\sigma^i. \tag{5}$$

Let us define

$$\begin{aligned} G^i_j &:= G^i_{.j}, & G^i_{jk} &:= G^i_{j.k}, \\ \bar{G}^i_j &:= \bar{G}^i_{.j}, & \bar{G}^i_{jk} &:= \bar{G}^i_{j.k}, \\ P_j &:= P_{.j}, & P_{jk} &:= P_{j.k}, \\ Q^i_j &:= Q^i_{.j}, & Q^i_{jk} &:= Q^i_{j.k}, & Q^i_{jkl} &:= Q^i_{jk.l}. \end{aligned}$$

Taking vertical derivations of (4) imply that

$$\bar{G}^i_j = G^i_j + P_j y^i + P\delta^i_j - Q^i_j, \tag{6}$$

$$\bar{G}^i_{jk} = G^i_{jk} + P_{jk}y^i + P_j\delta^i_k + P_k\delta^i_j - Q^i_{jk}, \tag{7}$$

$$\bar{B}^i_{jkl} = B^i_{jkl} + P_{jkl}y^i + P_{jk}\delta^i_l + P_{jl}\delta^i_k + P_{kl}\delta^i_j - Q^i_{jkl}, \tag{8}$$

The following hold

$$P_i = \sigma_i, \qquad P_{ij} = P_{ijk} = 0. \tag{9}$$

By (8) and (9), we get

$$\bar{B}^i{}_{jkl} = B^i{}_{jkl} - Q^i{}_{jkl}. \tag{10}$$

(10) gives us the proof. □

Lemma 3.2. *Let F be a weakly Landsberg metric on a manifold M. Then, $\bar{F} := \mathrm{e}^\sigma F$ is a weakly Landsberg metric if and only if*

$$\Big[2(y_pI_l + y_lI_p) + F^2(I_{p.l} + g_{sl}C^{ks}_{\ p.k})\Big]\sigma^p = 0. \tag{11}$$

Proof. Since

$$\bar{y}_i = \bar{F}\bar{F}_{.i} = \mathrm{e}^{2\sigma}y_i,$$

contracting (10) with $\bar{y}_i$ implies that

$$\bar{L}_{jkl} = \mathrm{e}^{2\sigma}\Big(L_{jkl} - \frac{1}{2}y_iQ^i{}_{jkl}\Big). \tag{12}$$

Multiplying (12) with $\bar{g}^{jk}$ implies that

$$\bar{J}_l = J_l - \frac{1}{2}y_ig^{jk}Q^i{}_{jkl}. \tag{13}$$

Thus, $\bar{F}$ is also weakly Landsberg metric if and only if

$$y_ig^{jk}Q^i{}_{jkl} = 0. \tag{14}$$

To complete the proof, we need to simplify (14). Using $(\sigma^i)_{.j} = -2C^i{}_{jm}\sigma^m$, we get

$$Q^i_j = y_j\sigma^i - F^2C^i{}_{jm}\sigma^m, \tag{15}$$

$$Q^i{}_{jk} = g_{jk}\sigma^i - 2\Big(y_j C^i_{km} + y_k C^i_{jm}\Big)\sigma^m - F^2\Big(C^i_{jp,k} - 2C^i_{jm}C^m_{kp}\Big)\sigma^p, \tag{16}$$

$$\begin{aligned} Q^i{}_{jkl} = {} & 2C_{jkl}\sigma^i - 2g_{jk}C^i{}_{lp}\sigma^p \\ & - 2\Big(g_{jl}C^i{}_{km} + y_j C^i{}_{km.l} + g_{kl}C^i{}_{jm} + y_k C^i{}_{jm.l}\Big)\sigma^m \\ & + 4\Big(y_j C^i{}_{km} + y_k C^i{}_{jm}\Big)C^m_{lp}\sigma^p - 2y_l\Big(C^i{}_{jp.k} - 2C^i{}_{jm}C^m_{kp}\Big)\sigma^p \\ & - F^2\Big(C^i{}_{jp.k.l} - 2C^i{}_{jm.l}C^m_{kp} - 2C^i{}_{jm}C^m_{kp.l}\Big)\sigma^p \\ & + 2F^2\Big(C^i{}_{jp.k} - 2C^i{}_{jm}C^m_{kp}\Big)C^p_{lt}\sigma^t. \end{aligned} \tag{17}$$

Using $y_i C^i{}_{jk} = 0$ and contracting (17) with y_i imply that

$$\begin{aligned} y_i Q^i{}_{jkl} = {} & 2\sigma_0 C_{jkl} - 2y_i(y_j C^i{}_{km.l} + y_k C^i{}_{jm.l})\sigma^m - 2y_l y_i C^i{}_{jp.k}\sigma^p \\ & - F^2 y_i(C^i{}_{jp.k.l} - 2C^i{}_{jm.l}C^m_{kp})\sigma^p + 2F^2 y_i C^i{}_{jp.k}C^p_{lq}\sigma^q. \end{aligned} \tag{18}$$

We have

$$y_i C^i{}_{jp.k} = -C_{jpk}. \tag{19}$$

Also, taking a vertical derivation of (19) with respect to y^l yields

$$y_i C^i{}_{jp.k.l} = -C_{jpk.l} - g_{il}C^i{}_{jp.k}. \tag{20}$$

Putting (19) and (20) in (18) gives

$$\begin{aligned} y_i Q^i{}_{jkl} = {} & 2\sigma_0 C_{jkl} + 2(y_j C_{lkm} + y_k C_{ljm} + y_l C_{jmk})\sigma^m \\ & + F^2(C_{jpk.l} + g_{il}C^i{}_{jp.k} - 2C_{jml}C^m_{kp})\sigma^p - 2F^2 C_{jpk}C^p_{lq}\sigma^q. \end{aligned} \tag{21}$$

Contracting (22) with g^{jk} implies that

$$\begin{aligned} y_i g^{jk} Q^i{}_{jkl} = \Big[& 2(y_p I_l + y_l I_p) \\ & + F^2(g^{jk}C_{jpk.l} + g^{jk}g_{il}C^i{}_{jp.k} - 2C^k{}_{ml}C^m_{kp} - 2I_q C^q{}_{lp})\Big]\sigma^p. \end{aligned} \tag{22}$$

We have

$$I_{p.l} = (g^{jk}C_{jpk})_{.l} = -2C^{jk}_l C_{jpk} + g^{jk}C_{jpk.l},$$

which yields

$$g^{jk}C_{jpk.l} = I_{p.l} + 2C^{jk}_l C_{jpk}. \tag{23}$$

Also, we get

$$C^{ki}_{p.k} = (C^{ki}_p)_{.k} = (g^{jk}C^i{}_{jp})_{.k} = -2I^j C^i{}_{jp} + g^{jk}C^i{}_{jp.k},$$

which implies that

$$g^{jk}C^i{}_{jp.k} = C^{ki}_{p.k} + 2I^j C^i{}_{jp}. \tag{24}$$

Contracting (24) with g_{il} yields

$$g_{il}g^{jk}C^i{}_{jp.k} = g_{il}C^{ki}_{p.k} + 2I^j C_{ljp}. \tag{25}$$

Putting (23) and (25) in (22) implies

$$y_i g^{jk} Q^i{}_{jkl} = \Big[2(y_p I_l + y_l I_p) + F^2(I_{p.l} + g_{sl} C^{ks}{}_{p.k})\Big]\sigma^p, \tag{26}$$

which completes the proof. □

Proof of Theorem 1.1. Suppose that (11) holds for every smooth function $\sigma = \sigma(x)$ on the manifold M. In this case, the following holds:

$$2(y_p I_l + y_l I_p) + F^2(I_{p.l} + g_{sl} C^{ks}{}_{p.k}) = 0. \tag{27}$$

We have

$$C^k{}_{lp.k} = (g_{sl} C^{ks}{}_p)_{.k} = 2C_{slk} C^{ks}{}_p + g_{sl} C^{ks}{}_{p.k}, \tag{28}$$

which yields

$$g_{sl} C^{ks}{}_{p.k} = C^k{}_{lp.k} - 2C_{lsk} C^{sk}{}_p. \tag{29}$$

Putting (29) in (27) yields

$$2(y_p I_l + y_l I_p) + F^2(I_{p.l} + C^k{}_{lp.k} - 2C_{lsk} C^{sk}{}_p) = 0. \tag{30}$$

We have

$$y^p C^k{}_{lp.k} = -I_l. \tag{31}$$

Considering (31) and multiplying (30) with y^p imply that $I_l = 0$. In this case, by Deicke's theorem, F reduces to a Riemannian metric [**3**]. □

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A. Tayebi, Department of Mathematics, Faculty of Science, University of Qom, Qom, Iran,
e-mail: `akbar.tayebi@gmail.com`

B. Najafi, Department of Mathematics and Computer Sciences, Amirkabir University of Technology (Tehran Polytechnique), Tehran, Iran,
e-mail: `behzad.najafi@aut.ac.ir`