

MATRIX EXTENSION OF SOME CLASSES OF FUNCTIONS

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ABSTRACT. In this paper, the notions of equal convergence (ec), uniform equal convergence ($u.ec$), discrete convergence (dc), and uniform discrete convergence ($u.dc$), which were defined for the sequences of real-valued functions, are generalized with regard to any regular matrix $A = (a_{n,k})$, and their generalized form are studied. The classical and generalized versions of these convergence concepts are compared, and some inclusions are given. Through constructed examples, it is shown that inclusions between them are strict.

Finally, as an application, a more general form of the famous Korovkin's theorem is presented.

1. INTRODUCTION

The concepts of discrete convergence and equal convergence for sequences of real-valued functions were defined and hard-labored by Császár and Laczkovich in [3]. The authors investigated the conditions under which these two definitions are equivalent, examined the properties they provide, and analyzed their relation to the Baire classes. In addition, the properties of discrete Baire class were studied in [4]. Subsequently, in [15], Papanastassiou defined the concepts of uniform equal convergence, uniform discrete convergence, and strong uniform equal convergence for sequences of real-valued functions. Later in [8], Das and Papanastassiou thoroughly studied the properties of classes of uniformly equally convergence functions and uniformly discretely convergence functions. Using equally convergence and uniformly equally convergence, the authors defined α -(ec), α -strong ($u.ec$) and α -($u.ec$), which are stronger than α -convergence. Das and Papanastassiou examined the properties of α -($u.ec$) and α -(ec), obtained a necessary and sufficient condition for a metric space to be compact using α -convergence and α -(uc). In addition, subsequences of α -(uc), α -strong ($u.ec$) and α -(ec) sequences of function were studied in [9]. Subsequently, in [14], Papanastassiou defined the concepts of semi- α convergence and semi-uniform convergence for sequences of functions,

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and studied connections among pointwise convergence, semi- α convergence, semi-uniform convergence, α -convergence, equal convergence and uniform equal convergence for sequences of functions.

Let us recall ideal notion for any set \mathcal{D} . A family $\mathcal{I} \subseteq \mathcal{P}(\mathcal{D})$ is called ideal if the following conditions are satisfied: (i) If $\mathcal{A} \in \mathcal{I}$ and $\mathcal{B} \subseteq \mathcal{A}$, then $\mathcal{B} \in \mathcal{I}$; and (ii) If $\mathcal{A}, \mathcal{B} \in \mathcal{I}$, then $\mathcal{A} \cup \mathcal{B} \in \mathcal{I}$.

In [7], the idea of (ec) for the sequences of functions was extended with the help of the ideal and compared with the concepts of $\mathcal{I}-(ec)$ and $\mathcal{I}-(uc)$. Also, $\mathcal{I}^*-(ec)$ and $\mathcal{I}^*-(u.ec)$ were defined for the sequences of real-valued functions, and Egoroff-type theorem was given for $\mathcal{I}^*-(ec)$. Later, in [6], the concepts of $\mathcal{I}^*-(u.dc)$ and \mathcal{I}^* -strong $(u.ec)$ for the sequences of functions were defined, and the set of these function classes was studied. Similarly, the $\Phi^{\mathcal{I}^*-\alpha-u.e.}$ and $\Phi^{\mathcal{I}^*-\alpha-s.u.e.}$ classes were defined and the properties of these classes were investigated in [1]. Also, in [1], it was indicated that \mathcal{I}^* -exhaustiveness, \mathcal{I}^* -uniform and \mathcal{I}^* - α -convergence of sequences of functions are conserved beneath the condition that uniform conjugacy.

In this study, we classify the equal convergence, uniform equal convergence, discrete convergence, and uniform discrete convergence for the sequences of functions with help of regular matrix.

In the first section, we compare these obtained new classes of functions with classes of equally convergent functions, classes of uniform equally convergent functions, classes of discretely convergent functions and classes of uniform discretely convergent functions.

In the second section, we examine the properties of the new classes of functions.

In the third section, we apply Korovkin-type approximation to the new classes of functions.

Let \mathcal{D} be a non-empty subset of \mathbb{R} , and let Φ be a real-valued arbitrary class of functions defined on \mathcal{D} . Let \mathcal{B} be any set, and denote by $|\mathcal{B}|$ the cardinality of \mathcal{B} .

Definition 1.1 ([8]). A sequence of functions $\tilde{\phi} = (\phi_n)$ in Φ is called:

- (i) equally convergent to a function ϕ (typed with $\phi_n \xrightarrow{e} \phi$) if there exists a sequence $(\varepsilon_n)_{n \in \mathbb{N}} > 0$ with $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ such that for each $t \in \mathcal{D}$, there exists an $n(t) \in \mathbb{N}$ such that

$$|\phi_n(t) - \phi(t)| < \varepsilon_n$$

holds for each $n \geq n(t)$;

- (ii) uniformly equally convergent to a function ϕ (typed with $\phi_n \xrightarrow{u.e} \phi$) if there exists a sequence $(\varepsilon_n)_{n \in \mathbb{N}} > 0$ with $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ and $n_0 \in \mathbb{N}$ such that

$$|\{n \in \mathbb{N} : |\phi_n(t) - \phi(t)| \geq \varepsilon_n\}| \leq n_0$$

for each $t \in \mathcal{D}$;

- (iii) discretely convergent to a function ϕ (typed with $\phi_n \xrightarrow{d} \phi$) if, for every $t \in \mathcal{D}$, there exists $n(t) \in \mathbb{N}$ such that $\phi_n(t) = \phi(t)$ holds for each $n \geq n(t)$;

- (iv) uniformly discretely convergent to a function ϕ (typed with $\phi_n \xrightarrow{u.d} \phi$) if there exists a $n_0 \in \mathbb{N}$ such that

$$|\{n \in \mathbb{N} : |\phi_n(t) - \phi(t)| > 0\}| \leq n_0$$

for each $t \in \mathcal{D}$.

From the definitions, we get the following function classes:

$$\Phi^m := \{\phi : \exists(\phi_n) \in \Phi \text{ such that } \phi_n \xrightarrow{m} \phi\},$$

where $m \in \{e, u.e, d, u.d\}$.

Let $A = (a_{n,k})$ be an infinite matrix with complex (or real) entries. A -transform of a sequence $\tilde{t} = (t_k)$ may be defined as

$$(A_n(\tilde{t})) = \left(\sum_{k=1}^{\infty} a_{n,k} t_k \right),$$

whenever $\sum_{k=1}^{\infty} a_{n,k} t_k < \infty$ holds for all $n \in \mathbb{N}$. The sequence \tilde{t} is called A -summable to a number t_0 if $A_n(\tilde{t}) \rightarrow t_0$ when $n \rightarrow \infty$.

A matrix A is called to be a regular matrix if $\lim_{n \rightarrow \infty} A_n(\tilde{t}) = \lim_{k \rightarrow \infty} t_k$ for every convergent sequence $\tilde{t} = (t_k)$.

Theorem 1.2 (Silverman-Toeplitz Theorem [13]). *A matrix $A = (a_{n,k})$ is regular if and only if*

- (i) $\sup_{n \in \mathbb{N}} \sum_{k=1}^{\infty} |a_{n,k}| < \infty$,
- (ii) $a_{n,k} \rightarrow 0$ ($n \rightarrow \infty, k$ fixed),
- (iii) $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} a_{n,k} = 1$.

It is well known that a regular matrix method converts at least one sequence that is divergent into a sequence that is convergent. For example, the bounded sequence $((-1)^k)$ is not convergent, however its Cesàro transformation converges to zero, that is, $(C, 1)\text{-}\lim_{n \rightarrow \infty} (-1)^n = 0$. This example shows that instead of directly examining the convergence of the sequences, it would be more effective to examine the convergence of their transform sequences under the matrix. The key question of this study is whether this advantage expressed for sequences also holds for sequence of functions.

Let $A = (a_{n,k})$ be a regular matrix, and $\tilde{\phi} = (\phi_k)_{k \in \mathbb{N}}$ be a sequence of functions in Φ . $A = (a_{n,k})$ transformation of $\tilde{\phi} = (\phi_n)_{n \in \mathbb{N}}$ is

$$(A_n(\tilde{\phi}))(t) := \left(\sum_{k=1}^{\infty} a_{n,k} \phi_k(t) \right)_{n \in \mathbb{N}}$$

when the series $\sum_{k=1}^{\infty} a_{n,k} \phi_k(t)$ is convergent for every $n \in \mathbb{N}$.

Definition 1.3. A sequence of functions $\tilde{\phi} = (\phi_k)_{k \in \mathbb{N}}$ in Φ is said to A -(ec) to a function ϕ if there exists a sequence $(\varepsilon_n)_{n \in \mathbb{N}} > 0$ with $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, and there is an $n(t) \in \mathbb{N}$ for every $t \in \mathcal{D}$ such that

$$|(A_n(\tilde{\phi}))(t) - \phi(t)| < \varepsilon_n$$

holds for every $n \geq n(t)$. This is denoted by $\phi_n \xrightarrow{A-ec} \phi$.

Definition 1.4. A sequence of functions $\tilde{\phi} = (\phi_k)_{k \in \mathbb{N}}$ in Φ is said to A -($u.ec$) to a function ϕ if there exists a sequence $(\varepsilon_n)_{n \in \mathbb{N}} > 0$ with $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, and there is a $n_0 \in \mathbb{N}$ such that

$$|\{n \in \mathbb{N} : |(A_n(\tilde{\phi}))(t) - \phi(t)| \geq \varepsilon_n\}| \leq n_0$$

holds for all $t \in \mathcal{D}$. This is denoted by $\phi_n \xrightarrow{A-u.ec} \phi$.

Definition 1.5. A sequence of functions $\tilde{\phi} = (\phi_k)_{k \in \mathbb{N}}$ in Φ is said to A -(dc) to a function ϕ if there exists a $n(t) \in \mathbb{N}$ for all $t \in \mathcal{D}$ such that

$$(A_n(\tilde{\phi}))(t) = \phi(t)$$

holds for each $n \geq n(t)$. This is denoted by $\phi_n \xrightarrow{A-dc} \phi$.

Definition 1.6. A sequence of functions $\tilde{\phi} = (\phi_k)_{k \in \mathbb{N}}$ in Φ is said to A -($u.dc$) to a function ϕ if there exists a $n_0 \in \mathbb{N}$ such that

$$|\{n \in \mathbb{N} : |(A_n(\tilde{\phi}))(t) - \phi(t)| > 0\}| \leq n_0$$

holds for each $t \in \mathcal{D}$. This is denoted by $\phi_n \xrightarrow{A-u.dc} \phi$.

Based on the definitions given above, the function classes $\Phi_A^e c$, $\Phi_A^{u.ec}$, $\Phi_A^d c$ and $\Phi_A^{u.dc}$ are obtained, where

$$\Phi_A^m := \{\phi : \exists (\phi_k) \in \Phi \text{ such that } \phi_k \xrightarrow{A-m} \phi\},$$

where $m \in \{ec, u.ec, dc, u.dc\}$.

Theorem 1.7. Let $A = (a_{n,k})$ be a regular matrix, and let $\tilde{\phi} = (\phi_k)_{k \in \mathbb{N}}$ be a sequence of function in Φ . Then $\phi_k \xrightarrow{m} \phi$ implies that $\phi_k \xrightarrow{A-m} \phi$, where $m \in \{ec, u.ec\}$.

Proof. We are going to demonstrate only the case $m = ec$. In the proof, the Silverman-Toeplitz theorem will play important role. The case $m = u.ec$ can be proved by following almost the same steps given in the first case.

Let sequence of functions (ϕ_k) be equally convergent to ϕ . Then there exists a sequence $(\varepsilon_n^*)_{n \in \mathbb{N}} > 0$, $\varepsilon_n^* \rightarrow 0$ ($n \rightarrow \infty$) and there exists a $n(t) \in \mathbb{N}$ for all $t \in \mathcal{D}$ such that

$$|\phi_k(t) - \phi(t)| < \varepsilon_n^*$$

holds for every $n \geq n(t)$. Now, assume that $\tilde{\phi}$ is not A -equally convergent to ϕ . Then, for all positive (ε_n) with $\varepsilon_n \rightarrow 0$ ($n \rightarrow \infty$), there exists $t_0 \in \mathcal{D}$ such that for all $n \in \mathbb{N}$ there exists $m_n > n$ such that

$$\left| \sum_{k=1}^{\infty} a_{m_n, k} \phi_k(t_0) - \phi(t_0) \right| \geq \varepsilon_{m_n}$$

holds. Without lost of generality, we can suppose that the sequence (m_n) is strictly increasing sequence of natural numbers. Let us consider $\tilde{A} = (a_{m_n, k})$, which is

submatrix of the regular matrix of $A = (a_{n,k})$ obtained by choosing m_n 's rows. Since $A = (a_{n,k})$ is a regular matrix, then $\tilde{A} = (a_{m_n,k})$ is also regular matrix.

Then we have the following inequality:

$$\begin{aligned} \varepsilon_{m_n} &\leq \left| \sum_{k=1}^{\infty} a_{m_n,k} \phi_k(t_0) - \phi(t_0) \right| \\ &= \left| \sum_{k=1}^{\infty} a_{m_n,k} \phi_k(t_0) - \sum_{k=1}^{\infty} a_{m_n,k} \phi(t_0) + \sum_{k=1}^{\infty} a_{m_n,k} \phi(t_0) - \phi(t_0) \right| \\ &\leq \sum_{k=1}^{\infty} |a_{m_n,k}| |(\phi_k(t_0) - \phi(t_0))| + |\phi(t_0)| \sum_{k=1}^{\infty} |a_{m_n,k} - 1| \\ &< \mathcal{M} \varepsilon_n^* + |\phi(t_0)| z_{m_n} \end{aligned}$$

for every $m_n \geq n$. By using regularity of the matrix $\tilde{A} = (a_{m_n,k})$, we can say that there exists $\mathcal{M} > 0$ such that $\sum_{k=1}^{\infty} |a_{m_n,k}| \leq \mathcal{M}$ holds. Also, regularity guaranties that the sequence $(z_{m_n}) = \left(\sum_{k=1}^{\infty} |a_{m_n,k} - 1| \right) \rightarrow 0$ as $n \rightarrow \infty$. Then if we choose $\varepsilon_{m_n} := \mathcal{M} \varepsilon_n^* + |\phi(t_0)| z_{m_n}$, then we meet a contradiction. Therefore, $\phi_n \xrightarrow{A-ec} \phi$. \square

Remark 1.8. The converse of the Theorem 1.7 is not true, in generally.

Example 1.9. Let a sequence of functions $\phi_n : [0, 1] \rightarrow \mathbb{R}$ be defined by $\phi_n(t) = (-1)^n t$ for each $n \in \mathbb{N}$. Let $(\varepsilon_n) > 0$ be any positive reals sequence such that $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, and let $\phi(t) = 0$. So,

$$|\phi_n(t) - \phi(t)| = |(-1)^n t| = t < \varepsilon_n$$

holds for each n when $t = 0$. If $t \neq 0$, then this is not true. Therefore, $\tilde{\phi} = (\phi_n)$ does not equally converge to $\phi = 0$.

Now, consider Cesàro matrix

$$C_1 = (a_{n,k}) := \begin{cases} \frac{1}{n}, & 1 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

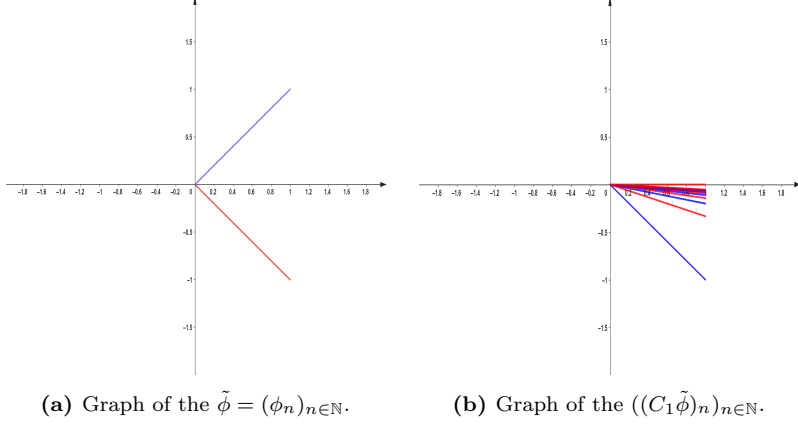
Hence, transformation sequence of functions $\tilde{\phi} = (\phi_n)$

$$(C_1 \tilde{\phi})_n(t) = \begin{cases} \frac{-t}{n}, & n \text{ odd and } 1 \leq k \leq n, \\ 0, & n \text{ even.} \end{cases}$$

Observe the sequence $(\varepsilon_n) = \left(\frac{1}{\sqrt{n}} \right)_{n \in \mathbb{N}}$. Then it is obvious that

$$|(C_1 \tilde{\phi})_n(t) - 0| = \frac{t}{n} < \frac{1}{\sqrt{n}}.$$

That is, it is enough to choose $n(t) := \lceil |t|^2 \rceil + 1$. Therefore, the sequence of functions $\tilde{\phi} = (\phi_n)$ is C_1 -(ec) to the zero function.



Example 1.10. Let $\phi_n: \mathbb{R} \rightarrow \mathbb{R}$ be a sequence of functions defined by

$$\phi_n(t) := \begin{cases} 0, & -\infty < t \leq n-1, \\ (-1)^n(-t+n-1), & n-1 \leq t \leq n, \\ (-1)^{n+1}, & n \leq t < \infty, \end{cases}$$

for every $n \in \mathbb{N}$. For the arbitrary $(\varepsilon_n) > 0$, $\varepsilon_n \rightarrow 0$ ($n \rightarrow \infty$), then for all $t \in \mathbb{R}$, we have

$$|\{n \in \mathbb{N} : |\phi_n(t)| \geq \varepsilon_n\}| \leq n(t),$$

where $n(t) := \lceil |t| \rceil + 1$ and $\sup_t n(t) = \infty$. Therefore, $\tilde{\phi} = \phi_n \xrightarrow{u.ec} 0$.

Now, let a matrix

$$A = (a_{n,k}) := \begin{cases} \frac{1}{2}, & k = n \text{ or } k = n+1, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $A = (a_{n,k})$ is a regular matrix. Then A transformation of $\tilde{\phi}$ is

$$(A_n(\tilde{\phi}))(t) = \begin{cases} \frac{(-1)^n(-t+n-1)}{2}, & t \in [n-1, n], \\ \frac{(-1)^{n+1}(t+n+1)}{2}, & t \in [n, n+1], \\ 0, & \text{otherwise.} \end{cases}$$

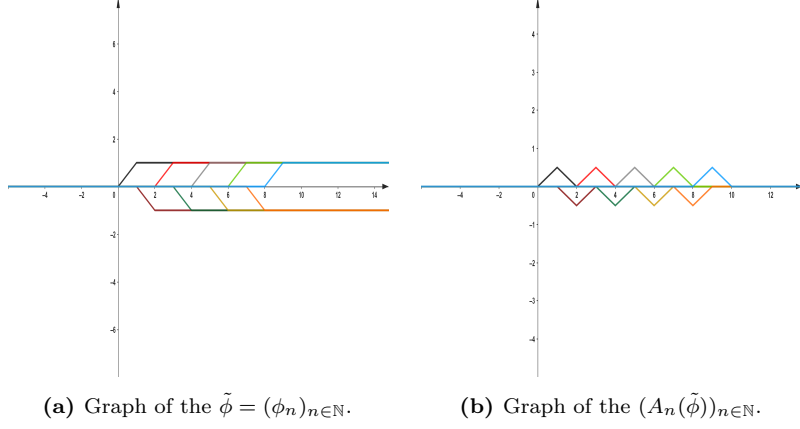
So, for every $(\varepsilon_n) > 0$ with $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ and $t \in \mathbb{R}$, we have

$$|\{n \in \mathbb{N} : |(A_n(\tilde{\phi}))(t)| \geq \varepsilon_n\}| \leq 3.$$

Therefore, (ϕ_n) is A -(*u.ec*) to 0.

Theorem 1.11. Let $A = (a_{n,k})$ be a regular matrix. Then the below inclusions are true:

$$(i) \Phi_A^{u.ec} \subseteq \Phi_A^{ec}, \quad (ii) \Phi_A^{u.dc} \subseteq \Phi_A^{dc}, \quad (iii) \Phi_A^{u.dc} \subseteq \Phi_A^{u.ec}.$$



Proof. (i) Let $\phi \in \Phi_A^{u.ec}$ be a function. Then there exists $(\phi_n) \in \Phi$ such that $\tilde{\phi} = \phi_n \xrightarrow{A-u.ec} \phi$. That is, there exists a sequence $(\varepsilon_n) > 0$, $\varepsilon_n \rightarrow 0$ ($n \rightarrow \infty$) and there is a $n_0 \in \mathbb{N}$ such that

$$(1.1) \quad |\{n \in \mathbb{N} : |(A_n(\tilde{\phi}))(t) - \phi(t)| \geq \varepsilon_n\}| \leq n_0,$$

holds for all $t \in \mathcal{D}$. Therefore, there is a $n_{n_0} \in \mathbb{N}$ which is maximal natural number satisfying (1.1), for all $t \in \mathcal{D}$, such that

$$|(A_n(\tilde{\phi}))(t) - \phi(t)| \leq \varepsilon_n$$

holds for each $n > n_{n_0}$. Hence, this fact implies that $\phi \in \Phi_A^{ec}$.

(ii) Assume that $\phi \in \Phi_A^{u.dc}$. Then there exists $(\phi_n) \in \Phi$ such that $\tilde{\phi} = \phi_n \xrightarrow{A-u.dc} \phi$. That is, there exists a $n_0 \in \mathbb{N}$ such that

$$|\{n \in \mathbb{N} : |(A_n(\tilde{\phi}))(t) - \phi(t)| > 0\}| \leq n_0$$

holds for all $t \in \mathcal{D}$. Therefore, there is a $n_{n_0} \in \mathbb{N}$ for every $t \in \mathcal{D}$ such that

$$(A_n(\tilde{\phi}))(t) = \phi(t)$$

for each $n > n_{n_0}$. Hence, $\phi \in \Phi_A^{ec}$.

(iii) Assume that $\phi \in \Phi_A^{u.dc}$. Then there exists $(\phi_n) \in \Phi$ such that $\tilde{\phi} = \phi_n \xrightarrow{A-u.dc} \phi$.

Namely, there exists a $n_0 \in \mathbb{N}$ such that

$$(1.2) \quad |\{n \in \mathbb{N} : |(A_n(\tilde{\phi}))(t) - \phi(t)| > 0\}| \leq n_0$$

holds for every $t \in \mathcal{D}$. So, the following inclusion

$$\{n \in \mathbb{N} : |(A_n(\tilde{\phi}))(t) - \phi(t)| \geq \varepsilon_n\} \subseteq \{n \in \mathbb{N} : |(A_n(\tilde{\phi}))(t) - \phi(t)| > 0\}$$

holds, then the inequality

$$|\{n \in \mathbb{N} : |(A_n(\tilde{\phi}))(t) - \phi(t)| \geq \varepsilon_n\}| \leq |\{n \in \mathbb{N} : |(A_n(\tilde{\phi}))(t) - \phi(t)| > 0\}|$$

is true. Hence, from (1.5), the following inequality

$$|\{n \in \mathbb{N} : |(A_n(\tilde{\phi}))(t) - \phi(t)| \geq \varepsilon_n\}| \leq n_0$$

holds. Thus, $\phi \in \Phi_A^{u.ec}$. \square

The following examples indicate that the converse inclusions of Theorem 1.11 are not correct, in general.

Example 1.12. Observe a sequence of functions $\phi_n: (0, 1] \rightarrow \mathbb{R}$ defined by

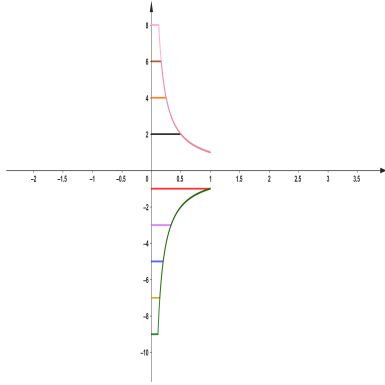
$$\phi_n(t) := \begin{cases} (-1)^n n, & 0 < t \leq \frac{1}{n}, \\ \frac{(-1)^n}{t}, & \frac{1}{n} \leq t \leq 1, \end{cases}$$

for each $n \in \mathbb{N}$, and let $A = (a_{n,k})$ be a regular matrix defined as

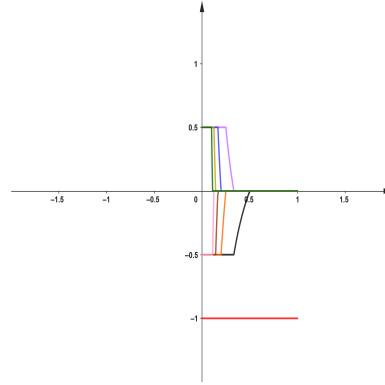
$$A = (a_{n,k}) := \begin{cases} 1, & k = n = 1, \\ \frac{1}{2}, & k = n \neq 1 \text{ or } k = n + 1, \\ 0, & \text{otherwise.} \end{cases}$$

A -transformation of the sequence of functions $\tilde{\phi} = (\phi_n)$ is

$$(A_n(\tilde{\phi}))(t) = \begin{cases} \frac{(-1)^{n+1}}{2}, & 0 < t \leq \frac{1}{n+1}, \\ \frac{(-1)^n}{2} \left(n - \frac{1}{t}\right), & \frac{1}{n+1} \leq t \leq \frac{1}{n}, \\ 0, & \frac{1}{n} \leq t \leq 1. \end{cases}$$



(a) Graph of the $\tilde{\phi} = (\phi_n)_{n \in \mathbb{N}}$.



(b) Graph of the $(A_n(\tilde{\phi}))_{n \in \mathbb{N}}$.

Let $(\varepsilon_n) > 0$, $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. So, there exists $n_0 \in \mathbb{N}$ for $t \in (0, 1]$ such that $\frac{1}{n_0} \leq t \leq \frac{1}{n_0+1}$ is true. Hence, for the arbitrary $(\varepsilon_n) > 0$, $\varepsilon_n \rightarrow 0$ ($n \rightarrow \infty$) then

$$|\{n \in \mathbb{N} : |(A_n(\tilde{\phi}))(t)| \geq \varepsilon_n\}| \leq n_0 + 1,$$

is satisfied. Therefore, $\tilde{\phi} \xrightarrow{A-u.ec} 0$.

However, there is a $n_{n_0} \in \mathbb{N}$ for every $t \in (0, 1]$ such that

$$|(A_n(\tilde{\phi}))(t)| < \varepsilon_n$$

holds for all $n > n_{n_0}$. Therefore, $\tilde{\phi} \xrightarrow{A-ec} 0$, $n \rightarrow \infty$ holds.

Example 1.13. Let a sequence of functions $\phi_n: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $\phi_n(t) = (-1)^{n+1}$ for all $n \in \mathbb{N}$. Consider the Cesàro matrix.

It is clear that

$$(C_1\phi)_n(t) = \begin{cases} \frac{1}{n}, & n \text{ odd,} \\ 0, & n \text{ even.} \end{cases}$$

Let $(\varepsilon_n) = (\frac{2}{n})$. Then

$$|\{n \in \mathbb{N} : |(C_1\phi)_n(t)| \geq \frac{2}{n}\}| = 0$$

holds. Thus, $\phi_n \xrightarrow{A-u.ec} 0$. But the following set

$$|\{n \in \mathbb{N} : |(C_1\phi)_n(t)| > 0\}|$$

is not finite. Therefore, $\phi_n \not\xrightarrow{A-u.dc} 0$.

Example 1.14. Observe the sequence of functions $\phi_n: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\phi_n(t) := \begin{cases} (-1)^n, & t \in (-\infty, n-1], \\ (-1)^n(t-n+2), & t \in [n-1, \infty), \end{cases}$$

for every $n \in \mathbb{N}$. Let us consider

$$A = (a_{n,k}) := \begin{cases} \frac{1}{2}, & k = n \text{ or } k = n+1, \\ 0, & \text{otherwise.} \end{cases}$$

So, the matrix A is regular and A -transformation of $\tilde{\phi} = (\phi_n)$ is

$$(A_n(\tilde{\phi}))(t) = \begin{cases} 0, & t \in (-\infty, n-1], \\ \frac{(-1)^n(t-n+1)}{2}, & t \in [n-1, n], \\ \frac{(-1)^n}{2}, & t \in [n, \infty). \end{cases}$$

Hence,

$$|\{n \in \mathbb{N} : |(A_n(\tilde{\phi}))(t)| > 0\}| \leq n(t),$$

where $n(t) := \lceil |t+1| \rceil$ for $t \in \mathbb{R}$. Therefore, $\phi_n \xrightarrow{A-u.dc} 0$. However, $(A_n(\tilde{\phi}))(t) = 0$ for each $n > n(t)$. Therefore, $\phi_n \not\xrightarrow{A-dc} 0$.

2. ALGEBRAIC PROPERTIES OF A -TYPES CONVERGENCE

In this section, we will analyze some algebraic properties of the sets Φ_A^m , where $m \in \{ec, u.ec, dc, u.dc\}$.

Theorem 2.1. Let $\phi, \psi \in \Phi_A^{ec}$ and $s \in \mathbb{R}$, and let $A = (a_{n,k})$ be a regular matrix. Then the following statements are true:

- (i) $\phi \mp \psi \in \Phi_A^{ec}$ and $s\phi \in \Phi_A^{ec}$, where s is a constant,
- (ii) $|\phi| \in \Phi_A^{ec}$ and $\max\{\phi, \psi\}, \min\{\phi, \psi\} \in \Phi_A^{ec}$.

Proof. Suppose that $\phi, \psi \in \Phi_A^{ec}$. Then there exist $(\phi_n), (\psi_n) \in \Phi$ such that $\tilde{\phi} = \phi_n \xrightarrow{A-ec} \phi$ and $\tilde{\psi} = \psi_n \xrightarrow{A-ec} \psi$ hold, respectively. That is, there exists $(\varepsilon_n^*) > 0$, $\varepsilon_n^* \rightarrow 0$ as $n \rightarrow \infty$, and there is a $n(t) \in \mathbb{N}$ for each $t \in \mathcal{D}$ such that for all $n > n(t)$,

$$(2.1) \quad |(A_n(\tilde{\phi}))(t) - \phi(t)| < \varepsilon_n^*$$

holds, and there exists $(\varepsilon_n^{**}) > 0$, $\varepsilon_n^{**} \rightarrow 0$ as $n \rightarrow \infty$, and there is a $m(t) \in \mathbb{N}$ for all $t \in \mathcal{D}$ such that for all $n > m(t)$,

$$(2.2) \quad |(A_n(\tilde{\psi}))(t) - \psi(t)| < \varepsilon_n^{**}$$

holds.

(i) Choose $r(t) := \max\{n(t), m(t)\}$. Then from (2.1) and (2.2), the following inequality

$$|((A_n(\tilde{\phi} + \tilde{\psi}))(t) - (\phi(t) + \psi(t)))| \leq |(A_n(\tilde{\phi}))(t) - \phi(t)| + |(A_n(\tilde{\psi}))(t) - \psi(t)| < \varepsilon_n$$

holds for all $n > r(t)$, where $(\varepsilon_n) := (\varepsilon_n^* + \varepsilon_n^{**})$. Therefore, $\phi + \psi \in \Phi_A^{ec}$. Similarly, it can be shown that $\phi - \psi \in \Phi_A^{ec}$.

(ii) Let $s \in \mathbb{R}$. From (2.1),

$$|(s(A_n(\tilde{\phi}))(t)) - (s\phi(t))| = |s| |(A_n(\tilde{\phi}))(t) - \phi(t)| < \varepsilon_n$$

for every $n > n(t)$, where $(\varepsilon_n) = (|s|\varepsilon_n^*)$. Hence, $c\phi \in \Phi_A^{ec}$.

(iii) From (2.1),

$$|(A_n(\tilde{\phi}))(t)| - |\phi(t)| \leq |(A_n(\tilde{\phi}))(t) - \phi(t)| < \varepsilon_n^*$$

for all $n > n(t)$. Then $|\phi| \in \Phi_A^{ec}$.

(iv) Recall that $\max\{\phi, \psi\} = \frac{\phi + \psi + |\phi - \psi|}{2}$ and $\min\{\phi, \psi\} = \frac{\phi + \psi - |\phi - \psi|}{2}$.

Let $r(t) = \max\{n(t), m(t)\}$ be chosen. Then from (2.1) and (2.2),

$$\begin{aligned} & \left| \max\{(A_n(\tilde{\phi}, \tilde{\psi}))(t)\} - \max\{\phi(t), \psi(t)\} \right| \\ &= \left| \frac{(A_n(\tilde{\phi}))(t) + (A_n(\tilde{\psi}))(t)}{2} + \frac{|(A_n(\tilde{\phi}))(t) - (A_n(\tilde{\psi}))(t)|}{2} - \frac{\phi(t) + \psi(t) + |\phi(t) - \psi(t)|}{2} \right| \\ &\leq \frac{|(A_n(\tilde{\phi}))(t) - \phi(t)|}{2} + \frac{|(A_n(\tilde{\psi}))(t) - \psi(t)|}{2} + \left| \frac{|(A_n(\tilde{\phi}))(t) - (A_n(\tilde{\psi}))(t)|}{2} - \frac{|\phi(t) - \psi(t)|}{2} \right| \\ &\leq |(A_n(\tilde{\phi}))(t) - \phi(t)| + |(A_n(\tilde{\psi}))(t) - \psi(t)| < \varepsilon_n \end{aligned}$$

for all $n > r(t)$, where $(\varepsilon_n) = (\varepsilon_n^* + \varepsilon_n^{**})$. Therefore, $\max\{\phi, \psi\} \in \Phi_A^{ec}$.

Similarly, we have

$$|\min\{(A_n(\tilde{\phi}, \tilde{\psi}))(t)\} - \min\{\phi(t), \psi(t)\}| < \varepsilon_n$$

for all $n > r(t)$. Then $\min\{\phi, \psi\} \in \Phi_A^{ec}$. \square

Theorem 2.2. Let $A = (a_{n,k})$ be a regular matrix. If $\phi, \psi \in \mathfrak{B}$ and $s \in \mathbb{R}$, then $\phi \mp \psi$, $s\phi$, $|\phi|$, $\max\{\phi, \psi\}$, $\min\{\phi, \psi\} \in \mathfrak{B}$, where $\mathfrak{B} \in \{\Phi_A^{u,ec}, \Phi_A^{dc}, \Phi_A^{u,dc}\}$.

Proof. It is proved by the same steps as in the proof of Theorem 2.1, taking into account the definitions. \square

Remark 2.3. Let $A = (a_{n,k})$ be a regular matrix, and let $\tilde{\phi} = \phi_n \xrightarrow{A-m} \phi$, where $m \in \{ec, u.ec, dc, u.dc\}$. If ϕ is bounded on \mathcal{D} , then $(A_n(\tilde{\phi}))$ is bounded on \mathcal{D} .

Proof. We are going to demonstrate only the case $m = ec$. The other cases can be proved by following the same steps given in the first case.

Suppose that ϕ be bounded on \mathcal{D} . Then there exists $\mathcal{M} > 0$ such that $|\phi(t)| \leq \mathcal{M}$ for every $t \in \mathcal{D}$. Since $\phi_n \xrightarrow{A-ec} \phi$, there exists a sequence $(\varepsilon_n) > 0$, $\varepsilon_n \rightarrow 0$ ($n \rightarrow \infty$) and there exists a $n(t) \in \mathbb{N}$ for all $t \in \mathcal{D}$ such that

$$|(A_n(\tilde{\phi}))(t) - \phi(t)| < \varepsilon_n$$

holds for each $n \geq n(t)$. Since $\varepsilon_n \rightarrow 0$ ($n \rightarrow \infty$), (ε_n) is bounded for each $n \in \mathbb{N}$. That is, there exists $\mathcal{K} > 0$ such that $|\varepsilon_n| \leq \mathcal{K}$ for each $n \in \mathbb{N}$.

Then, for each $n \geq n(t)$,

$$\left| \sum_{k=1}^{\infty} a_{n,k} \phi_k(t) \right| \leq \left| \sum_{k=1}^{\infty} a_{n,k} \phi_k(t) - \phi(t) \right| + |\phi(t)| < \varepsilon_n + |\phi(t)| < \mathcal{K} + \mathcal{M}.$$

On the other hand, for $1 \leq n < n(t)$,

$$\left| \sum_{k=1}^{\infty} a_{n,k} \phi_k(t) \right| \leq \mathcal{L} := \max \left\{ \left| \sum_{k=1}^{\infty} a_{1,k} \phi_k(t) \right|, \dots, \left| \sum_{k=1}^{\infty} a_{n(t)-1,k} \phi_k(t) \right| \right\}.$$

Hence, $(A_n(\tilde{\phi}))$ is bounded on \mathcal{D} by $\max\{\mathcal{L}, \mathcal{K} + \mathcal{M}\}$ for all $n \in \mathbb{N}$ and for all $t \in \mathcal{D}$. \square

Remark 2.4. Let $A = (a_{n,k})$ be a regular matrix, and let $\tilde{\phi} = \phi_n \xrightarrow{A-m} \phi$, where $m \in \{ec, u.ec, dc, u.dc\}$. Assume that $\phi(t) \neq 0$ and $(\phi_n(t)) \neq 0$ for every $t \in \mathcal{D}$, $n \in \mathbb{N}$. If $\frac{1}{\phi}$ is bounded on \mathcal{D} , then $(A_n(\frac{1}{\tilde{\phi}}))$ is bounded on \mathcal{D} .

Remark 2.5. Let $A = (a_{n,k})$ be a regular matrix, and let $\tilde{\phi} = (\phi_n) \in \Phi$. If $(A_n(\tilde{\phi}))$ is bounded on \mathcal{D} , (ϕ_n) is bounded on \mathcal{D} for every n .

Proof. Assume that (ϕ_n) is unbounded on \mathcal{D} but $(A_n(\tilde{\phi}))$ is bounded on \mathcal{D} . Since (ϕ_n) is unbounded on \mathcal{D} , there exists a $n_m \in \mathbb{N}$ and $t_0 \in \mathcal{D}$ for all $m > 0$ such that

$$|\phi_{n_m}(t_0)| > m.$$

Set $m = (\frac{1}{a_{n,k_m}})$ such that $(a_{n,k_m}) \neq 0$ for every $n, m \in \mathbb{N}$. Then

$$\left| \sum_{m=1}^{\infty} a_{n,k_m} \phi_{k_m}(t_0) \right| > \left| \sum_{m=1}^{\infty} a_{n,k_m} \frac{1}{a_{n,k_m}} \right| > \infty.$$

This is a contradiction. Therefore, (ϕ_n) is bounded on \mathcal{D} for every n and for every $t \in \mathcal{D}$. \square

Remark 2.6. Let $A = (a_{n,k})$ be a regular matrix, and let $\tilde{\phi} = (\phi_n) \in \Phi$. Assume that $\phi(t) \neq 0$ and $(\phi_n(t)) \neq 0$ for every $t \in \mathcal{D}$, $n \in \mathbb{N}$. If $(A_n(\frac{1}{\tilde{\phi}}))$ is bounded on \mathcal{D} , $(\frac{1}{\phi_n})$ is bounded on \mathcal{D} for every n .

Theorem 2.7. *Let $\phi, \psi: \mathcal{D} \rightarrow \mathbb{R}$ be functions, let $A = (a_{n,k})$ be a regular matrix, and let $\phi, \psi \in \Phi_A^{ec}$ be arbitrary functions. If the functions ϕ and ψ are bounded on \mathcal{D} , then $\phi \cdot \psi \in \Phi_A^{ec}$.*

Proof. Let $A = (a_{n,k})$ be a regular matrix. Since $\phi, \psi \in \Phi_A^{ec}$, there exists a sequences of functions $(\phi_n), (\psi_n) \in \Phi$ such that $\tilde{\phi} = \phi_n \xrightarrow{A-ec} \phi$ and $\tilde{\psi} = \psi_n \xrightarrow{A-ec} \psi$. Suppose that the functions ϕ and ψ are bounded on \mathcal{D} . Then there exists $\mathcal{M} > 0$ such that $|\phi(t)| \leq \mathcal{M}$ holds for every $t \in \mathcal{D}$. Also, (ψ_n) is bounded on \mathcal{D} from Remark 2.3 and from Remark 2.5. Then there exists $\mathcal{K} > 0$ such that $|\psi_n(t)| \leq \mathcal{K}$ holds for all $n \in \mathbb{N}$ and $t \in \mathcal{D}$.

Since $\phi_n \xrightarrow{A-ec} \phi$ and $\psi_n \xrightarrow{A-ec} \psi$, there exists $(\varepsilon_n^*), (\varepsilon_n^{**}) > 0$, $\varepsilon_n^* \rightarrow 0$ and $\varepsilon_n^{**} \rightarrow 0$ as $n \rightarrow \infty$, and there exists $n(t), m(t) \in \mathbb{N}$ for each $t \in \mathcal{D}$ such that

$$|(A_n(\tilde{\phi}))(t) - \phi(t)| < \frac{\varepsilon_n^*}{\mathcal{K}}$$

holds for all $n \geq n(t)$, and

$$|(A_n(\tilde{\psi}))(t) - \psi(t)| < \frac{\varepsilon_n^{**}}{\mathcal{M}}$$

holds for all $n \geq m(t)$.

Set $b(t) = \max\{n(t), m(t)\}$. So, for every $n \geq b(t)$,

$$\begin{aligned} & \left| \sum_{k=1}^{\infty} a_{n,k} \phi_k(t) \psi_k(t) - \phi(t) \psi(t) \right| \\ &= \left| \sum_{k=1}^{\infty} a_{n,k} \phi_k(t) \psi_k(t) - \phi(t) \sum_{k=1}^{\infty} a_{n,k} \psi_k(t) + \phi(t) \sum_{k=1}^{\infty} a_{n,k} \psi_k(t) - \phi(t) \psi(t) \right| \\ &\leq \left| \sum_{k=1}^{\infty} a_{n,k} \phi_k(t) \psi_k(t) - \phi(t) \sum_{k=1}^{\infty} a_{n,k} \psi_k(t) \right| + |\phi(t)| \left| \sum_{k=1}^{\infty} a_{n,k} \psi_k(t) - \psi(t) \right| \\ &\leq \mathcal{K} \left| \sum_{k=1}^{\infty} a_{n,k} \phi_k(t) - \phi(t) + \phi(t) - \phi(t) \sum_{k=1}^{\infty} a_{n,k} \right| + \mathcal{M} \left| \sum_{k=1}^{\infty} a_{n,k} \psi_k(t) - \psi(t) \right| \\ &\leq \mathcal{K} \left| \sum_{k=1}^{\infty} a_{n,k} \phi_k(t) - \phi(t) \right| + \mathcal{K} \left| \sum_{k=1}^{\infty} a_{n,k} - 1 \right| + \mathcal{M} \left| \sum_{k=1}^{\infty} a_{n,k} \psi_k(t) - \psi(t) \right| \\ &< \varepsilon_n \end{aligned}$$

inequality is obtained, where $(\varepsilon_n) = (\varepsilon_n^* + \mathcal{K}z_n + \varepsilon_n^{**})$ and $(z_n) = (\sum_{k=1}^{\infty} a_{n,k} - 1)$. Therefore, $\phi \cdot \psi \in \Phi_A^{ec}$. \square

Theorem 2.8. *Let $\phi, \psi: \mathcal{D} \rightarrow \mathbb{R}$ be functions. Assume that $\phi, \psi \in \mathfrak{B}$. If the functions ϕ and ψ are bounded on \mathcal{D} , then $\phi \cdot \psi \in \mathfrak{B}$, where $\mathfrak{B} \in \{\Phi_A^{u.ec}, \Phi_A^{dc}, \Phi_A^{u.de}\}$.*

Proof. It is proved by the same steps as the proof of Theorem 2.7, taking into account the definitions. \square

Now, with the following example, we show that one of the functions ϕ and ψ given in the Theorem 2.7 and Theorem 2.8 cannot be thrown out of the boundedness condition.

Example 2.9. Let $\phi_n: (0, \infty) \rightarrow \mathbb{R}$, $\psi_n: [0, 1] \rightarrow \mathbb{R}$, and $\xi_n: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $\tilde{\phi} = (\phi_n) = (\ln(\frac{t(n+1)}{n}))$, $\tilde{\psi} = (\psi_n) = ((-1)^{n+1}t)$, and $\tilde{\xi} = (\xi_n) = (t + (-1)^n)$ for every $n \in \mathbb{N}$. Let

$$A = (a_{n,k}) = \begin{cases} \frac{1}{2}, & k = n \text{ or } k = n + 1, \\ 0, & \text{otherwise.} \end{cases}$$

The multiplications of the sequences of functions $\tilde{\phi}$, $\tilde{\psi}$ and $\tilde{\xi}$ with the A regular matrix are

$$(A_n(\tilde{\phi}))(t) = \left(\frac{1}{2} \ln \left(\frac{t^2(n+2)}{n}\right)\right), \quad (A_n(\tilde{\psi}))(t) = (0) \quad \text{and} \quad (A_n(\tilde{\xi}))(t) = (t).$$

Hence, $\phi_n \xrightarrow{A-u.e} \frac{1}{2} \ln t^2 = \phi(t)$ and $\psi_n \xrightarrow{A-u.ec} 0 = \psi(t)$. Obviously, ϕ is not a bounded function on $(0, \infty)$. Also, $\phi_n \xrightarrow{A-ec} \frac{1}{2} \ln t^2$ and $\psi_n \xrightarrow{A-ec} 0$ from Theorem 1.11(i). Since $(\tilde{\phi}\tilde{\psi})(t) = ((-1)^{n+1}t \ln(\frac{t(n+1)}{n}))$, the multiplication of $(\tilde{\phi}\tilde{\psi})(t)$ with A regular matrix is $(A_n(\tilde{\phi}\tilde{\psi}))(t) = \left(\frac{t}{2} \ln \left(\frac{(n+1)^2}{n(n+2)}\right)\right)$. But $\tilde{\phi}\tilde{\psi} \xrightarrow{A-ec} 0$ because

$$\varepsilon_n = \ln \left(\frac{(n+1)^2}{n(n+2)}\right) \leq \frac{t}{2} \ln \left(\frac{(n+1)^2}{n(n+2)}\right) \leq t \ln \left(\frac{(n+1)^2}{n(n+2)}\right).$$

If we take $t \rightarrow \infty$ in the rightmost expression of this last inequality, $t \ln \left(\frac{(n+1)^2}{n(n+2)}\right) \rightarrow \infty$. That is, as t approaches infinity, $n_0 \in \mathbb{N}$ cannot be determined such that $\left|\frac{t}{2} \ln \left(\frac{(n+1)^2}{n(n+2)}\right)\right| < \varepsilon_n$. Since $\tilde{\phi}\tilde{\psi} \xrightarrow{A-ec} 0$, $\tilde{\phi}\tilde{\psi} \xrightarrow{A-u.ec} 0$.

In addition, $\xi_n \xrightarrow{A-u.dc} t = \xi(t)$ and $\psi_n \xrightarrow{A-u.dc} 0 = \psi(t)$. Obviously, ξ is not a bounded function on \mathbb{R} . Also, $\xi_n \xrightarrow{A-dc} t$ and $\psi_n \xrightarrow{A-dc} 0$ from Theorem 1.11(ii). Since $(\tilde{\xi}\tilde{\psi})(t) = ((-1)^{n+1}t^2 - t)$, the multiplication of $(\tilde{\xi}\tilde{\psi})(t)$ with A regular matrix is $(A_n(\tilde{\xi}\tilde{\psi}))(t) = (-t)$. But $\tilde{\xi}\tilde{\psi} \xrightarrow{A-dc} 0$ because there is no $n \in \mathbb{N}$ such that

$$(A_n(\tilde{\xi}\tilde{\psi}))(t) = -t = 0 = (\xi\psi)(t).$$

Also, from Theorem 1.11(ii), $\tilde{\xi}\tilde{\psi}$ is not A -uniformly discretely convergent to $\xi\psi$.

Theorem 2.10. Let $\phi, \phi_n: \mathcal{D} \rightarrow \mathbb{R}$ be functions, and let $\phi_n \xrightarrow{A-ec} \phi$ be bounded on \mathcal{D} such that $\phi(t) \neq 0$ and $(\phi_n(t)) \neq 0$ for each $t \in \mathcal{D}$ and for all $n \in \mathbb{N}$. If the function $\frac{1}{\phi}$ is bounded on \mathcal{D} , then $\frac{1}{\phi} \in \Phi_A^{ec}$.

Proof. Assume that the function $\frac{1}{\phi}$ is bounded on \mathcal{D} . Then there exists $\mathcal{M} > 0$ such that $\left|\frac{1}{\phi(t)}\right| \leq \mathcal{M}$ for all $t \in \mathcal{D}$. Let $A = (a_{n,k})$ be a regular matrix. Since $\phi \in \Phi_A^{ec}$, there exists $(\phi_n) \in \Phi$ such that $\tilde{\phi} = \phi_n \xrightarrow{A-ec} \phi$. Also, $\left(\frac{1}{\phi_n}\right)$ is bounded on \mathcal{D} from Remark 2.4 and from Remark 2.6. Then there exists $\mathcal{K} > 0$ such that

$\left| \frac{1}{\phi_n(t)} \right| \leq \mathcal{K}$ for every $t \in \mathcal{D}$ and for all $n \in \mathbb{N}$. Since $\phi_n \xrightarrow{A-ec} \phi$, there exists $(\varepsilon_n^*) > 0$, $\varepsilon_n^* \rightarrow 0$ ($n \rightarrow \infty$) and there exists $n(t) \in \mathbb{N}$ for all $t \in \mathcal{D}$ such that

$$|(A_n(\tilde{\phi}))(t) - \phi(t)| < \frac{\varepsilon_n^*}{\mathcal{K}\mathcal{M}}$$

holds for all $n \geq n(t)$. So, for every $n \geq n(t)$,

$$\begin{aligned} & \left| A_n \left(\frac{1}{\tilde{\phi}} \right) (t) - \frac{1}{\phi(t)} \right| \\ &= \left| \sum_{k=1}^{\infty} a_{n,k} \frac{1}{\phi_k(t)} - \frac{1}{\phi(t)} \right| = \left| \sum_{k=1}^{\infty} \frac{a_{n,k} \phi(t) - \phi_k(t)}{\phi_k(t) \phi(t)} \right| \\ &= \left| \sum_{k=1}^{\infty} \frac{\phi(t)[a_{n,k} - 1] + \phi_k(t)[a_{n,k} - 1] + \phi(t) - a_{n,k} \phi_k(t)}{\phi_k(t) \phi(t)} \right| \\ &\leq (\mathcal{K} + \mathcal{M}) \left| \sum_{k=1}^{\infty} a_{n,k} - 1 \right| + \mathcal{K}\mathcal{M} \left| \sum_{k=1}^{\infty} a_{n,k} \phi_k(t) - \phi(t) \right| < \varepsilon_n, \end{aligned}$$

where $(z_n) = (\sum_{k=1}^{\infty} a_{n,k} - 1)$ and $(\varepsilon_n) = ((\mathcal{K} + \mathcal{M})z_n + \varepsilon_n^*)$. Hence, $\frac{1}{\phi} \in \Phi_A^{ec}$. \square

Theorem 2.11. *Let $\phi, \phi_n : \mathcal{D} \rightarrow \mathbb{R}$ be a function, and let $\phi \in \mathfrak{B}$ be bounded on \mathcal{D} such that $\phi(t) \neq 0$ and $(\phi_n(t)) \neq 0$ for each $t \in \mathcal{D}$ and for all $n \in \mathbb{N}$. If the function $\frac{1}{\phi}$ is bounded on \mathcal{D} , then $\frac{1}{\phi} \in \mathfrak{B}$, where $\mathfrak{B} \in \{\Phi_A^{u.ec}, \Phi_A^{dc}, \Phi_A^{u.dc}\}$.*

Proof. It is proved by the same steps as in the proof of Theorem 2.10, taking the definitions into account. \square

In the following examples, we show that the contrary of the Theorem 2.10 and Theorem 2.11 is not correct.

Example 2.12. Observe $\phi_n : (0, 1) \rightarrow \mathbb{R}$, defined by $\tilde{\phi} = (\phi_n(t)) = (t + \frac{(-1)^n}{n})$ for every $n \in \mathbb{N}$. Let

$$A = (a_{n,k}) := \begin{cases} -1, & k = n, \\ 2, & k = 2n, \\ 0, & \text{otherwise.} \end{cases}$$

The multiplication of sequences of function (ϕ_n) with the A regular matrix is

$$(A_n(\tilde{\phi}))(t) = \begin{cases} t + \frac{2}{n}, & n \text{ odd,} \\ t, & n \text{ even.} \end{cases}$$

Obviously, $\phi_n \xrightarrow{A-u.ec} t = \phi$, ϕ is bounded on $(0, 1)$ and $\phi(t) \neq 0$ for every $t \in (0, 1)$. However, $\frac{1}{\phi}$ is not bounded on $(0, 1)$. Additively, $\phi_n \xrightarrow{A-ec} \phi$ from Theorem 1.11. The multiplication of sequences of function $(\frac{1}{\phi_n})$ with the A regular matrix is

$$\left(A_n \left(\frac{1}{\tilde{\phi}} \right) \right)_n (t) = \begin{cases} \frac{t - \frac{5}{2n}}{(t - \frac{1}{n})(t + \frac{1}{2n})}, & n \text{ odd,} \\ \frac{t + \frac{3}{2n}}{(t + \frac{1}{n})(t + \frac{1}{2n})}, & n \text{ even.} \end{cases}$$

If n is odd, then

$$\varepsilon_n^* = \frac{|4n-1|}{2n^2} \leq \left| \frac{t - \frac{5}{2n}}{\left(t - \frac{1}{n}\right)\left(t + \frac{1}{2n}\right)} - \frac{1}{t} \right| \leq \left| \frac{-4tn+1}{t(nt-1)} \right| = b_n(t).$$

If n is even, then

$$\varepsilon_n^{**} = \frac{1}{2n^2 + 3n + 1} \leq \left| \frac{t + \frac{3}{2n}}{\left(t + \frac{1}{n}\right)\left(t + \frac{1}{2n}\right)} - \frac{1}{t} \right| \leq \frac{1}{t(nt+1)} = s_n(t).$$

If we take $t \rightarrow 0$ in the rightmost expression of this inequalities, $s_n, b_n \rightarrow \infty$. That is, as t approaches zero, $n_0 \in \mathbb{N}$ cannot be determined such that

$$\left| \left(A_n \left(\frac{1}{\tilde{\phi}} \right) \right)_n (t) - \frac{1}{\phi(t)} \right| < \varepsilon_n.$$

Therefore, $\frac{1}{\phi_n} \xrightarrow{A-ec} \frac{1}{\phi}$. So, $\frac{1}{\phi_n} \xrightarrow{A-u,ec} \frac{1}{\phi}$.

Example 2.13. Let $\phi_n: (0, 1) \rightarrow \mathbb{R}$ be defined by $(\phi_n(t)) = \tilde{\phi} = (t + (-1)^n)$ for every $n \in \mathbb{N}$.

Let

$$A = (a_{n,k}) := \begin{cases} \frac{1}{2}, & k = n \text{ or } k = n + 1, \\ 0, & \text{otherwise.} \end{cases}$$

The multiplication of sequences of function (ϕ_n) with the A regular matrix is

$$(A_n(\tilde{\phi}))(t) = (t).$$

Obviously, $\phi_n \xrightarrow{A-u,dc} t = \phi$, ϕ is bounded on $(0, 1)$ and $\phi(t) \neq 0$ for every $t \in (0, 1)$.

Additively, from Theorem 1.11(ii), $\phi_n \xrightarrow{A-dc} \phi$. However, $\frac{1}{\phi}$ is not bounded on $(0, 1)$.

A transformation of $\left(\frac{1}{\phi_n}\right)$ is

$$\left(A_n \left(\frac{1}{\tilde{\phi}} \right) \right)_n (t) = \left(\frac{t}{(t+1)(t-1)} \right).$$

Assume that $\left(\frac{1}{\phi_n}\right)$ is A -discretely convergent to $\frac{1}{\phi}$. Then there is a $n(t) \in \mathbb{N}$ such that

$$\left(A_n \left(\frac{1}{\tilde{\phi}} \right) \right)_n (t) = \frac{1}{\phi(t)}$$

holds for every $n \geq n(t)$. Given the above equation,

$$\begin{aligned} \frac{t}{(t-1)(t+1)} &= \frac{1}{t}, \\ t^2 &= t^2 - 1, \\ 0 &= -1. \end{aligned}$$

It is contradiction. Therefore, $\frac{1}{\phi_n} \xrightarrow{A-dc} \frac{1}{\phi}$. From Theorem 1.11(ii), $\frac{1}{\phi_n} \xrightarrow{A-u,dc} \frac{1}{\phi}$.

3. KOROVKIN TYPE THEOREMS

Let (L_n) be a sequence of positive linear operators defined on

$$C(\mathcal{K}) := \{\phi \mid \phi: \mathcal{K} \rightarrow \mathbb{R} \text{ continuous}\},$$

where \mathcal{K} is a compact subset of the set of real numbers.

Korovkin [12] defined the necessary properties for the uniform convergence of $L_n(\phi)$ to a function ϕ by utilising $1, t, t^2$ test functions. Furthermore, there have been many studies of the Korovkin type approximation theorem, as in [2, 5, 10, 11, 16] and [17].

In this section, we will prove that Korovkin-type approximation theorems hold for the sequence of functions for A -equal convergence, A -uniform equal convergence, A -discrete convergence and A -uniform discrete convergence in $C(\mathcal{K})$.

Theorem 3.1. *Let (L_n) be a sequence of positive linear operators from $C(\mathcal{K})$ to $C(\mathcal{K})$, and let $A = (a_{n,k})$ be a regular matrix. Then $L_n(\phi) \xrightarrow{A-m} \phi$ holds for every $\phi \in C(\mathcal{K})$ if and only if $L_n(e_i) \xrightarrow{A-m} e_i$ for $i = 0, 1, 2$, where $e_i(t) = t^i$ and $m \in \{ec, u.ec, dc, u.dc\}$.*

Proof. We are going to bestow upon only the proof the case $m = ec$. Other cases of m can be proved by following the same steps with small differences given in $m = ec$.

Assume that $L_n(\phi) \xrightarrow{A-ec} \phi$ for every $\phi \in C(\mathcal{K})$. Then it is provided in $\phi(t) = e_0(t), \phi(t) = e_1(t), \phi(t) = e_2(t)$. Therefore, $L_n(e_i) \xrightarrow{A-ec} e_i$ for $i = 0, 1, 2$, where $e_i(t) = t^i$.

For the converse, assume that $L_n(e_i) \xrightarrow{A-ec} e_i$ for $i = 0, 1, 2$, where $e_i(t) = t^i$. Let $A = (a_{n,k})$ be a regular matrix. Then there exists $(\epsilon_n^i) > 0, \epsilon_n^i \rightarrow 0$ ($n \rightarrow \infty$) and there is a $n(t) \in \mathbb{N}$ for every $t \in \mathcal{K}$ such that

$$\left| \sum_{k=1}^{\infty} a_{n,k} L_k(e_i; t) - e_i(t) \right| < \epsilon_n^i, \quad \text{for } i = 0, 1, 2$$

holds for each $n \geq n(t)$.

Since $\phi \in C(\mathcal{K})$, there exists $\mathcal{M} > 0$ such that

$$|\phi(t)| \leq \mathcal{M}$$

for each $t \in \mathcal{K}$. So,

$$(3.1) \quad |\phi(t) - \phi(t_0)| \leq 2\mathcal{M}$$

for each $t, t_0 \in \mathcal{K}$. Also, since $\phi \in C(\mathcal{K})$ for every $\epsilon > 0$, there is a $\delta(\epsilon, t_0) > 0$ such that $|t_0 - t| < \delta$ implies

$$(3.2) \quad |\phi(t_0) - \phi(t)| < \epsilon$$

holds for every $t \in \mathcal{K}$. Set $\psi = \psi(t_0) = (t - t_0)^2$ and from (3.1) and (3.2)

$$-\epsilon - \frac{2\mathcal{M}}{\delta^2}\psi < \phi(t_0) - \phi(t) < \epsilon + \frac{2\mathcal{M}}{\delta^2}\psi$$

inequality is satisfied. Since $L_n(\phi; t)$ is positive and linear,

$$\begin{aligned} - \sum_{k=1}^{\infty} a_{n,k} L_k(e_0; t) \left(\varepsilon + \frac{2\mathcal{M}}{\delta^2} \psi \right) &< \sum_{k=1}^{\infty} a_{n,k} L_k(e_0; t) (\phi(t_0) - \phi(t)) \\ &< \sum_{k=1}^{\infty} a_{n,k} L_k(e_0; t) \left(\varepsilon + \frac{2\mathcal{M}}{\delta^2} \psi \right) \end{aligned}$$

holds. Consider t is fixed, then $\phi(t)$ is a constant. Hence,

$$\begin{aligned} (3.3) \quad & - \varepsilon \sum_{k=1}^{\infty} a_{n,k} L_k(e_0; t) - \frac{2\mathcal{M}}{\delta^2} \sum_{k=1}^{\infty} a_{n,k} L_k(\psi(t_0); t) \\ & < \sum_{k=1}^{\infty} a_{n,k} L_k(\phi; t) - \phi(t) \sum_{k=1}^{\infty} a_{n,k} L_k(e_0; t) \\ & < \varepsilon \sum_{k=1}^{\infty} a_{n,k} L_k(e_0; t) + \frac{2\mathcal{M}}{\delta^2} \sum_{k=1}^{\infty} a_{n,k} L_k(\psi(t_0); t). \end{aligned}$$

From (3.3), we have

$$\begin{aligned} (3.4) \quad & \sum_{k=1}^{\infty} a_{n,k} L_k(\phi; t) - \phi(t) = \sum_{k=1}^{\infty} a_{n,k} L_k(\phi; t) - \phi(t) \sum_{k=1}^{\infty} a_{n,k} L_k(e_0; t) \\ & \quad + \phi(t) \left[\sum_{k=1}^{\infty} a_{n,k} L_k(e_0; t) - 1 \right] \\ & < \varepsilon + \varepsilon \left[\sum_{k=1}^{\infty} a_{n,k} L_k(e_0; t) - e_0(t) \right] \\ & \quad + \frac{2\mathcal{M}}{\delta^2} \sum_{k=1}^{\infty} a_{n,k} L_k(\psi(t_0); t) \\ & \quad + \phi(t) \left[\sum_{k=1}^{\infty} a_{n,k} L_k(e_0; t) - e_0(t) \right]. \end{aligned}$$

Let us calculate $\sum_{k=1}^{\infty} a_{n,k} L_k(\psi(t_0); t)$,

$$\begin{aligned} \sum_{k=1}^{\infty} a_{n,k} L_k(\psi(t_0); t) &= \sum_{k=1}^{\infty} a_{n,k} L_k((t_0 - t)^2; t) \\ &= \left[\sum_{k=1}^{\infty} a_{n,k} L_k(e_2; t) - e_2(t) \right] - 2t \left[\sum_{k=1}^{\infty} a_{n,k} L_k(e_1; t) - e_1(t) \right] \\ & \quad + t^2 \left[\sum_{k=1}^{\infty} a_{n,k} L_k(e_0; t) - e_0(t) \right]. \end{aligned}$$

Then, with (3.4),

$$\begin{aligned} \sum_{k=1}^{\infty} a_{n,k} L_k(\phi; t) - \phi(t) &< \varepsilon + \left(\varepsilon + \phi(t) + \frac{2\mathcal{M}}{\delta^2} t^2 \right) \left[\sum_{k=1}^{\infty} a_{n,k} L_k(e_0; t) - e_0(t) \right] \\ &\quad - \frac{4\mathcal{M}}{\delta^2} t \left[\sum_{k=1}^{\infty} a_{n,k} L_k(e_1; t) - e_1(t) \right] \\ &\quad + \frac{2\mathcal{M}}{\delta^2} \left[\sum_{k=1}^{\infty} a_{n,k} L_k(e_2; t) - e_2(t) \right]. \end{aligned}$$

Since ε is arbitrary positive real number, we write

$$\begin{aligned} \sum_{k=1}^{\infty} a_{n,k} L_k(\phi; t) - \phi(t) &< \left(\phi(t) + \frac{2\mathcal{M}}{\delta^2} t^2 \right) \left[\sum_{k=1}^{\infty} a_{n,k} L_k(e_0; t) - e_0(t) \right] \\ (3.5) \quad &\quad - \frac{4\mathcal{M}}{\delta^2} t \left[\sum_{k=1}^{\infty} a_{n,k} L_k(e_1; t) - e_1(t) \right] \\ &\quad + \frac{2\mathcal{M}}{\delta^2} \left[\sum_{k=1}^{\infty} a_{n,k} L_k(e_2; t) - e_2(t) \right]. \end{aligned}$$

By using (3.5)

$$\begin{aligned} \left| \sum_{k=1}^{\infty} a_{n,k} L_k(\phi; t) - \phi(t) \right| &< \left| \left(\phi(t) + \frac{2\mathcal{M}}{\delta^2} t^2 \right) \sum_{k=1}^{\infty} a_{n,k} L_k(e_0; t) - e_0(t) \right| \\ &\quad + \left| \left(\frac{-4\mathcal{M}}{\delta^2} t \right) \sum_{k=1}^{\infty} a_{n,k} L_k(e_1; t) - e_1(t) \right| \\ &\quad + \left| \left(\frac{2\mathcal{M}}{\delta^2} \right) \sum_{k=1}^{\infty} a_{n,k} L_k(e_2; t) - e_2(t) \right| \\ &< \left(\mathcal{M} + \frac{2\mathcal{M}}{\delta^2} t^2 \right) \left| \sum_{k=1}^{\infty} a_{n,k} L_k(e_0; t) - e_0(t) \right| \\ &\quad + \left(\frac{4\mathcal{M}}{\delta^2} |t| \right) \left| \sum_{k=1}^{\infty} a_{n,k} L_k(e_1; t) - e_1(t) \right| \\ &\quad + \left(\frac{2\mathcal{M}}{\delta^2} \right) \left| \sum_{k=1}^{\infty} a_{n,k} L_k(e_2; t) - e_2(t) \right| < \varepsilon_n, \end{aligned}$$

where $(\varepsilon_n) = \left(\left(\mathcal{M} + \frac{2\mathcal{M}}{\delta^2} t^2 \right) \varepsilon_n^0 + \left(\frac{4\mathcal{M}}{\delta^2} |t| \right) \varepsilon_n^1 + \left(\frac{2\mathcal{M}}{\delta^2} \right) \varepsilon_n^2 \right)$. Therefore,

$$L_n(\phi) \xrightarrow{A-\varepsilon\zeta} \phi. \quad \square$$

Example 3.2. Let the sequence of functions $\phi_n: [0, 1] \rightarrow \mathbb{R}$ be defined by $\phi_n(t) = (-1)^{n+1}$. Considering the Szász polynomials on $C([0, 1])$,

$$S_n(\phi; t) := e^{-nt} \sum_{k=0}^{\infty} \phi\left(\frac{k}{n}\right) \frac{(nt)^k}{k!},$$

and define the positive linear operators by

$$L_n(\phi; t) := (1 + \phi_n(t))S_n(\phi; t),$$

where $e_i(t) = t^i$, $i = 0, 1, 2$ and $t \in [0, 1]$. Then, by simple calculation, we obtain

$$L_n(e_0; t) = \begin{cases} 2, & n \text{ odd,} \\ 0, & n \text{ even,} \end{cases} \quad L_n(e_1; t) = \begin{cases} 2t, & n \text{ odd,} \\ 0, & n \text{ even,} \end{cases}$$

and

$$L_n(e_2; t) = \begin{cases} 2\left(t^2 + \frac{t}{n}\right), & n \text{ odd,} \\ 0, & n \text{ even.} \end{cases}$$

It is clear that $L_n(e_i; t)$ is not equally convergent to $e_i(t)$ for $i = 0, 1, 2$.

Let

$$A = a_{n,k} := \begin{cases} \frac{1}{2}, & k = n \text{ or } k = n + 1, \\ 0, & \text{otherwise.} \end{cases}$$

A -transformations of $L_n(e_i; t)$ for $i = 0, 1, 2$ are

$$\begin{aligned} (AL)_n(e_0; t) &= 1, \\ (AL)_n(e_1; t) &= t, \\ (AL)_n(e_2; t) &= \begin{cases} t^2 + \frac{t}{n}, & n \text{ odd,} \\ t^2 + \frac{t}{n+1}, & n \text{ even.} \end{cases} \end{aligned}$$

Then $(\varepsilon_n) > 0$ with $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ such that

$$\begin{aligned} |(AL)_n(e_0; t) - e_0(t)| &= |1 - 1| = 0 < \varepsilon_n, \\ |(AL)_n(e_1; t) - e_1(t)| &= |t - t| = 0 < \varepsilon_n, \end{aligned}$$

hold for every $n \in \mathbb{N}$. Hence, $L_n(e_0; t)$ is A -equally convergent to $e_0(t)$ and $L_n(e_1; t)$ is A -equally convergent to $e_1(t)$.

Also, if n is odd, then

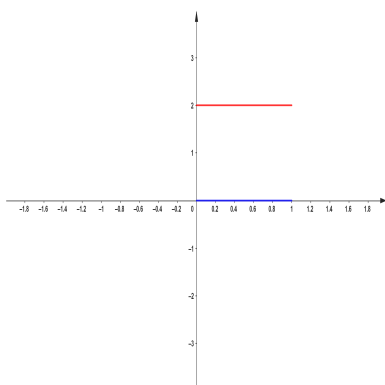
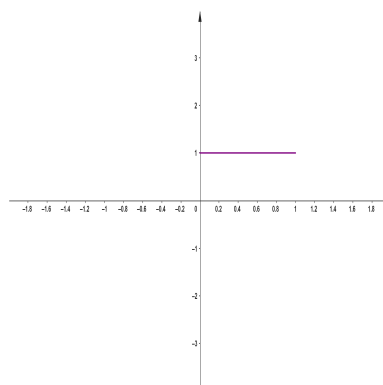
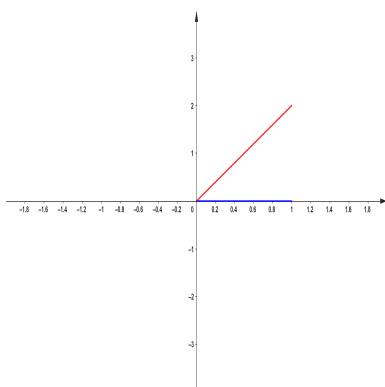
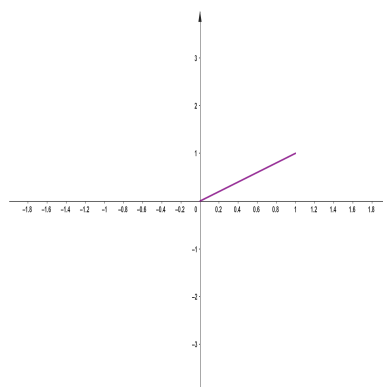
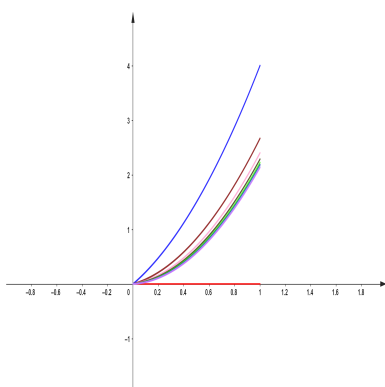
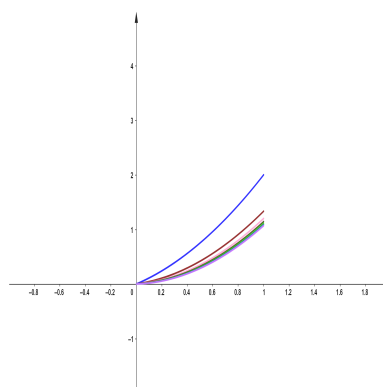
$$|(AL)_n(e_2; t) - e_2(t)| = \left|t^2 + \frac{t}{n} - t^2\right| = \frac{t}{n} < \frac{1}{\sqrt{n}},$$

holds. It is enough to choose $n(t) = \lceil |t^2| \rceil$.

If n is even, then

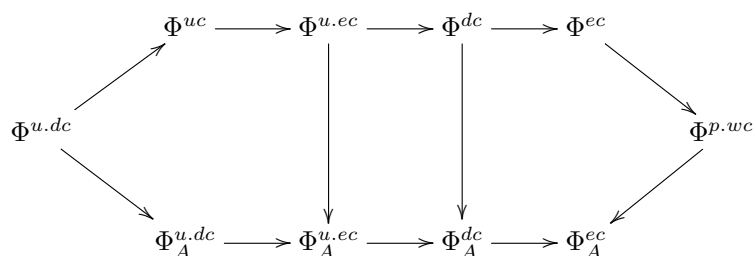
$$|(AL)_n(e_2; t) - e_2(t)| = \left|t^2 + \frac{t}{n+1} - t^2\right| = \frac{t}{n+1} < \frac{1}{n},$$

holds for every $n \in \mathbb{N}$. Therefore, $L_n(e_2; t)$ is A -equally convergent to $e_2(t)$.

(a) Graph of the $(L_n(e_0; t))_{n \in \mathbb{N}}$.(b) Graph of the $((AL)_n(e_0; t))_{n \in \mathbb{N}}$.(a) Graph of the $(L_n(e_1; t))_{n \in \mathbb{N}}$.(b) Graph of the $((AL)_n(e_1; t))_{n \in \mathbb{N}}$.(a) Graph of the $(L_n(e_2; t))_{n \in \mathbb{N}}$.(b) Graph of the $((AL)_n(e_2; t))_{n \in \mathbb{N}}$.

Conclusion 3.3. In this study, we expanded the concepts of equally convergence, uniformly equally convergence, discretely convergence and uniformly discretely convergence given in study [8]. We have obtained the diagram given below.

(Note that $F \rightarrow G$ means that $F \subseteq G$, and $\Phi^{p.wc}$ and Φ^{uc} are classes of pointwise convergence and uniformly convergence, respectively):



Open Problem 3.4. Let \mathcal{D} and \mathcal{C} be arbitrary sets, and let $\phi, \phi_n : \mathcal{D} \rightarrow \mathcal{C}$ be functions for all $n \in \mathbb{N}$. The sequence of functions $(\phi_n)_{n \in \mathbb{N}}$ α -converges to ϕ if and only if for each $t \in \mathcal{D}$ and for all sequence $(t_n)_{n \in \mathbb{N}}$ of points of \mathcal{D} converging to t , the sequence $(\phi_n(t_n))_{n \in \mathbb{N}}$ converges to $\phi(t)$. It is written as $\phi_n \xrightarrow{\alpha} \phi$ and it is known as continuous convergence.

The sequence $(\phi_n)_{n \in \mathbb{N}}$ strongly uniformly equally converges to ϕ if there exists a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ of positive reals with $\sum_{k=1}^{\infty} \varepsilon_k < \infty$ and $n_0 \in \mathbb{N}$ such that cardinality of the set

$$\{n \in \mathbb{N} : |\phi_n(t) - \phi(t)| \geq \varepsilon_n\}$$

is at most n_0 for each $t \in \mathcal{D}$. It is written as $\phi_n \xrightarrow{s.u.\varepsilon} \phi$.

In [8], α -(*u.ec*), α -(*ec*) and α -strong (*ec*) are defined by expanding α -(*c*).

Similar to the definitions we have given, an *A* regular matrix transformation of the concepts of strongly uniformly convergence, α -(*c*), α -strongly (*uc*), α -(*u.ec*) and α -(*ec*) can be given.

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