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AN ORDER INDUCED BY THE DIRECT PRODUCT OF T-NORMS

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Abstract. In this paper, we study the direct product of triangular norms (t-norms) on bounded lattices. We define an order induced by t-norms, which is a direct product of two t-norms on bounded lattices, and properties of introduced order are deeply investigated. We explore some properties of the direct product of two t-norms on bounded lattices. We investigate the properties of the set of comparable and incomparable elements with respect to the $\preceq_T$ partial order.

1. Introduction and motivation

1.1. A brief review on the development of t-norms and bounded lattice

Triangular norms (t-norms) and triangular conorms (t-conorms) were introduced in 1963 by Schweizer and Sklar [**29**] within the framework of probabilistic metric spaces. More specifically, they are based on a notion used by Menger [**25**] in order to extend the triangle inequality in the definition of metric spaces towards probabilistic metric spaces. In fuzzy set theory, they were introduced for the first time by Alsina, Trillas and Valverde [**1**], and Prade [**27**], who used them for the definition of new classes of fuzzy union and intersection operators. They are isotonic, associative and commutative binary operators on the unit interval, that furthermore satisfy a few special boundary conditions. T-norms and t-conorms have been used in many applications such as fuzzy sets, fuzzy logic, fuzzy system modeling, expert systems, neural networks and approximate reasoning [**16, 17, 18, 19**]. T-norms and t-conorms have been examined by many authors, both theoretically and in terms of application. Some theoretical results were analyzed in detail by [**2, 4, 8**]. Concerning applications, recall, e.g., fuzzy logics [**26**], expert systems [**15**], non-additive integrals [**22**], fuzzy rule-based systems [**32**].

In 1986, Mitsch defined a natural order for semigroups [**24**]. In 1999, De Baets and Mesiar introduced direct product of t-norms on product lattices and investigated some of the algebraic properties of introduced t-norms [**14**]. In 2011, Karaçal and Kesicioğlu defined an order induced by t-norms on bounded lattices [**20**].

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1.2. The motivation

Nowadays, the orders induced by t-norms, uninorms and nullnorms on bounded lattices have been studied extensively. Among these, an order induced by t-norms on a bounded lattice M has always been a challenging problem due to the poor structures of M compared with $[0,1]$.

In this paper, we study the direct product of t-norms on bounded lattices. The paper consists of four main parts. Firstly, in Preliminaries, we give some necessary definitions we will work with. In Section 3, we define an order induced by t-norms, which is a direct product of two t-norms on bounded lattices. We also define the set of comparable and incomparable elements with respect to the $T_1 \times T_2$-partial order, denoted $\preceq_{T_1 \times T_2}$, and we obtain some interesting results related to direct product of t-norms on $[0,1]^2$. In Section 4, some concluding remarks are added.

2. Preliminaries

A lattice [**10**] is a partially ordered set $(M, \leq)$, in which each two element subset $\{a, b\}$ has an infimum, denoted as $a \wedge b$, and a supremum, denoted as $a \vee b$. A bounded lattice $(M, \leq, 0_M, 1_M)$ is a lattice that has the bottom and top elements written as 0_M and 1_M, respectively. Given a bounded lattice $(M, \leq, 0_M, 1_M)$ and $a, b \in M$, if a and b are incomparable, in this case, we use the notation $a \parallel b$.

Definition 2.1 ([**14**])**.** Let $(M_1, \leq_1, 0_1, 1_1)$ and $(M_2, \leq_2, 0_2, 1_2)$ be bounded lattices. Then $M_1 \times M_2 = (M_1 \times M_2, \leq, (0_1, 0_2), (1_1, 1_2))$ is a bounded lattice with partial order relation $\leq$, $\wedge$ and $\vee$ defined by

$$(a_1, b_1) \leq (a_2, b_2) \iff a_1 \leq_1 a_2 \text{ and } b_1 \leq_2 b_2.$$
$$(a_1, b_1) \wedge (a_2, b_2) = (a_1 \wedge_1 a_2, b_1 \wedge_2 b_2).$$
$$(a_1, b_1) \vee (a_2, b_2) = (a_1 \vee_1 a_2, b_1 \vee_2 b_2).$$

We use M_1 instead of $(M_1, \leq_1, 0_1, 1_1)$, M_2 instead of $(M_2, \leq_2, 0_2, 1_2)$ and $M_1 \times M_2$ instead of $(M_1 \times M_2, \leq, \wedge, \vee, (0_1, 0_2), (1_1, 1_2))$.

Definition 2.2 ([**28**])**.** Let M be a bounded lattice. A *triangular norm* T (briefly t-norm) is a binary operation on M that is commutative, associative, monotone and has neutral element 1_M.

Definition 2.3 ([**28**])**.** Let M be a bounded lattice. A *triangular conorm* S (briefly t-conorm) is a binary operation on M that is commutative, associative, monotone and has neutral element 0_M.

Definition 2.4 ([**14**])**.** Let M_1 and M_2 be bounded lattices and T_1 and T_2 be t-norms on M_1 and M_2, respectively. Then the direct product $T_1 \times T_2$ of T_1 and T_2, defined by

$$T_1 \times T_2((a_1, b_1), (a_2, b_2)) = (T_1(a_1, a_2), T_2(b_1, b_2)),$$

is a t-norm on the product lattice $M_1 \times M_2$.

Definition 2.5 ([**11**])**.** A t-norm T on M is *divisible* if the following condition holds:

$$\forall\, a, b \in M \text{ with } a \leq b, \text{ there is a } m \in M \text{ such that } a = T(b, m).$$

Definition 2.6 ([**28**])**.** Let $(a_i, b_i)_{i\in I}$ be a family of pairwise disjoint open subintervals of $[0, 1]$ and let $(T_i)_{i\in I}$ be a family of t-norms. Then the ordinal sum $T = (< a_i, b_i, T_i >)_{i\in I} \colon [0, 1]^2 \to [0, 1]$ is given by

$$T(a, b) = \begin{cases} a_i + (b_i - a_i)T_i(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}), & (a, b) \in [a_i, b_i]^2, \\ \min(a, b), & \text{otherwise.} \end{cases}$$

Definition 2.7 ([**20**])**.** Let M be a bounded lattice and T be a t-norm on M. The order called a T*-partial order* (triangular order) for t-norm T is defined as follows:

$$a \preceq_T b \colon \iff T(\ell, b) = a \text{ for some } \ell \in M.$$

3. $\preceq_{T_1\times T_2}$-PARTIAL ORDER AND PROPERTIES

In this section, we introduce an order induced by t-norms which is a direct product of two t-norms on bounded lattices. Additionally, some propositions presented here are derived from [**3, 6**], and as such, certain proofs are omitted.

Definition 3.1. Let M_1 and M_2 be bounded lattices, T_1 and T_2 be t-norms on M_1 and M_2, respectively, and consider their direct product $T_1 \times T_2$ on $M_1 \times M_2$. Let $\preceq_{T_1}$ and $\preceq_{T_2}$ be partial orders induced by t-norms T_1 and T_2, respectively. Then the relation $\preceq_{T_1\times T_2}$ is defined by

$$(a, b) \preceq_{T_1\times T_2} (c, d) \iff a \preceq_{T_1} c \quad \text{and} \quad b \preceq_{T_2} d$$

for all $(a, b), (c, d) \in M_1 \times M_2$.

Example 3.2.
Consider the bounded lattice $(M_1 = M_2 = \{0_{M_1}, k, p, t, m, s, 1_{M_1}\}, \leq, 0_{M_1}, 1_{M_1})$ given in Figure 1 and the t-norms T_1 and T_2 on $M_1 = M_2$ defined in Tables 1 and 2, respectively.

Table 1. The t-norm T_1 on $M_1 = M_2$

T_1	0_{M_1}	k	p	t	m	s	1_{M_1}
0_{M_1}	0_{M_1}	0_{M_1}	0_{M_1}	0_{M_1}	0_{M_1}	0_{M_1}	0_{M_1}
k	0_{M_1}	k	k	k	k	k	k
p	0_{M_1}	k	p	k	p	p	p
t	0_{M_1}	k	k	k	k	k	t
m	0_{M_1}	k	p	k	m	m	m
s	0_{M_1}	k	p	k	m	s	s
1_{M_1}	0_{M_1}	k	p	t	m	s	1_{M_1}

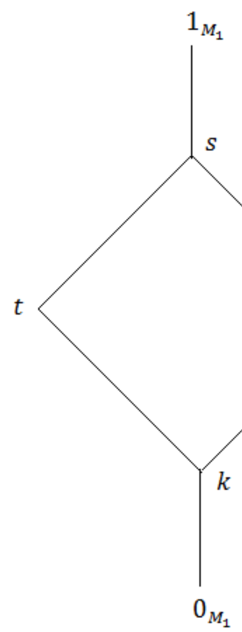

Figure 1. The lattice M_1

Table 2. The t-norm T_2 on $M_1 = M_2$

T_2	0_{M_1}	k	p	t	m	s	1_{M_1}
0_{M_1}	0_{M_1}	0_{M_1}	0_{M_1}	0_{M_1}	0_{M_1}	0_{M_1}	0_{M_1}
k	0_{M_1}	0_{M_1}	0_{M_1}	0_{M_1}	0_{M_1}	0_{M_1}	k
p	0_{M_1}	0_{M_1}	0_{M_1}	0_{M_1}	0_{M_1}	0_{M_1}	p
t	0_{M_1}	0_{M_1}	0_{M_1}	0_{M_1}	0_{M_1}	0_{M_1}	t
m	0_{M_1}	0_{M_1}	0_{M_1}	0_{M_1}	m	m	m
s	0_{M_1}	0_{M_1}	0_{M_1}	0_{M_1}	m	s	s
1_{M_1}	0_{M_1}	k	p	t	m	s	1_{M_1}

Since $T_1(m,s) = m$ and $T_2(m,s) = m$, it follows that $m \preceq_{T_1} s$ and $m \preceq_{T_2} s$. Consequently, by Definition 3.1, we have $(m,m) \preceq_{T_1 \times T_2} (s,s)$. Next, we want to show that $(k,k) \not\preceq_{T_1 \times T_2} (p,p)$. We assume that $(k,k) \preceq_{T_1 \times T_2} (p,p)$. By Definition 3.1, this would imply $k \preceq_{T_1} p$ and $k \preceq_{T_2} p$. Then there exists an element $\ell \in M_1$ such that $T_2(p,\ell) = k$. According to Table 2, we obtain $T_2(p,\ell) = k$, a contradiction, because there does not exist an element $\ell \in M_1$ such that $T_2(p,\ell) = k$. Thus, $k \not\preceq_{T_2} p$. So, we have that $(k,k) \not\preceq_{T_1 \times T_2} (p,p)$ by Definition 3.1.

Proposition 3.3. Let M_1 and M_2 be bounded lattices, T_1 be a t-norm on M_1 and T_2 be a t-norm on M_2 and consider their direct product $T_1 \times T_2$ on $M_1 \times M_2$. Then the relation $\preceq_{T_1 \times T_2}$ defined in Definition 3.1 is a partial order on $M_1 \times M_2$.

Proposition 3.4. Let M_1 and M_2 be bounded lattices, T_1 be a t-norm on M_1 and T_2 be a t-norm on M_2 and consider their direct product $T_1 \times T_2$ on $M_1 \times M_2$. Then $M_1 \times M_2$ is a bounded partially ordered set with respect to the $\preceq_{T_1 \times T_2}$ partial order.

Remark. Let M_1 and M_2 be bounded lattices, T_1 be a t-norm on M_1 and T_2 be a t-norm on M_2 and consider their direct product $T_1 \times T_2$ on $M_1 \times M_2$. Then

$$(a, b) \preceq_{T_1 \times T_2} (c, d) \implies a \leq_1 c \text{ and } b \leq_2 d$$

for all $(a, b), (c, d) \in M_1 \times M_2$.

We now explore some properties of direct product of t-norms on bounded lattices. We define comparable and incomparable elements with respect to the $T_1 \times T_2$-partial order on bounded lattices. Using these definitions, we obtain some interesting results for direct product of t-norms on $[0, 1]^2$.

Definition 3.5. Let M be a bounded lattice and T be a t-norm on bounded lattice M. The set C_T is defined as follows:

$$C_T = \Big\{a \in M \mid \text{there exist } b, b' \in M \setminus \{0_M, a, 1_M\},\ a \preceq_T b \text{ and } b' \preceq_T a\Big\}.$$

Remark. It is clear that $\{0_M, 1_M\} \notin C_T$. If we take $b, b' \in \{0_M, a, 1_M\}$, then it is trivial that all elements in M satisfy the condition of Definition 3.5. So, we have to take $b, b' \notin \{0_M, a, 1_M\}$ in Definition 3.5.

Remark. For any t-norm T on bounded lattice, if $\mid M \mid \leq 4$, then it is obtained that $C_T = \emptyset$.

Example 3.6.
Consider the lattice $(M_2 = \{0_{M_2}, n, e, k, r, p, s, 1_{M_2}\}, \leq, 0_{M_2}, 1_{M_2})$, which is depicted by Hasse diagram in Figure 2, and consider the t-norm $T(a, b) = a \wedge b$ on M_2.

Since $T(e, k) = e$ and $T(e, n) = n$, it follows that $e \preceq_T k$ and $n \preceq_T e$. So, $e \in C_T$. Similarly, since $T(p, s) = p$ and $T(p, r) = r$, we obtain that $p \preceq_T s$ and $r \preceq_T p$. So, $p \in C_T$. Thus, it is obtained that $C_T = \{e, p\}$.

Example 3.7. Consider the t-norm T^{nM} on $[0, 1]$ is defined by

$$T^{nM}(a, b) = \begin{cases} 0, & a + b \leq 1, \\ \min(a, b), & \text{otherwise.} \end{cases}$$

Then $C_{T^{nM}} = (\frac{1}{2}, 1)$. Now, let us prove this statement. Let $a \in (\frac{1}{2}, 1)$. Then $T^{nM}(a, b) = a$ for all $b \in (a, 1)$. Therefore, $a \preceq_{T^{nM}} b$. Since $T^{nM}(a, b') = b'$ for all $b' \in (\frac{1}{2}, a]$, it follows that $b' \preceq_{T^{nM}} a$. So, we obtain that $a \in C_{T^{nM}}$, i.e., $(\frac{1}{2}, 1) \subseteq C_{T^{nM}}$. Conversely, let $a \in C_{T^{nM}}$. We want to show that $a \in (\frac{1}{2}, 1)$. Suppose that $a \notin (\frac{1}{2}, 1)$. Then, it must be $a = 1$ or $a \in [0, \frac{1}{2}]$. According to

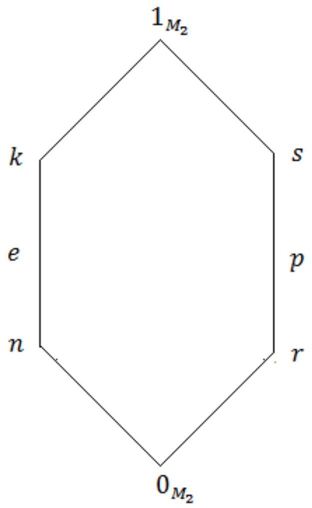

Figure 2. The lattice M_2

Remark 3, it can not be $a = 1$. So, it must be $a \in [0, \frac{1}{2}]$. Since $a \in C_{T^{nM}}$, there exist elements $b, b' \in (0,1) \setminus \{a\}$ such that $a \preceq_{T^{nM}} b$ and $b' \preceq_{T^{nM}} a$. Let $b' \preceq_{T^{nM}} a$. Then there exists $k \in [0,1]$ such that $T^{nM}(a,k) = b'$. Since $b' \neq 0$, it follows that $T^{nM}(a,k) = \min(a,k) = b'$. Since $a \neq b'$, it holds $k = b'$. So, it is obtained $a + k \leq 1$. Thus, we obtained $T^{nM}(a,k) = b' = 0$, a contradiction.

Proposition 3.8. Let $(M, \leq, 0_M, 1_M)$ be a bounded lattice such that $| M |> 4$ and consider the weakest t-norm

$$T_W(x,y) = \begin{cases} x, & y = 1_M, \\ y, & x = 1_M, \\ 0_M, & \text{otherwise}, \end{cases}$$

on M. Then $C_{T_W} = \emptyset$.

Proof. Let $a \in C_{T_W}$. Then there exist $b, b' \in L \setminus \{0_M, a, 1_M\}$ such that $a \preceq_{T_W} b$ and $b' \preceq_{T_W} a$. Let $a \preceq_{T_W} b$. Then there exist $\ell \in M$ such that $T_W(b,\ell) = a$. If $\ell = 1$, then it must be $a = b$, a contradiction. If $\ell \neq 1$, then it must be $a = 0$, a contradiction. So, it can not be $a \preceq_{T_W} b$. Similarly, it can be shown that $b' \npreceq_{T_W} a$. So, $C_{T_W} = \emptyset$. □

Proposition 3.9. Let $(M, \leq, 0_M, 1_M)$ be a bounded chain such that $| M |= n$, $n > 4$ and consider the greatest t-norm $T_\wedge(x,y) = x \wedge y$ on M. Then $| C_{T_\wedge} |= n-4$.

Example 3.10. Let $(M = \{0_M, p, q, r, s, t, 1_M\}, \leq, 0_M, 1_M)$ be a chain such that $0_M < p < q < r < s < t < 1_M$. It is clear that $C_{T_\wedge} = \{q, r, s\}$. So, $| C_{T_\wedge} |= 3 = 7 - 4$.

Proposition 3.11. Let T_1 and T_2 be t-norms on the bounded lattice M. If $\preceq_{T_1} \subseteq \preceq_{T_2}$, then $C_{T_1} \subseteq C_{T_2}$.

Example 3.12. Let $(M = \{0_M, p, q, r, s, t, 1_M\}, \leq, 0_M, 1_M)$ be a chain such that $0_M < p < q < r < s < t < 1_M$. Consider the t-norms T_W and $T_\wedge$ on M. It is clear that $\preceq_{T_W} \subseteq \preceq_{T_\wedge}$. According to Proposition 3.8 and Example 3.10, it is obtained that $C_{T_W} = \emptyset$ and $C_{T_\wedge} = \{q, r, s\}$, respectively. So, it is clear that $C_{T_W} \subseteq C_{T_\wedge}$.

Corollary 3.13. *Let T_1 and T_2 be t-norms on the bounded lattice M. If $\preceq_{T_1} = \preceq_{T_2}$, then $C_{T_1} = C_{T_2}$.*

Remark. The converse of Corollary 3.13 may not be true. Now, we will show that this claim holds.

Example 3.14. Consider the t-norm $T^\star$ on $[0,1]$ is defined as follows:

$$T^*(a,b) = \begin{cases} 0, & (a,b) \in (0, \frac{1}{2})^2, \\ \min(a,b), & \text{otherwise}, \end{cases}$$

and consider the t-norm T^{nM} on $[0,1]$. It can be shown that $C_{T^*} = (\frac{1}{2}, 1)$. According to Example 3.7, it is obtained that $C_{T^{nM}} = C_{T^*}$. But, it does not need to be $\preceq_{T^{nM}} = \preceq_{T^*}$.

Now, we will show that this claim holds. Since $T^*(\frac{1}{3}, \frac{2}{3}) = \frac{1}{3}$, it must be $\frac{1}{3} \preceq_{T^*} \frac{2}{3}$. On the other hand, $\frac{1}{3} \npreceq_{T^{nM}} \frac{2}{3}$. We suppose that $\frac{1}{3} \preceq_{T^{nM}} \frac{2}{3}$. Then there exists an element $k \in [0,1]$ such that $T^{nM}(\frac{2}{3}, k) = \frac{1}{3}$. By definition of T^{nM} , it is obtained that $T^{nM}(\frac{2}{3}, k) = \min(\frac{2}{3}, k) = \frac{1}{3}$. Then it must be $k = \frac{1}{3}$. Since $\frac{2}{3} + \frac{1}{3} = 1$, it must be $T^{nM}(\frac{2}{3}, \frac{1}{3}) = \frac{1}{3} = 0$, a contradiction. So, $\frac{1}{3} \npreceq_{T^{nM}} \frac{2}{3}$. Consequently, $\preceq_{T^{nM}} \neq \preceq_{T^*}$.

The set C_T allows us to introduce the next equivalence relation on the class of all t-norms on bounded lattices.

Definition 3.15. Define a relation δ on the class of all t-norms on bounded lattices by $T_1 \delta T_2$

$$T_1 \delta T_2 : \iff C_{T_1} = C_{T_2}.$$

Lemma 3.16. *The relation δ given in* Definition 3.15 *is an equivalence relation.*

Definition 3.17. For a given t-norm T on bounded lattice M, we denote by $\overline{T}$ the δ equivalence class linked to T, i.e,

$$\overline{T} = \{T' \mid T' \delta T\}.$$

Example 3.18. Consider the t-norms $T^\star$ and T^{nM} on $[0,1]$. Since $C_{T^*} = (\frac{1}{2}, 1) = C_{T^{nM}}$, it is clear that the t-norms T^* and T^{nM} are equivalent according to the relation δ.

Proposition 3.19. Let $(M, \leq, 0_M, 1_M)$ be a bounded lattice. Equivalence class of the infimum t-norm $T_\wedge$ is the set of all divisible t-norms on M.

Proof. Let $T' \in \overline{T_\wedge}$, $x \le y$ and $y' \le x$ for $x, y, y' \in M$. Since $T_\wedge(x,y) = x$ and $T_\wedge(x,y') = y'$, it follows that $x \preceq_{T_\wedge} y$ and $y' \preceq_{T_\wedge} x$. That is $x \in C_{T_\wedge}$. Since $C_{T_\wedge} = C_{T'}$, it must be $x \in C_{T'}$. Then for some $z, z' \in M$, $x \preceq_{T'} z$ and $z' \preceq_{T'} x$. Let $x \preceq_{T'} z$. Then there exist $m \in M$ such that $T'(z,m) = x$. So, it is obtained that T' is a divisible t-norm. Similarly, it can be shown that T' is a divisible t-norm for $z' \preceq_{T'} x$. Conversely, it is easy to show that $T' \delta T_\wedge$, when T' is a divisible t-norm on M. □

When $M = [0,1]$, it is clear that the equivalence class of the infimum t-norm T_M is the set of all continuous t-norms on $[0,1]$.

Proposition 3.20. Let M_1 and M_2 be bounded lattices, T_1 be a t-norm on M_1 and T_2 be a t-norm on M_2 and consider their direct product $T_1 \times T_2$ on $M_1 \times M_2$. Then

$$C_{T_1 \times T_2} = C_{T_1} \times C_{T_2}.$$

Definition 3.21 ([**7**])**.** Let M be a bounded lattice, T be a t-norm on M and let K_T be defined by

$$K_T = \{a \in M \mid \text{for some } b \in M,\ [a < b \text{ and } a \not\preceq_T b] \text{ or } [b < a \text{ and } b \not\preceq_T a]\}.$$

Proposition 3.22 ([**8**])**.** Let $(M, \le, 0_M, 1_M)$ be a bounded lattice and T be a t-norm on M. If there exist two elements of M such that these are incomparable, then $K_T \ne \emptyset$.

Remark. Proposition 3.22 can not be true for set C_T. That is, if there exist two elements of M such that these are incomparable, then it does not need to be $C_T \ne \emptyset$. To illustrate this claim we shall give the following example.

Example 3.23. Consider the lattice $(M_3 = \{0_{M_3}, k, m, n, 1_{M_3}\}, \le, 0_{M_3}, 1_{M_3})$ given in Figure 3. Consider the t-norm $T(a,b) = a \wedge b$ on M_3. It is clear that $C_T = \emptyset$.

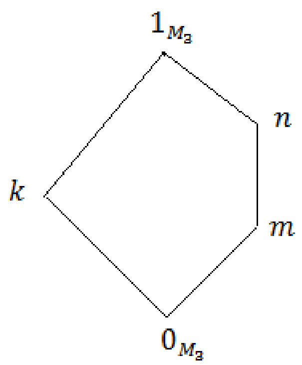

Figure 3. The lattice M_3

Proposition 3.24. Let M_1 and M_2 be bounded lattices, T_1 be a t-norm on M_1 and T_2 be a t-norm on M_2 and consider their direct product $T_1 \times T_2$ on $M_1 \times M_2$. Then

$$K_{T_1} \times K_{T_2} \subseteq K_{T_1 \times T_2}.$$

Remark. The converse of Proposition 3.24 may not be true. Here is an example illustrating such a case.

Example 3.25. Consider the t-norms T^{nM} and $T^{\star}$ on $[0,1]$. We want to demonstrate that it need not to be $K_{T^{nM} \times T^{\star}} \subseteq K_{T^{nM}} \times K_{T^{\star}}$. We will show that $(\frac{4}{5}, \frac{4}{5}) \in K_{T^{nM} \times T^{\star}}$ and $(\frac{4}{5}, \frac{4}{5}) \notin K_{T^{nM}} \times K_{T^{\star}}$. We claim that $\frac{1}{5} \npreceq_{T^{nM}} \frac{4}{5}$. Suppose that $\frac{1}{5} \preceq_{T^{nM}} \frac{4}{5}$. Then there exists an element $k \in [0,1]$ such that $T^{nM}(\frac{4}{5}, k) = \frac{1}{5}$. Since $\min(\frac{4}{5}, k) = \frac{1}{5}$, it follows that $k = \frac{1}{5}$. In this case, $\frac{1}{5} + \frac{4}{5} = 1$, it is obtained that $\frac{1}{5} = 0$, a contradiction. So, it holds $\frac{1}{5} \npreceq_{T^{nM}} \frac{4}{5}$. Thus, $(\frac{1}{5}, x) \npreceq_{T^{nM} \times T^{\star}} (\frac{4}{5}, \frac{4}{5})$ for all $x \in [0, \frac{4}{5}]$. So, $(\frac{4}{5}, \frac{4}{5}) \in K_{T^{nM} \times T^{\star}}$. On the other side, since $K_{T^{\star}} = (0, \frac{1}{2})$ by Aşıcı (see [**7**]), then we have that $\frac{4}{5} \notin K_{T^{\star}}$. So, $(\frac{4}{5}, \frac{4}{5}) \notin K_{T^{nM}} \times K_{T^{\star}}$. Consequently, $K_{T^{nM} \times T^{\star}} \subseteq K_{T^{nM}} \times K_{T^{\star}}$ does not hold.

Remark. If we take t-norms T_1 and T_2 to be equal, then the converse of Proposition 3.24 is true, i.e., equality is satisfied.

Remark. The converse of Proposition 3.24 may be true for some special t-norms on the unit interval $[0,1]$. Here is an example illustrating such a case.

Example 3.26. Consider the t-norm defined as follows:

$$T(a,b) = \begin{cases} \frac{ab}{2}, & (a,b) \in [0,1)^2, \\ \min(a,b), & \text{otherwise}, \end{cases}$$

and consider the t-norm T^{nM}. By [**23**], we know that $K_T = K_{T^{nM}} = (0,1)$. So, it is clear that $K_{T \times T^{nM}} = (0,1) \times (0,1) = K_T \times K_{T^{nM}}$.

Definition 3.27. Let M be a bounded lattice, T be a t-norm on M and $K_T^{\star}$ defined by

$$K_T^{\star} = \{a \in K_T \mid \text{for some } b, b' \in M,\ [a < b \text{ but } a \npreceq_T b] \text{ and } [b' < a \text{ but } b' \npreceq_T a]\}.$$

Proposition 3.28. Let M_1 and M_2 be bounded lattices, T_1 be a t-norm on M_1 and T_2 be a t-norm on M_2 and consider their direct product $T_1 \times T_2$ on $M_1 \times M_2$. Then

$$K_{T_1}^{\star} \times K_{T_2}^{\star} \subseteq K_{T_1 \times T_2}^{\star}.$$

Remark. The converse of Proposition 3.28 may not be true. Here is an example illustrating such a case.

Example 3.29. Consider the t-norms T^{nM} and $T^{\star}$ on $[0,1]$. We want to demonstrate that $K_{T^{nM} \times T^{\star}}^{\star} \nsubseteq K_{T^{nM}}^{\star} \times K_{T^{\star}}^{\star}$. We will show that $(\frac{1}{4}, \frac{2}{3}) \in K_{T^{nM} \times T^{\star}}^{\star}$ and $(\frac{1}{4}, \frac{2}{3}) \notin K_{T^{nM}}^{\star} \times K_{T^{\star}}^{\star}$. We claim that $\frac{1}{5} \npreceq_{T^{nM}} \frac{1}{4}$. Suppose that $\frac{1}{5} \preceq_{T^{nM}} \frac{1}{4}$. Then there exists an element $k \in [0,1]$ such that $T^{nM}(\frac{1}{4}, k) = \frac{1}{5}$. Since $\min(\frac{1}{4}, k) =$

$\frac{1}{5}$, it follows that $k = \frac{1}{5}$. In this case , $\frac{1}{4} + \frac{1}{5} < 1$, it is obtained that $\frac{1}{5} = 0$, a contradiction. So, $\frac{1}{5} \not\preceq_{T^{nM}} \frac{1}{4}$ and $(\frac{1}{5}, x) \not\preceq_{T^{nM} \times T^\star} (\frac{1}{4}, \frac{2}{3})$ for some $x \in (0, \frac{2}{3})$.

We also want to show that $\frac{1}{4} \not\preceq_{T^{nM}} \frac{3}{4}$. Suppose that $\frac{1}{4} \preceq_{T^{nM}} \frac{3}{4}$. Then there exists an element $\ell \in [0,1]$ such that $T^{nM}(\frac{3}{4}, \ell) = \frac{1}{4}$. Since $\min(\frac{3}{4}, \ell) = \frac{1}{4}$, it follows that $\ell = \frac{1}{4}$. In this case, $\frac{3}{4} + \frac{1}{4} = 1$, it is obtained that $\frac{1}{4} = 0$, a contradiction. So, $\frac{1}{4} \not\preceq_{T^{nM}} \frac{3}{4}$ and $(\frac{1}{4}, \frac{2}{3}) \not\preceq_{T^{nM} \times T^\star} (\frac{3}{4}, y')$ for some $y' \in (\frac{2}{3}, 1)$. Consequently, we have that $(\frac{1}{4}, \frac{2}{3}) \in K^\star_{T^{nM} \times T^\star}$. On the other side, it is clear that $K^\star_{T^{nM}} = (0, \frac{1}{2})$. So, we have that $\frac{2}{3} \notin K^\star_{T^{nM}}$ and $(\frac{1}{4}, \frac{2}{3}) \notin K^\star_{T^{nM}} \times K^\star_{T^\star}$. Consequently, $K^\star_{T^{nM} \times T^\star} \subseteq K^\star_{T^{nM}} \times K^\star_{T^\star}$ does not hold.

Remark. If we take the t-norms T_1 and T_2 to be equal, then the converse of Proposition 3.28 is true, i.e., equality is satisfied.

Remark. The converse of Proposition 3.28 may be true for some special t-norms on the unit interval $[0, 1]$. Here is an example illustrating such a case.

Example 3.30. Consider the t-norms T^{nM} and $T^\star$ on $[0, 1]$. It is clear that $K_{T^{nM}} = K_{T^\star} = (0, \frac{1}{2})$. So, it is apparent that $K_{T^{nM} \times T^\star} = (0, \frac{1}{2}) \times (0, \frac{1}{2}) = K_{T^{nM}} \times K_{T^\star}$.

4. Concluding remarks

We have introduced and studied t-norms on bounded lattices. We have defined and discussed $T_1 \times T_2$-partial order, denoted by $\preceq_{T_1 \times T_2}$. Additionally, we have defined the set of comparable and incomparable elements with respect to the $T_1 \times T_2$-partial order, denoted $\preceq_{T_1 \times T_2}$, and we have obtained some interesting results related to direct product of t-norms on $[0, 1]^2$. The theoretical developments in this paper provide a more systematic choice of t-norms and t-conorms under given conditions.

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