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ALTERNATING DIRECTION EXPLICIT METHODS FOR CONVECTION DIFFUSION EQUATIONS

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Abstract. In this work, we investigate the stability and consistency properties of *alternating direction explicit* (ADE) finite difference schemes applied to convection-diffusion-reaction equations. Employing different discretization strategies of the convection term, we obtain various ADE schemes and study their stability and consistency properties. An ADE scheme consists of two sub steps (called upward and downward sweeps) where already computed values at the new time level are used in the discretization stencil. For linear convection-diffusion-reaction equations, the consistency of the single sweeps is of order $\mathrm{O}(k^2+h^2+k/h)$, but the average of these two sweeps has a consistency of order $\mathrm{O}(k^2+h^2)$, where k, h denote the step size in time and space.

1. Introduction

In this work, we consider the alternating direction explicit (ADE) method for the solution of convection diffusion reaction equations. This scheme strongly uses boundary data in the solution algorithm and hence, it is very sensible to incorrect treatment of boundary conditions. The ADE scheme consists of two steps (upward and downward sweeps).

The ADE method for linear partial differential equations (PDEs) is an unconditionally stable explicit scheme of second order and thus can compete with the Crank-Nicolson scheme, the alternating direction implicit (ADI) and the locally one-dimensional (LOD) splitting methods. The ADI methods and splitting methods are examples of the Multiplicative Operator Scheme (MOS), which is rather difficult to parallelise. In contrast, ADE methods belong to the group of Additive Operator Scheme (AOS) which is easier to parallelise.

The structure of this work is as follows: In Section 2, we present the considered PDEs and explain the basic idea of the ADE scheme and its modified difference

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quotients. Next, the numerical analysis studying stability and consistency of the method is presented in Sections 3 and 4, respectively.

2. Solving PDEs with the ADE method

We start considering the partial differential equation (PDE)

$$v_t = a\, v_{xx} + b\, v_x - c\, v, \qquad t \geq 0 \text{ and for all } x \in \mathbb{R}, \tag{1}$$

with the constant coefficients $a \geq$ const. > 0, $b \geq 0$, $c \geq 0$ and supplied with smooth initial data. We denote the analytical classical solution of (1) by $v := v(x,t)$ and use subscripts to abbreviate partial differentiation, e.g., $v_{xx} := \partial^2 v / \partial x^2$.

Secondly, we consider the classical linear Black-Scholes (BS) equation

$$v_t = \frac{1}{2}\sigma^2 S^2 v_{SS} + rSv_S - rv, \qquad t \geq 0 \text{ and for all } S \in \mathbb{R}, \tag{2}$$

which is a generalization of the PDE (1) to space dependent coefficients. In computational finance, a solution $v(S,t)$ of the PDE (2) represents a European option price. A European option is a contract between the holder of the option and the future buyer, that at a time instance T, the expiration time, the underlying asset (stock) can be sold or bought (call or put option) for a fixed strike price K. Using the BS formula, the option price is calculated for the corresponding underlying asset price S (stock price) in a time interval $t \in (0,T)$.

Let us note that the BS equation (2) is derived under quite restrictive market assumptions which are not very realistic. Relaxing these assumptions leads to new models (e.g., including transaction costs, illiquidity on the market) that are strongly nonlinear BS equations that can only be solved analytically in very simple cases.

While there exist analytical tools to solve explicitly (1) and (2), the interest in studying the ADE method for these simple 1D cases is the fact that we want to extend this approach in a subsequent work to nonlinear PDEs and to higher dimensions. Applying the ADE to the nonlinear BS equations, we need to solve only a scalar nonlinear equation (instead of a nonlinear system of equations for a standard implicit method). Thus, the computational effort using ADE instead of an implicit scheme is highly reduced. Also, for higher space dimensions, the number of ADE sweeps does not increase, it remains two. These facts make the ADE methods an attractive candidate to study in more detailed way.

2.1. The Idea of the ADE scheme

The ADE scheme consists of two explicit sub steps called sweeps. A sweeping step is constructed from one boundary to another and vice versa. Figure 1 is an illustrative example of an upward sweep (analogous to the downward sweep in Figure 2).

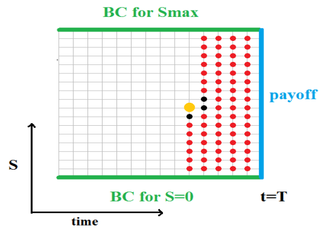

Figure 1. Upward sweep.

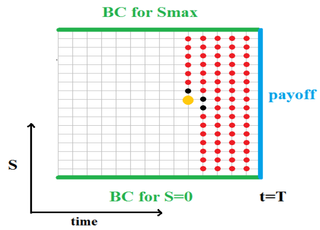

Figure 2. Downward sweep.

Figures 1 and 2 display grid for calculation price of call option in the Black-Scholes model. Blue line represents payoff as an initial condition and the green lines are given by Dirichlet boundary conditions for small and big asset values. Calculation is provided backward in time.

To calculate the value of the yellow point, we use the black values. We can see that we do not use only values from the previous time level but also already known values from the current time level, which preserve explicitness of the scheme. After each time,we combine the solutions from the upward and downward sweep by averaging.

To introduce the ADE method systematically, we follow the lines of of Leung & Osher [**3**], and Duffy [**1**]. The computational spatial interval $(x_{\min}, x_{\max})$, or $(0, S_{\max})$, respectively, is divided into J subintervals, i.e., the space step is $h = (x_{\max} - x_{\min})/J$ and the grid points $x_j = jh$, or $h = S_{\max}/J$, $S_j = jh$, respectively. Thus for the coefficients of the BS equation (2), we get $a(S_j) = \frac{1}{2}\sigma^2(jh)^2$, $b(S_j) = rjh$, $c(S_j) = r$.

We consider the resulting spatial semidiscretization to the PDE (1), i.e., the following system of ODEs

$$v' = A(v)v, \qquad t > 0, \tag{3}$$

with $v(t) \in \mathbb{R}^{J-1}$. For simplicity, let us consider a uniform grid; the time interval $[0, T]$ divided uniformly into N sub-intervals with the step size $k = T/N$, i.e., we have the grid points $t_n = nk$. Applying the trapezoidal rule to (3) leads to the Crank-Nicolson scheme

$$v^{n+1} = \big[I - kA(v^n)\big]^{-1}\big[I + kA(v^n)\big]v^n, \tag{4}$$

where $v^n \approx v(t_n)$. While this classical scheme (4) is unconditionally stable and of second order in time and space, it becomes computationally expensive to invert the operator $I - kA(v^n)$, especially, in higher space dimensions. In order to obtain an efficient scheme while keeping the other desirable properties, this operator is split additively by the matrix decomposition $A = L + D + U$, where L is lower diagonal, D is diagonal and U denotes an upper-diagonal matrix. Next, following the notation of [**3**], we further define the *symmetric splitting*

$$B = L + \frac{1}{2}D, \qquad C = U + \frac{1}{2}D. \tag{5}$$

Then we can formulate the three steps of the ADE scheme with its upward/downward sweeps and the combination (also for higher dimensions) as

$$\text{(6)} \qquad \text{UP} \qquad u^{n+1} = \big[I - kB(v^n)\big]^{-1}\big[I + kC(v^n)\big]v^n,$$

$$\text{(7)} \qquad \text{DOWN} \qquad d^{n+1} = \big[I - kC(v^n)\big]^{-1}\big[I + kB(v^n)\big]v^n,$$

$$\text{(8)} \qquad \text{COMB} \qquad v^{n+1} = \frac{1}{2}\big[u^{n+1} + d^{n+1}\big].$$

In other words, in the two sweeps above, we assign solution values that are already computed at the new time level to the operator to be inverted. Hence, the resulting scheme is *explicit*, i.e., efficient. It remains the questions if we could preserve the unconditional stability and second order accuracy. This will be our main topic in the sequel.

Let us summarize the procedure for one space dimension. The approximation to the solution $v(x,t)$ at the grid point (x_j,t_n) is $c(x_j,t_n) =: c_j^n$ given as an average of upward sweep u_j^n and downward sweep d_j^n. This combination c_j^n contains the initial data at the beginning. For $n = 0,1,\dots,N-1$, we repeat the following steps:

1. Initialization: $u_j^n = c_j^n, \quad d_j^n = c_j^n, \qquad j = 1,\dots,J-1$
2. **U**pward sweep: $u_j^{n+1}, \qquad j = 1,\dots,J-1$
3. **D**ownward sweep: $d_j^{n+1}, \qquad j = J-1,\dots,1$
4. **C**ombination: $\quad c^{n+1} = (u^{n+1} + d^{n+1})/2$

Using different approximation strategies for the convection, diffusion and reaction terms, we obtain different variations of the ADE schemes, which were proposed by Saul'ev [**4**].

2.2. The modified difference quotients for the ADE method

In this subsection, we want to illustrate the outcome of the previous Section 2.1. Thus, we select some spatial discretization and investigate which ADE scheme will result.

For the discretization of the *diffusion term*, we use, cf. [**4**],

$$\text{(9)} \qquad \begin{aligned} \frac{\partial^2 v(x_j,t_n)}{\partial x^2} &\approx \frac{u_{j+1}^n - u_j^n - u_j^{n+1} + u_{j-1}^{n+1}}{h^2}, \qquad && j = 1,\dots,J-1 \\ \frac{\partial^2 v(x_j,t_n)}{\partial x^2} &\approx \frac{d_{j+1}^{n+1} - d_j^{n+1} - d_j^n + d_{j-1}^n}{h^2}, \qquad && j = J-1,\dots,1. \end{aligned}$$

In order to obtain a symmetric scheme, we use the following approximations of the *reaction term*, the same for the upward and downward sweep

$$\text{(10)} \qquad \begin{aligned} v(x_j,t_n) &\approx \frac{u_j^{n+1} + u_j^n}{2}, \qquad && j = 1,\dots,J-1, \\ v(x_j,t_n) &\approx \frac{d_j^{n+1} + d_j^n}{2}, \qquad && j = J-1,\dots,1. \end{aligned}$$

Different approximations of the *convection term* are possible [**3**], [**2**]. In the following we state three of them. First, Towler and Yang [**7**] used special kind of centered differences

$$
\begin{aligned}
\frac{\partial v(x_j,t_n)}{\partial x} &\approx \frac{u_{j+1}^n - u_{j-1}^{n+1}}{2h}, && j = 1,\dots,J-1,\\
\frac{\partial v(x_j,t_n)}{\partial x} &\approx \frac{d_{j+1}^{n+1} - d_{j-1}^{n}}{2h}, && j = J-1,\dots,1.
\end{aligned}
\tag{11}
$$

More accurate approximations were proposed by Roberts and Weiss [**6**], Piacsek and Williams [**5**]

$$
\begin{aligned}
\frac{\partial v(x_j,t_n)}{\partial x} &\approx \frac{u_{j+1}^n - u_j^n + u_j^{n+1} - u_{j-1}^{n+1}}{2h}, && j = 1,\dots,J-1,\\
\frac{\partial v(x_j,t_n)}{\partial x} &\approx \frac{d_{j+1}^{n+1} - d_j^{n+1} + d_j^n - d_{j-1}^{n}}{2h}, && j = J-1,\dots,1.
\end{aligned}
\tag{12}
$$

As a third option we will use *upwind* approximations combined with the ADE technique. Since we have in mind financial applications, we will focus on left going waves, i.e., $b > 0$ in (1). Right going waves $b < 0$ are treated analogously.

The well-known first order approximation reads

$$
\frac{\partial v(x_j,t)}{\partial x} \approx \frac{v_{j+1}(t) - v_j(t)}{h} \qquad j = J-1,\dots,1,
\tag{13}
$$

and the forward difference of second order [**8**]

$$
\frac{\partial v(x_j,t)}{\partial x} \approx \frac{-v_{j+2}(t) + 4v_{j+1}(t) - 3v_j(t)}{2h}, \qquad j = J-1,\dots,1.
\tag{14}
$$

Applying the ADE time splitting idea of Section 2.1, we obtain for the upwind strategy (13)

$$
\begin{aligned}
\frac{\partial v(x_j,t_{n+1})}{\partial x} &\approx \frac{u_{j+1}^n - u_j^n}{h}, && j = 1,\dots,J-1,\\
\frac{\partial v(x_j,t_{n+1})}{\partial x} &\approx \frac{d_{j+1}^n - d_j^n + d_{j+1}^{n+1} - d_j^{n+1}}{2h}, && j = J-1,\dots,1,
\end{aligned}
\tag{15}
$$

and for the second order approximation

$$
\begin{aligned}
\frac{\partial v(x_j,t_{n+1})}{\partial x} &\approx \frac{-u_{j+2}^n + 4u_{j+1}^n - 3u_j^n}{2h}, && j = 1,\dots,J-1,\\
\frac{\partial v(x_j,t_{n+1})}{\partial x} &\approx \frac{-d_{j+2}^n + 4d_{j+1}^n - 3d_j^n}{4h} \\
&\quad + \frac{-d_{j+2}^{n+1} + 4d_{j+1}^{n+1} - 3d_j^{n+1}}{4h}, && j = J-1,\dots,1.
\end{aligned}
\tag{16}
$$

We will show that this upwind approximation (15) leads to a stable scheme.

3. Stability of the ADE method

In this section, we investigate the stability of the proposed ADE method using the matrix approach in Section 3.1 and the classical von-Neumann method in Section 3.2. For the convection-diffusion-reaction equation (1), we obtain unconditional stability using the matrix approach. This stability analysis can be extended by adding homogeneous BCs, without affecting the stability results. This is our motivation to deal with the matrix approach.

3.1. Stability analysis using Matrix approach

We are motivated by [**3**], where the authors claim and proof that "if A is symmetric negative definite, the ADE scheme is unconditionally stable". We have to define symmetric discretization quotients to get symmetric discrete operators. For reaction-diffusion equation, applying central difference quotients, we get symmetric operator A. We can follow the ideas for the proof for the heat equation from [**3**].

Using upwind discretization formulas instead of central differencing leads also to an unconditionally stable scheme. "If A is lower-triangular with all diagonal elements negative, the ADE scheme is unconditionally stable" is generally claimed and proved in [**3**]. In the following we choose suitable differentiating approximations, we formulate theorems about stability properties and prove it.

Theorem 3.1. *The ADE scheme applied to the reaction-diffusion PDE* (1) *(with $b = 0$) is unconditionally stable.*

Proof. Without loss of generality, we focus on the upward sweep

$$\frac{u_j^{n+1} - u_j^n}{k} = a\frac{u_{j+1}^n - u_j^n - u_j^{n+1} + u_{j-1}^{n+1}}{h^2} - c\frac{u_j^{n+1} + u_j^n}{2}.$$

Let us denote the parabolic mesh ratio $\alpha := a\frac{k}{h^2}$, $\gamma := ck$; where a, c are constants.

$$u_j^{n+1} = u_j^n + \alpha\left(u_{j+1}^n - u_j^n - u_j^{n+1} + u_{j-1}^{n+1}\right) - \frac{\gamma}{2}\left(u_j^{n+1} + u_j^n\right) \tag{17}$$

$$\left(1 + \alpha + \frac{\gamma}{2}\right) u_j^{n+1} + (-\alpha)u_{j-1}^{n+1} = \left(1 - \alpha - \frac{\gamma}{2}\right) u_j^n + \alpha u_{j+1}^n$$

We follow roughly the train of thoughts of Leung and Osher [**3**] and write the upward sweep (17) with homogeneous BCs in matrix notation

$$A_u u^{n+1} = B_u u^n, \qquad n \geq 0,$$

with A_u, $B_u \in R^{(J-1)\times(J-1)}$ given by

$$A_u = \begin{pmatrix} 1+\alpha+\frac{\gamma}{2} & 0 & \dots & 0 \\ -\alpha & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & -\alpha & 1+\alpha+\frac{\gamma}{2} \end{pmatrix} = I + \begin{pmatrix} \alpha+\frac{\gamma}{2} & 0 & \dots & 0 \\ -\alpha & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & -\alpha & \alpha+\frac{\gamma}{2} \end{pmatrix}$$

$$A_u =: I + E,$$

$$B_u = \begin{pmatrix} 1-\alpha-\frac{\gamma}{2} & \alpha & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \alpha \\ 0 & \dots & 0 & 1-\alpha-\frac{\gamma}{2} \end{pmatrix} = I - \begin{pmatrix} \alpha+\frac{\gamma}{2} & -\alpha & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -\alpha \\ 0 & \dots & 0 & \alpha+\frac{\gamma}{2} \end{pmatrix}$$

$$B_u =: I - E^\top.$$

Next, we consider the matrices

$$A_u^\top + A_u = 2I + D,$$

$$\text{where} \qquad D := E + E^\top = \begin{pmatrix} 2\alpha+\gamma & -\alpha & \dots & 0 \\ -\alpha & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -\alpha \\ 0 & \dots & -\alpha & 2\alpha+\gamma \end{pmatrix}.$$

The matrix D is positive definite and thus, we can define the induced D-norm as

$$\|C\|_D^2 := \sup_{x\neq 0} \frac{\|C_x\|_D^2}{\|x\|_D^2} = \sup_{x\neq 0} \frac{x^\top C^\top DCx}{x^\top Dx},$$

and the upward sweep can be written as

$$U^{n+1} = A_u^{-1} B_u U^n.$$

Next, we consider the D-norm for the upward sweep matrix $A_u^{-1}B_u$

$$\|A_u^{-1}B_u\|_D^2 := \sup_{x\neq 0} \frac{x^\top B_u^\top A_u^{-\top} D A_u^{-1} B_u x}{x^\top Dx}$$

The numerator $B_u^\top A_u^{-\top} D A_u^{-1} B_u$ can be easily rewritten after a few algebraic steps as $D - 2\gamma(A_u^{-1}D)^\top(A_u^{-1}D)$. From our notation $A_u = I + E$ and $B_u = I - E^\top$, it follows

$$B_u^\top A_u^{-\top} D A_u^{-1} B_u = (I - E^\top)^\top A_u^{-\top} D A_u^{-1} (I - E^\top),$$

where $E^\top = D - E$. An expression in terms of matrices A_u and D gets the following form

$$\begin{aligned} &(A_u^\top - D)^\top A_u^{-\top} D A_u^{-1} (A_u^\top - D) \\ &\qquad = D - DA_u^{-1}D - D^\top A_u^{-\top} D + DA_u^{-\top} D A_u^{-1} D \\ &\qquad = D - DA_u^{-\top} A_u^\top A_u^{-1} D - DA_u^{-\top} A_u A_u^{-1} D + DA_u^{-\top} D A_u^{-1} D \\ &\qquad = D + DA_u^{-\top}\left[-A_u^{-\top} - A_u + D\right] A_u^{-1} D \\ &\qquad = D - 2(A_u^{-1}D)^\top (A_u^{-1}D) \end{aligned}$$

and hence, it follows

$$\|A_u^{-1}B_u\|_D^2 = 1 - 2\sup_{x\neq 0} \frac{\|A_u^{-1}Dx\|_2^2}{\|x\|_D^2}.$$

Thus, the spectral radius of the upward sweep matrix $A_u^{-1}B_u$ reads

$$\rho(A_u^{-1}B_u) \leq \|A_u^{-1}B_u\|_D < 1,$$

and we can conclude that the upward sweep is unconditionally stable.

An analogous result holds for the downward step. In the corresponding equation

$$A_d d^{n+1} = B_d d^n, \qquad n \geq 0 \tag{18}$$

the matrices A_d and B_d are defined as $A_d = A_u^\top$ and $B_d = B_u^\top$. The analysis is done analogous: we can define a positive definite matrix and follow again the steps from the previous proof. Consequently, also the combination as an arithmetic average of these two sub steps is also unconditionally stable. □

The stability analysis using the matrix approach according to [**3**] worked for reaction-diffusion equations with constant coefficients. However, this proof is not transferable for the stability analysis of methods with non-symmetric terms, e.g., the difference quotients for the convection term proposed by Towler and Yang (eq. 11), or Roberts and Weiss (eq. 12), cf. Section 3.2.

As a remedy we can apply a modified upwind discretization of the convection term. The resulting structure of the matrices A_u, B_u is different, but we can do a similar proof.

Theorem 3.2. *ADE scheme, using upwind discretization in convection term, applied to the reaction-diffusion-convection equation* (1) *is unconditionally stable in the upward sweep and unconditionally stable in the downward one.*

Proof. Again, without loss of generality, we focus on the upward sweep and consider an upwind discretization for a left-going wave, i.e., $b \geq 0$ (since later we would like to extend this approach for Black-Scholes model, where $b \geq 0$). In the upward sweep we use difference quotients using values just from the old time level (13)

$$\frac{u_j^{n+1} - u_j^n}{k} = a\frac{u_{j+1}^n - u_j^n - u_j^{n+1} + u_{j-1}^{n+1}}{h^2} + b\frac{u_{j+1}^n - u_j^n}{h} - c\frac{u_j^{n+1} + u_j^n}{2}.$$

Using the abbreviations $\alpha := a\frac{k}{h^2}$, $\beta := b\frac{k}{h} \geq 0$, $\gamma := ck$, we can write

$$-\alpha u_{j-1}^{n+1} + \Bigl(1 + \alpha + \frac{\gamma}{2}\Bigr)u_j^{n+1} = \Bigl(1 - \alpha - \beta - \frac{\gamma}{2}\Bigr)u_j^n + \Bigl(\alpha + \beta\Bigr)u_{j+1}^n. \tag{19}$$

We follow again roughly the ideas of Leung and Osher [**3**] and consider the upward sweep (19) with homogeneous BCs

$$A_u u^{n+1} = B_u u^n, \qquad n \geq 0,$$

with the system matrices A_u, $B_u \in R^{(J-1)\times(J-1)}$ given by

$$A_u = \begin{pmatrix} 1+\alpha+\frac{\gamma}{2} & 0 & \dots & 0 \\ -\alpha & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & -\alpha & 1+\alpha+\frac{\gamma}{2} \end{pmatrix}$$

$$= I + \begin{pmatrix} \alpha+\frac{\gamma}{2} & 0 & \dots & 0 \\ -\alpha & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & -\alpha & \alpha+\frac{\gamma}{2} \end{pmatrix} =: I + E,$$

$$B_u = \begin{pmatrix} 1-\alpha-\beta-\frac{\gamma}{2} & \alpha+\beta & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \alpha+\beta \\ 0 & \dots & 0 & 1-\alpha-\beta-\frac{\gamma}{2} \end{pmatrix}$$

$$= I - \begin{pmatrix} \alpha+\beta+\frac{\gamma}{2} & -\alpha-\beta & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -\alpha-\beta \\ 0 & \dots & 0 & \alpha+\beta+\frac{\gamma}{2} \end{pmatrix} =: I - F,$$

where $$D := E + F = \begin{pmatrix} 2\alpha+\beta+\gamma & -\alpha-\beta & \dots & 0 \\ -\alpha & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -\alpha-\beta \\ 0 & \dots & -\alpha & 2\alpha+\beta+\gamma \end{pmatrix}.$$

The matrix D is not symmetric but obviously positive definite.

In the sequel we have just outlined the steps which differ from the previous proof. The numerator $B_u^\top A_u^{-\top} D A_u^{-1} B_u$ can be easily rewritten after a few algebraic steps as $D - 2\gamma(A_u^{-1}D)^\top(A_u^{-1}D)$.

From our notation $A_u = I + E$ and $B_u = I - F$, it follows

$$B_u^\top A_u^{-\top} D A_u^{-1} B_u = (I-F)^\top A_u^{-\top} D A_u^{-1} (I-F),$$

where $F := D - E$. An expression in terms of matrices A_u and D gets the following form

$$(I+E-D)^\top A_u^{-\top} D A_u^{-1} (I+E-D) = (A_u - D)^\top A_u^{-\top} D A_u^{-1} (A_u - D)$$

and we proceed the same way as in previous proof.

For the *downward sweep*, we have

$$\frac{d_j^{n+1}-d_j^n}{k} = a\frac{d_{j+1}^{n+1}-d_j^{n+1}-d_j^n+d_{j-1}^n}{h^2} + b\frac{d_{j+1}^n+d_{j+1}^{n+1}-d_j^n-d_j^{n+1}}{2h} - c\frac{d_j^{n+1}+d_j^n}{2}.$$

Using the abbreviations $\alpha := a\frac{k}{h^2}$, $\beta := b\frac{k}{h} \le 0$, $\gamma := ck$, we can write

$$\Big(1+\alpha+\frac{\beta}{2}+\frac{\gamma}{2}\Big)d_j^{n+1} + \Big(-\alpha-\frac{\beta}{2}\Big)d_{j+1}^{n+1} = \alpha d_{j-1}^n + \Big(1-\alpha-\frac{\beta}{2}-\frac{\gamma}{2}\Big)d_j^n + \frac{\beta}{2}d_{j+1}^n \tag{20}$$

$$A_D d^{n+1} = B_D d^n, \qquad n \ge 0,$$

with A_D, $B_D \in R^{(J-1)\times(J-1)}$ given by matrices A_D, B_D. The matrix A_D is upper-diagonal $A_D = \mathrm{diag}(1+\alpha+\frac{\beta}{2}+\frac{\gamma}{2}, -\alpha-\frac{\beta}{2})$. The matrix B_D is tridiagonal with diagonal terms $B_D = \mathrm{diag}(\alpha, 1-\alpha-\frac{\beta}{2}-\frac{\gamma}{2}, \frac{\beta}{2})$. Likewise, we construct matrices $D = \mathrm{diag}(-\alpha, 2\alpha+\beta+\gamma, -\alpha-\beta)$ as a tridiagonal positive definite matrix. We can follow the same way of proof, and thus we conclude the unconditional stability of the downward sweep. □

3.2. Von Neumann stability analysis for the convection-diffusion-reaction equation

Since analysis using matrix approach was suitable for upwind kind of approximation in convection term, here we investigate stability properties of the ADE schemes, where discretization of convection term is provided according to [**7**] and [**6**].

We consider the convection-diffusion-reaction equation (1) and focus on the upward sweep of the ADE procedure in the sequel. An appropriate choice for the approximation of the convection term is the one due to Roberts and Weiss [**6**] since performing just a downward sweep leads to the unconditionally stable solution.

Theorem 3.3. *The ADE scheme with the Roberts and Weiss approximation in the convection term, applied to the PDE* (1) *is conditionally stable in the upward sweep and unconditionally stable for the downward one.*

Proof. Using Roberts and Weiss discretization in convection term, we get

$$\frac{u_j^{n+1}-u_j^n}{k} = a\frac{u_{j+1}^n-u_j^n-u_j^{n+1}+u_{j-1}^{n+1}}{h^2} + b\frac{u_{j+1}^n-u_j^n+u_j^{n+1}-u_{j-1}^{n+1}}{2h} - c\frac{u_j^{n+1}+u_j^n}{2}.$$

Let us denote the parabolic mesh ratio $\alpha := a\frac{k}{h^2}$, the hyperbolic mesh ratio $\beta := b\frac{k}{h}$ and $\gamma := ck$; where a, b, c are nonnegative constants.

$$u_j^{n+1} = u_j^n + \alpha\Big(u_{j+1}^n - u_j^n - u_j^{n+1} + u_{j-1}^{n+1}\Big) + \frac{\beta}{2}\Big(u_{j+1}^n - u_j^n + u_j^{n+1} - u_{j-1}^{n+1}\Big) - \frac{\gamma}{2}\Big(u_j^{n+1} + u_j^n\Big).$$

Applying von Neumann Ansatz $u_j^n := \mathrm{e}^{n\alpha k}\,\mathrm{e}^{\mathrm{i}\,j\lambda h}$ amplification factor A_1 reads

$$A_1 = \frac{A + B\,\mathrm{e}^{\mathrm{i}\,\lambda h}}{C + D\,\mathrm{e}^{-\,\mathrm{i}\,\lambda h}}$$

where $A = 1-\alpha-\beta/2-\gamma/2$; $B = \alpha+\beta/2$; $C = 1+\alpha-\beta/2+\gamma/2$; $D = -\alpha+\beta/2$.

For stability we require $|A_1|^2 \le 1$.

$$|A_1|^2 = A_1\overline{A_1} = \frac{\big(A + B\,\mathrm{e}^{\mathrm{i}\,\lambda h}\big)\big(A + B\,\mathrm{e}^{-\,\mathrm{i}\,\lambda h}\big)}{\big(C + D\,\mathrm{e}^{-\,\mathrm{i}\,\lambda h}\big)\big(C + D\,\mathrm{e}^{\mathrm{i}\,\lambda h}\big)} \le 1$$

$$A^2 + B^2 + 2AB\cos(\lambda h) \le C^2 + D^2 + 2CD\cos(\lambda h)$$

$$2\,(AB - CD)\cos(\lambda h) \le C^2 + D^2 - A^2 - B^2$$

$$(4\alpha - 4\alpha\beta - \beta\gamma)\cos(\lambda h) \le 4\alpha - 4\alpha\beta - \beta\gamma + 2\gamma. \tag{21}$$

Here we can distinguish 2 cases with respect to the sign of $(4\alpha - 4\alpha\beta - \beta\gamma)$.

- <u>*Case 1:*</u> By substituing α, β, γ into $(4\alpha - 4\alpha\beta - \beta\gamma) > 0$, we get the following condition

$$\alpha < \frac{a}{2Pe} - \frac{ck}{4}. \tag{22}$$

  In this case, equation (21) can be rewritten as

$$\cos(\lambda h) \le \frac{4\alpha - 4\alpha\beta - \beta\gamma + 2\gamma}{4\alpha - 4\alpha\beta - \beta\gamma} \tag{23}$$

$$1 \le 1 + \frac{2\gamma}{4\alpha - 4\alpha\beta - \beta\gamma}$$

$$0 \le \frac{2\gamma}{4\alpha - 4\alpha\beta - \beta\gamma}. \tag{24}$$

  We can notice that condition (24) is satisfied for all the possible values of parameters since $\gamma > 0$ and $4\alpha - 4\alpha\beta - \beta\gamma > 0$.

- <u>*Case 2:*</u> We consider $(4\alpha - 4\alpha\beta - \beta\gamma) < 0$, what is equivalent to the condition

$$\alpha > \frac{a}{2Pe} - \frac{ck}{4}. \tag{25}$$

  In this case, equation (21) can be rewritten as

$$\cos(\lambda h) \ge \frac{4\alpha - 4\alpha\beta - \beta\gamma + 2\gamma}{4\alpha - 4\alpha\beta - \beta\gamma} \tag{26}$$

$$-1 \ge 1 + \frac{2\gamma}{4\alpha - 4\alpha\beta - \beta\gamma}$$

$$0 \geq \frac{8\alpha - 8\alpha\beta - 2\beta\gamma + 2\gamma}{4\alpha - 4\alpha\beta - \beta\gamma}. \tag{27}$$

Since $(4\alpha - 4\alpha\beta - \beta\gamma) < 0$, then $(8\alpha - 8\alpha\beta - 2\beta\gamma + 2\gamma) \geq 0$. After substituing α, β, γ and after elementary algebraic steps, we get

$$\alpha \leq \frac{a}{2Pe} - \frac{ck}{4} + \frac{ch}{4b}. \tag{28}$$

Case 2 leads to conditions (25) and (28) what means that

$$\alpha \in \left(\frac{a}{2Pe} - \frac{ck}{4}, \frac{a}{2Pe} - \frac{ck}{4} + \frac{ch}{4b} \right] \tag{29}$$

To sum up case 1 and case 2, we can claim that conditions (22) and (29), and also considering the situation where $(4\alpha - 4\alpha\beta - \beta\gamma) = 0$, we get

$$\alpha \leq \frac{a}{2Pe} - \frac{ck}{4} + \frac{ch}{4b}, \tag{30}$$

where $Pe = \frac{bh}{2}$ is the so-called Peclet number.

For the downward sweep, we get the following amplification factor

$$A_2 = \frac{\left[1 - \alpha + \frac{\beta}{2} - \frac{\gamma}{2}\right] + [\alpha - \frac{\beta}{2}]\, \mathrm{e}^{-\mathrm{i}\lambda h}}{\left[1 + \alpha + \frac{\beta}{2} + \frac{\gamma}{2}\right] + [-\alpha - \frac{\beta}{2}]\, \mathrm{e}^{\mathrm{i}\lambda h}} \tag{31}$$

Stability condition $|A_2|^2 \leq 1$ leads to the formula

$$\cos(\lambda h) \leq \frac{4\alpha + 4\alpha\beta + \beta\gamma + 2\gamma}{4\alpha + 4\alpha\beta + \beta\gamma}.$$

Let us note that the last condition can be simplified to the condition

$$\frac{2\gamma}{4\alpha + 4\alpha\beta + \beta\gamma} \geq 0. \tag{32}$$

The coefficients α, β, γ are positive, i.e., the condition (32) is satisfied, and thus we have the unconditional stability for the downward sweep using the Roberts and Weiss approximation, which completes the proof. □

In case of the Roberts and Weiss approximation we propose to use only the unconditional stable downward sweep.

Theorem 3.4. *ADE scheme, using Towler and Yang approximation in the convection term, applied to the PDE* (1) *is conditionally stable in both sweeps.*

Proof. For the Towler and Yang approximation, the stability condition for the upward sweep reads

$$(4\alpha - 2\alpha\beta - \beta\gamma)\cos(\lambda h) \leq 4\alpha - 2\alpha\beta + 2\gamma, \tag{33}$$

where again we can distinguish 2 cases with respect to the sign of left hand side of the equation (33).

- *Case 1:* If $(4\alpha - 2\alpha\beta - \beta\gamma) > 0$, it means

$$\alpha < \frac{a}{Pe} - \frac{ck}{2}. \tag{34}$$

In this case, equation (33) can be rewritten as

$$\cos(\lambda h) \leq \frac{4\alpha - 4\alpha\beta - \beta\gamma + 2\gamma}{4\alpha - 2\alpha\beta - \beta\gamma} \tag{35}$$

$$1 \leq 1 + \frac{\gamma(2+\beta)}{4\alpha - 2\alpha\beta - \beta\gamma}$$

$$0 \leq \frac{\gamma(2+\beta)}{4\alpha - 2\alpha\beta - \beta\gamma}. \tag{36}$$

We can notice that condition (36) is satisfied for all the possible values of parameters since $\gamma \geq 0$ and $(2+\beta) > 0$ and $4\alpha - 2\alpha\beta - \beta\gamma > 0$.

- *Case 2:* We consider $(4\alpha - 2\alpha\beta - \beta\gamma) < 0$, what is equivalent to the condition

$$\alpha > \frac{a}{Pe} - \frac{ck}{2}. \tag{37}$$

In this case, equation (33) can be rewritten as

$$\cos(\lambda h) \geq \frac{4\alpha - 4\alpha\beta - \beta\gamma + 2\gamma}{4\alpha - 2\alpha\beta - \beta\gamma} \tag{38}$$

$$-2 \geq \frac{\gamma(2+\beta)}{4\alpha - 2\alpha\beta - \beta\gamma}$$

After substituing α, β, γ and simplification, it leads to the condition

$$\alpha \leq \frac{a}{Pe} - \frac{ck}{2} + \frac{ch}{2b} + \frac{1}{2}. \tag{39}$$

In case 2, we obtain two conditions (37) and (39), namely

$$\alpha \in \left(\frac{a}{2Pe} - \frac{ck}{2}, \frac{a}{Pe} - \frac{ck}{2} + \frac{ch}{2b} + \frac{1}{2}\right] \tag{40}$$

From case 1, condition (34) and case 2 condition (40) in Towler and Yang case and considering also possibility of $(4\alpha - 2\alpha\beta - \beta\gamma) = 0$, we can sum up

$$\alpha \leq \frac{a}{Pe} - \frac{ck}{2} + \frac{ch}{2b} + \frac{1}{2}. \tag{41}$$

For the downward sweep, the stability condition is

$$\cos(\lambda h) \leq \frac{4\alpha + 2\alpha\beta + 2\gamma}{4\alpha + 2\alpha\beta + \beta\gamma},$$

which leads to the condition

$$\frac{k}{h^2} \leq \frac{1}{Pe} \tag{42}$$

Both sweeps in Towler and Yang discretization of convection term in reaction-diffusion-convection equation are conditionally stable under the conditions (41) and (42). □

4. Consistency Analysis of the ADE methods

In this section, we provide a consistency analysis of the ADE methods for solving the convection-diffusion-reaction equation (1) and for the BS model (2).

4.1. Consistency of the ADE scheme for convection-diffusion-reaction equations

We study the consistency of the following ADE discretization

$$\frac{u_j^{n+1}-u_j^n}{k} = a\frac{u_{j+1}^n - u_j^n - u_j^{n+1} + u_{j-1}^{n+1}}{h^2} + b\frac{u_{j+1}^n - u_j^n + u_j^{n+1} - u_{j-1}^{n+1}}{2h} - c\frac{u_j^{n+1}+u_j^n}{2}$$

to the convection-diffusion-reaction equation (1). The local truncation error (LTE) of the upward sweep is given by

$$\begin{aligned} LTE_{\mathrm{up}} = k\Big(-\frac{1}{2}v_{tt} + \frac{1}{2}av_{xxt} + \frac{1}{2}bv_{xt}\Big) \\ + k^2\Big(-\frac{1}{6}v_{ttt} + \frac{1}{4}av_{xxtt} + \frac{1}{4}bv_{xtt}\Big) + h^2\Big(\frac{1}{12}av_{xxxx}\frac{1}{6}bv_{xxx}\Big) \\ - kh\Big(\frac{1}{6}av_{xxxt} + \frac{1}{4}bv_{xxt}\Big) - \frac{k}{h}av_{xt} - \frac{k^2}{h}\Big(\frac{1}{2}av_{xtt}\Big) - \frac{k^3}{h}\Big(\frac{1}{6}av_{xttt}\Big), \end{aligned}$$

and analogously, the LTE for the downward sweep reads

$$\begin{aligned} LTE_{\mathrm{down}} = k\Big(-\frac{1}{2}v_{tt} + \frac{1}{2}av_{xxt} + \frac{1}{2}bv_{xt}\Big) \\ + k^2\Big(-\frac{1}{6}v_{ttt} + \frac{1}{4}av_{xxtt} + \frac{1}{4}bv_{xtt}\Big) + h^2\Big(\frac{1}{12}av_{xxxx}\frac{1}{6}bv_{xxx}\Big) \\ + kh\Big(\frac{1}{6}av_{xxxt} + \frac{1}{4}bv_{xxt}\Big) + \frac{k}{h}av_{xt} + \frac{k^2}{h}\Big(\frac{1}{2}av_{xtt}\Big) + \frac{k^3}{h}\Big(\frac{1}{6}av_{xttt}\Big). \end{aligned}$$

Thus, we end up for the LTE for the combined sweep

$$\begin{aligned} LTE_{\mathrm{ADE}} = k\Big(-\frac{1}{2}v_{tt} + \frac{1}{2}av_{xxt} + \frac{1}{2}bv_{xt}\Big) \\ + k^2\Big(-\frac{1}{6}v_{ttt} + \frac{1}{4}av_{xxtt} + \frac{1}{4}bv_{xtt}\Big) + h^2\Big(\frac{1}{12}av_{xxxx}\frac{1}{6}bv_{xxx}\Big), \end{aligned}$$

Assuming a constant parabolic mesh ratio k/h^2, the first order term in k can be written in the form $\mathrm{O}(k) = \mathrm{O}(h^2)$ and hence, we get

$$\begin{aligned} LTE_{\mathrm{ADE}} = k^2\Big(-\frac{1}{6}v_{ttt} + \frac{1}{4}av_{xxtt} + \frac{1}{4}bv_{xtt}\Big) \\ + h^2\Big(\frac{1}{12}av_{xxxx}\frac{1}{6}bv_{xxx} - \frac{1}{2}v_{tt} + \frac{1}{2}av_{xxt} + \frac{1}{2}bv_{xt}\Big), \end{aligned}$$

Hence, the order of consistency of the ADE method for the PDE (1) is $\mathrm{O}(k^2+h^2)$.

4.2. The Consistency of the ADE method for the linear BS model

As an extension of the PDE (1), we consider the linear BS equation (2).

Theorem 4.1. *The order of consistency of the ADE method for the linear BS equation is* $\mathrm{O}(k^2 + h^2)$ *in both sweeps and in the final combined solution.*

Proof. The linear BS PDE is a special case of (1) with the space-dependent coefficients $a(S) = \frac{1}{2}\sigma^2 S^2$, $b(S) = rS$, $c(S) = r$. The LTE for the upward sweep reads

$$\begin{aligned} LTE_{\mathrm{BS}} = {} & k\Big(-\frac{1}{2}v_{tt} + \frac{1}{2}av_{xxt} + \frac{1}{2}bv_{xt}\Big) \\ & + k^2\Big(-\frac{1}{6}v_{ttt} + \frac{1}{4}av_{xxtt} + \frac{1}{4}bv_{xtt}\Big) + h^2\Big(\frac{1}{12}av_{xxxx}\frac{1}{6}bv_{xxx}\Big) \\ & + kh\Big(-\frac{1}{6}av_{xxxt} - \frac{1}{4}bv_{xxt}\Big) + \frac{k}{h}\Big(-av_{xt}\Big) \\ & + \frac{k^2}{h}\Big(-\frac{1}{2}av_{xtt}\Big) + \frac{k^3}{h}\Big(-\frac{1}{6}av_{xttt}\Big). \end{aligned}$$

If we assume a constant parabolic mesh ratio $\alpha = k/h^2$, then we get

$$LTE = k\Big(-\frac{1}{2}v_{tt}\Big) + k^2\Big(-\frac{1}{6}v_{ttt}\Big) = \alpha h^2\Big(-\frac{1}{2}v_{tt}\Big) + k^2\Big(-\frac{1}{6}v_{ttt}\Big),$$

where we neglected higher order terms. A similar result holds for the downward sweep. We have shown that consistency for the linear BS model is $\mathrm{O}(k^2 + h^2)$ in downward, upward and hence, also in the combination. □

4.3. Application and numerical experiments with the linear model

We apply the ADE method and calculate a price for a vanilla European call option in a classic linear BS model with constant coefficients. Choosing the following set of parameters $r = 0.03$ (interest rate); $q = 0$ (continuous dividend yield); $\sigma = 0.2$ (volatility); $T = 1$ (maturity time in years); $S_{\max} = 90$ (maximal stock price); $K = 30$ (strike price); and defining a grid with $N = 50$ time steps; $J = 200$ space steps, we get an option price, which is shown in Figure 3.

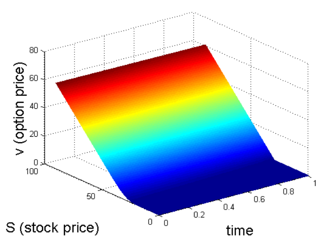

Figure 3. Option price.

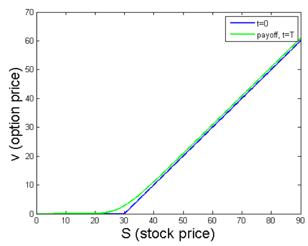

Figure 4. Solution at time $t = 0$ and $t = T$.

In this subsection, we analyze the computational and theoretical order of convergence. In Table 1, there is recorded an error as a difference between numerical solution using ADE method and the closed form BS formula for different meshes with fixed mesh ratio 0.23. In Table 2, ratios of errors from the Table 1 are calculated. One can observe that using double space steps, ratio of errors converges to the number 4, what confirms that the theoretical order of convergence is 2.

N	**J**	**mesh ratio**	**error**
3	50	0.23	0.2458
12	100	0.23	0.0855
50	200	0.23	0.0208
200	400	0.23	0.0052
800	800	0.23	0.0013

Table 1. Error as a difference between exact solution and approximation.

ratio of errors	
error50/error100	2.87
error100/error200	4.11
error200/error400	4
error400/error800	4

Table 2. Ratio of errors.

Figures 5–9 show an error on different grids as a difference between numerical solution and the exact one (from the BS formula). Table 1 records the maximum value of the error from the time $t = 0$, it means that we observe the maximal value of the errors whole calculation in the current time. At the beginning of the calculation (nearby maturity time) we can observe the highest error, which is caused by the non-smooth initial data. This error decreases during the calculation. The finer the mesh, the faster the decrease of the error (5)–(9).

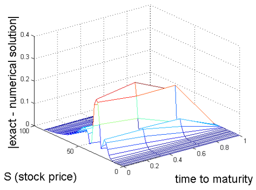

Figure 5. Error, $N = 3$, $J = 50$.

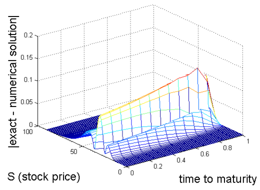

Figure 6. Error, $N = 12$, $J = 100$.

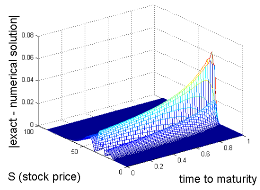

Figure 7. Error, $N = 50$, $J = 200$.

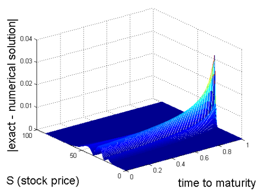

Figure 8. Error, $N = 200$, $J = 400$.

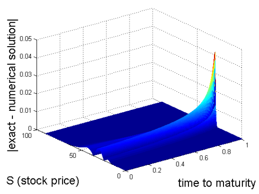

Figure 9. Error, $N = 800$, $J = 800$.

Conclusion and Outlook

We provided a numerical analysis for ADE methods solving linear convection-diffusion-reaction equations. Stability was investigated by two different approaches. The matrix approach yields unconditional stability in the downward sweep using upwind discretization. The von-Neumann analysis yields unconditional stability of the downward sweep using the Roberts and Weiss approximation.

It turned out that the order of consistency is $O(k^2 + h^2 + k/h)$ for the upward or downward sweeps, but its combination exhibits an increase order of consistency $O(k^2 + h^2)$. Next, for the BS model, as an application in computational finance, we obtained an order of consistency $O(k^2 + h^2)$ for both downward and upward sweeps.

Our aim is to extend our numerical analysis, esp. the stability analysis, to PDEs with non-constant space dependent coefficients and provide experiments with the BS PDE. We will investigate the stability properties of ADE methods for solving nonlinear BS models and provide illustrative numerical experiments. Finally, extensions of the ADE method to higher dimensions will be considered.

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