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ON THE CLASS OF LIMITED OPERATORS

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Abstract. We introduce the classes of weak* Dunford-Pettis and order limited operators and characterize Banach lattices on which order limited and weak* Dunford-Pettis operators coincide with those of limited operators.

1. Introduction

It is well known that the class of (weak) Dunford-Pettis operators plays an important role in the theory of Banach spaces in general. This paper focuses on the various interrelationships between weak* Dunford-Pettis, order limited and limited operators on Banach lattices. The article is organized as follows. In Section 2.1, we give a definition of the so-called weak* Dunford-Pettis operators acting among Banach spaces. In Section 2.2, we introduce the notion of an order limited operator from a Banach lattice to a Banach space. All the above definitions are stronger versions of those previously studied which are well known. We conclude the paper with the section containing the main results we prove under appropriate assumptions, the equivalence among limited operators, order limited and weak* Dunford-Pettis operators.

2. Preliminaries

Let us recall that a norm bounded subset A of a Banach space X is said to be Dunford-Pettis* set (abbr. DP* set; limited set [**3**]) if every weak* null sequence (f_n) of X' converges uniformly on A, that is, $\lim\limits_{n\to\infty} \sup_{x\in A} |\langle f_n, x\rangle| = 0$. Note that every relatively compact set is DP* but the converse is not true in general. Indeed, the set $\{e_n : n \in \mathbb{N}\}$ of unit coordinate vectors is a DP* set in ℓ^∞ which is not relatively compact. If every DP* set of a Banach space X is relatively compact, then X is said to have the Gelfand-Phillips property (abbr. GP-property [**8**]).

According to this terminology, a Banach space X has the Dunford-Pettis property (abbr. DP property) if and only if $\lim\limits_{n\to\infty} f_n(x_n) = 0$ for every weakly null pair of sequences $((x_n), (f_n))$ in $X \times X'$.

A stronger version of DP property was introduced by Borwein et al. [**2**]. Indeed, a Banach space X has the Dunford-Pettis* property (abbr. DP* property) if every

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relatively weakly compact subset of X is DP*. It turns out that a Banach space X has the DP* property if and only if $\lim_{n\to\infty} f_n(x_n) = 0$ for every weakly null sequence (x_n) in X and every weak* convergent sequence (f_n) in X'. Furthermore, if X is a Grothendieck space (see [**5**, p. 179]), then these properties are equivalent in X.

Recall from [**8**] that a bounded linear operator $T\colon X \to Y$ between two Banach spaces is limited completely continuous (abbr. *lcc*) if it carries limited and weakly null sequences in X to norm null ones in Y. Also, we recall from [**3**] that an operator T from a Banach space X into another Y is said to be limited operator if it carries the closed unit ball of X into a DP* set of Y. It is easy to see that $T\colon X \to Y$ is limited if and only if $T'\colon Y' \to X'$ takes weak* null sequences to norm null ones.

An operator $T\colon X \to Y$ between two Banach spaces is called a Dunford-Pettis operator if T carries weakly convergent sequences into norm convergent sequences (equivalently, for each weakly null sequence (x_n), we have $\lim\limits_{n\to\infty} \|T(x_n)\| = 0$). Alternatively, an operator $T\colon X \to Y$ is a Dunford-Pettis operator if and only if T carries relatively weakly compact sets into compact sets. An operator T from a Banach space X into another Y is said to be weak Dunford-Pettis if $S \circ T$ is a Dunford-Pettis operator for every weakly compact operator S from Y into an arbitrary Banach space Z. Alternatively, T is weak Dunford-Pettis if the sequence $f_n(T(x_n))$ converges to 0 whenever (x_n) and (f_n) are weakly null sequences in X and Y', respectively [**1**].

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. If E is a Banach lattice, its topological dual E' endowed with the dual norm is also a Banach lattice. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, the sequence (x_α) converges to 0 for the norm $\|\cdot\|$, where the notation $x_\alpha \downarrow 0$ means that the sequence (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$.

The lattice operations in E' are called weak* sequentially continuous if the sequence $(|f_n|)$ converges to 0 in the weak* topology whenever the sequence (f_n) converges weak* to 0 in E'. A Banach space X has the Schur property if each weakly null sequence in X converges to zero in the norm topology. A Banach lattice E is said to be an AM-space if for each $x, y \in E$ such that $\inf(x, y) = 0$, we have $\|x + y\| = \max\{\|x\|, \|y\|\}$. We refer the reader to [**1**] for unexplained terminology on Banach lattice theory. Some useful and additional properties of limited sets and Banach spaces with the GP property can be found in [**3, 6, 8**].

2.1. Weak* Dunford-Pettis operators

Definition 2.1. An operator $T\colon X \to Y$ between two Banach spaces is said to be a weak* Dunford-Pettis operator (abbr. w*DP) whenever $x_n \to 0$ in the $\sigma(X, X')$-topology of X and $f_n \to 0$ in the $\sigma(Y', Y)$-topology of Y' imply $\lim_{n\to\infty} f_n(T(x_n)) = 0$.

This class of w*DP operators is bigger than the class of Dunford-Pettis operators, but smaller than the class of weak Dunford-Pettis operators, that is, every Dunford-Pettis operator is w*DP and every w*DP operator is weak Dunford-Pettis, but the converses are false. Indeed, the identity operator of the Banach space ℓ^∞ is w*DP (because ℓ^∞ is a space with the DP* property) but fails to be a Dunford-Pettis operator (because ℓ^∞ does not have the Schur property). Also, the identity operator of the Banach space c_0 is weak Dunford-Pettis (because c_0 has the Dunford-Pettis property) but it is not a w*DP operator (because c_0 does not have the DP* property). Furthermore, if Y is a Grothendieck space, then the notions of weak Dunford-Pettis and w*DP operators coincide.

Note that the space of all w*DP operators is a two-sided ideal in the space of all operators on a Banach space. That is, if S, T are two operators between Banach spaces such that S or T is w*DP, then the product operator $S \circ T$ is w*DP.

Proposition 2.1. *Let X be a Banach space. Then the following statements are equivalent:*

1. *X has the DP* property;*
2. *Every operator $T\colon X \to X$ is w*DP;*
3. *The identity operator of X is w*DP.*

Proof. (1) $\Rightarrow$ (2) Let $x_n \to 0$ in the $\sigma(X, X')$ topology of X and $f_n \to 0$ in the $\sigma(X', X)$ topology of X'. Then $T'(f_n) \to 0$ for the topology $\sigma(X', X)$ and since X has the DP* property, then $f_n(Tx_n) = T'(f_n)(x_n) \to 0$. And we are done.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Obvious. □

As a consequence, we have the following characterization.

Corollary 2.1. *Let X be a Banach space. Then the following statements are equivalent:*

1. *X has the DP* property;*
2. *For an arbitrary Banach space Y, every operator $T\colon X \to Y$ is w*DP;*
3. *For an arbitrary Banach space Y, every operator $T\colon Y \to X$ is w*DP.*

2.2. Order limited operators

Definition 2.2. An operator T from a Banach lattice E into a Banach space X is said to be order limited if it carries each order bounded subset of E into a DP* set of X, i.e., for each $x \in E^+$, the subset $T([-x, x])$ is DP* in X.

Clearly, every limited operator is order limited but the converse is not true in general. Indeed, the identity operator of the Banach lattice ℓ^2 is order limited (because the lattice operations of the Banach lattice ℓ^2 are weak* sequentially continuous) but it is not a limited operator (because the closed unit ball of the Banach space ℓ^2 is not DP*) [**5**].

Furthermore, if the Banach lattice E is an AM-space with order unit $u > 0$, then its closed unit ball coincides with the order interval $[-u, u]$ ([**1**, p. 194]), and hence the class of order limited operators coincides with that of limited operators on E.

It is easy to see that the class of order limited operators is a two-sided ideal in the space of all order bounded operators between Banach lattices. That is, if S and T are two order bounded operators between Banach lattices such that S or T is order limited, then the product operator $S \circ T$ is order limited.

Proposition 2.2. *An operator $T: E \to X$ defined from a Banach lattice E into a Banach space X is order limited if and only if for every weak* null sequence (f_n) of X', we have $|T'(f_n)| \to 0$ in the $\sigma(E', E)$ topology of E'.*

Proof. Given $x \in E^+$. Let (f_n) be a weak* null sequence of X', we consider the operator $S: X \to c_0$ defined by $S(x) = (f_n(x))_{n=1}^{\infty}$ for all $x \in X$.

It is clear that S is well defined and is an *lcc* operator (because c_0 has the GP-property).

Since T is an order limited operator, then $T([-x, x])$ is a DP* subset of X, and hence it follows from [**8**, Theorem 2.1] that $S \circ T([-x, x])$ is a relatively compact subset of c_0 for each $x \in E^+$. So, it follows from the [**1**, Exercise 14 of Section 3.2] that

$$\begin{aligned}|T'(f_n)|(x) &= \sup\{|T'(f_n)(y)|; y \in [-x, x]\}\\ &= \sup\{|f_n(z)|; z \in T([-x, x])\} \to 0.\end{aligned}$$

Conversely, for each $x \in E^+$, we have $|T'(f_n)|(x) = \sup\{|f_n(z)|; z \in T([-x, x])\} \to 0$ for each weak* null sequence (f_n) of X'. This proves that $T([-x, x])$ is a DP* subset of X for each $x \in E^+$, and hence T is order limited. □

As a consequence, we obtain the following characterization.

Corollary 2.2. *Let E be a Banach lattice. Then the following statements are equivalent:*

1. *The lattice operations of E' are weak* sequentially continuous;*
2. *Every operator $T: E \to E$ is order limited;*
3. *The identity operator of E is order limited.*

3. Main results

To establish our main results, we need the following Lemma which is a direct consequence of the Josefson-Nissenzweig Theorem.

Lemma 3.1. *Let F be an infinite dimensional Banach lattice, then there exist a weak* null sequence $(f_n) \subset F'$ and a sequence $(y_n) \subset F^+$ such that $\|f_n\| = 1$, $\|y_n\| \leq 1$ and $|f_n(y_n)| \geq \frac{1}{4}$ for all n.*

Proof. If F is infinite dimensional, then it follows from the Josefson-Nissenzweig Theorem that there exists a weak* null sequence $(f_n) \subset F'$ such that $\|f_n\| = 1$ for all n.

As $\|f_n\| = \sup\{|f_n(y)| : y \in F, \|y\| \leq 1\}$ for each n, then there exists $y_n \in B_F$ such that $|f_n(y_n)| \geq \frac{1}{2}\|f_n\| = \frac{1}{2}$, where B_F is the closed unit ball of F. By observing that $|f_n(y_n^+)| \geq \frac{1}{4}$ or $|f_n(y_n^-)| \geq \frac{1}{4}$ and replacing y_n by y_n^+ or by y_n^- we may assume that for each n, there exists $y_n \in F^+$ with $\|y_n\| \leq 1$ and $|f_n(y_n)| \geq \frac{1}{4}$. □

Proposition 3.1. *Let T be an operator between two Banach spaces X and Y. If T is limited, then T is w*DP.*

Proof. Let (x_n) be a weakly null sequence in X and (f_n) be a weak* null sequence in Y'. We have to prove that $\lim_{n\to\infty} f_n(Tx_n) = 0$. As (f_n) is a sequence of Y' such that $f_n \to 0$ in the $\sigma(Y',Y)$-topology of Y' and since T is a limited operator, then $\|T'(f_n)\| \to 0$.

On the other hand, since (x_n) is a weakly null sequence in X, then (x_n) is norm bounded and by the inequality $|f_n(Tx_n)| = |T'(f_n)(x_n)| \leq (\sup_m \|x_m\|)\|T'(f_n)\|$, we conclude that T is w*DP. □

The converse of the above proposition is not true. Indeed, the identity operator on the Banach space ℓ^∞ is w*DP but fails to be a limited operator (because the closed unit ball of the Banach space ℓ^∞ is not a DP* set).

As shown in the paragraph after Definition 2.2, there exist an order limited operators which are not limited. Furthermore, there exist order limited and w*DP operators which are not limited. Indeed, the identity operator of the Banach lattice ℓ^1 is order limited and w*DP but fails to be limited operator (because the closed unit ball of the Banach space ℓ^1 is not DP* set).

Note that if E or F is a finite dimensional Banach space, then every operator $T\colon E \to F$ is limited. For infinite dimensional spaces, we have the following theorem.

Theorem 3.1. *Let E and F be two Banach lattices such that F is infinite dimensional. Then the following assertions are equivalent:*

1. *Each order limited and w*DP operator $T\colon E \to F$, T is limited;*
2. *The norm of E' is order continuous.*

Proof. (1) ⇒ (2) Assume that (2) is false, i.e., the norm of E' is not order continuous. We will construct an operator $T\colon E \to F$ which is w*DP and order limited but not limited.

Since the norm of E' is not order continuous, by [**7**, Theorem 2.4.14], we may assume that ℓ^1 is a closed sublattice of E and it follows from [**7**, Proposition 2.3.11] that there is a positive projection P from E onto ℓ^1.

On the other hand, since F is infinite dimensional, then it follows from Lemma 3.1 there exists a weak* null sequence $(f_n) \subset F'$ and a sequence $(y_n) \subset F^+$ such that $\|f_n\| = 1$, $\|y_n\| \leq 1$ and $|f_n(y_n)| \geq \frac{1}{4}$ for all n.

Now, we consider the operator $T = S \circ P\colon E \to \ell^1 \to F$, where S is the operator defined by $S\colon \ell^1 \to F$, $(\alpha_n) \mapsto \sum_{n=1}^{\infty} \alpha_n y_n$.

Since ℓ^1 has the Schur property, then the operator T is Dunford-Pettis, and hence T is w*DP and order limited.

But the operator T is not limited. Indeed, the sequence (f_n) is weak* null in F' and as the projection $P\colon E \to \ell^1$ is the identity on ℓ^1, we have $B_{\ell^1} \subset P(B_H)$, where B_H is the closed unit ball of $H(= E \text{ or } \ell^1)$.

Since

$$\begin{aligned}\|T'(f_n)\| &= \sup_{x\in B_E} |f_n(Tx)| = \sup_{x\in B_E} |f_n(S(Px))| \\ &\geq \sup_{(\alpha_i)_i\in B_{\ell^1}} |f_n(S(\alpha_i))| \geq |f_n(S(e_n))| \geq |f_n(y_n)| > \frac{1}{4},\end{aligned}$$

where $(e_n)_{n=1}^{\infty}$ is the canonical basis of ℓ^1, then the operator T is not limited.

(2) $\Rightarrow$ (1) Let (f_n) be a weak* null sequence in F', we have to prove that $T'(f_n)$ converges to 0 for the norm of E'. Since the norm of E' is order continuous by [**4**, Corollary 2.7], it suffices to show that $|T'(f_n)| \to 0$ in the $\sigma(E', E)$-topology of E' and $T'(f_n)(x_n) \to 0$ for every norm bounded disjoint sequence (x_n) in E^+. Indeed, since the norm of E' is order continuous, it follows from [**4**, Corollary 2.9] that (x_n) is a weakly null sequence in E. Hence, as T is w*DP, we obtain $T'(f_n)(x_n) = f_n(Tx_n) \to 0$.

On the other hand, as (f_n) is a weak* null sequence in F' and since T is order limited, then it follows from Proposition 2.2 that $|T'(f_n)| \to 0$ in the $\sigma(E', E)$-topology of E'. Finally, we deduce that the operator T is limited. □

As a consequence of Theorem 3.1, we obtain the following result

Corollary 3.1. *Let E and F be two Banach lattices such that E has the DP* property (resp., E' has weak* sequentially continuous lattice operations) and F is infinite dimensional. Then the following assertions are equivalent:*

1. *Each order limited (resp., w*DP) operator T from E into F is limited;*
2. *The norm of E' is order continuous.*

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